Simplicity and tracial weights on non-unital reduced crossed products

Yuhei Suzuki (Hokkaido University)

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Defn G : locally cpct group is

1. C^*-simple if

$$C_r^*(G) := \overline{\lambda(C_c(G))}^{\|\cdot\|} \subset \mathbb{B}(L^2G)$$

is simple (no closed $*$ ideal)

2. Unique trace property (UTP)

if the extension of

$ev_e : C_c(G) \longrightarrow \mathbb{C}$

(Plancherel weight)

is the only (proper)

tracial weight on $C^*_\lambda(G)$

(up to scalar, G: unimodular)

<u>Examples</u> · $\mathbb{F}_d$ ($d \geq 2$)

C^*-simple, UTP (Powers '75)

· For discrete groups nice characterizations are known (Kalantar – Kennedy '14)

(e.g. Γ: C^*-simple (resp. UTP)

$\iff \Gamma \curvearrowright \partial_F \Gamma$ free (resp. faithful))

· Non-discrete examples

(S'17, S'21)

First examples

e.g.

$\bigoplus_{\mathbb{N}} \mathbb{F}_2 \rtimes \prod_{\mathbb{N}} \mathbb{Z}_2$

(de la Harpe's problem)

Any l.c.t.d.

$\overset{}{\underset{\text{open}}{\hookrightarrow}}$ C^*-simple

Applications of C^*-simplicity/UTP

Thm (BKKO '17)

- Γ : C^*-simple discrete group $\quad$ A : simple unital C^*-alg.

Then $A \rtimes_r \Gamma$ simple

- Γ : UTP

Then
$$T(A)^\Gamma \xrightarrow{\sim} T(A \rtimes_r \Gamma)$$
$$\tau \mapsto \tau \circ E$$

MT of this talk: Generalize to NON-UNITAL case.

$1_A \in A$: very special

- Invariant under ANY automorphism.
- Compactness phenomena
- traces are bounded

Crucial in the proof of BKKO theorem.

We have to find suitable replacements of 1_A (and new techniques)

Why NON-UNITAL case IMPORTANT?

- Packer–Raeburn stabilization trick unavoidably involves non-unital C^*-algs.

Twisted C^*-dynamical system :

$$1 \to N \to G \to Q \to 1$$

exact seq.

Want to understand (decompose)

$$C_r^* G \quad , \quad A \rtimes_r G$$

by N and Q.

Special Case: $G = N \rtimes Q$

$$\rightsquigarrow \quad C^*_r G = (C^*_r N) \rtimes_r Q$$

$$A \rtimes_r G = (A \rtimes_r N) \rtimes_r Q$$

General case (Mackey, Green, Busby-Smith)

We have a

twisted action

$(\alpha, u) : Q \curvearrowright A \rtimes_r N$ s.t

$$A \rtimes_r G \simeq (A \rtimes_r N) \rtimes_{r,\alpha,u} Q$$

$$\left(\begin{array}{l} u : \text{2-cocycle} \\ \quad \text{associated to} \\ \qquad \text{a cross section} \\ \qquad s : Q \to G \end{array}\right)$$

- Twisted crossed product :

$$\boxed{\begin{array}{ll} \text{Twisted action :} & \alpha : G \to \mathrm{Aut}(A) \\ & u : G \times G \to \mathcal{U}(M(A)) \\ & \qquad\qquad \alpha\text{-2-cocycle} \end{array}}$$

$\longrightarrow A \rtimes_{r,\alpha,u} G$

Similar structures to

the usual crossed product

but more complicated

e.g.) $G \to M(A \rtimes_{r,\alpha,u} G)$

multiplicative only

up to $\mathcal{U}(M(A))$.

The following trick is

quite powerful.

Thm (Packer–Raeburn '89)

$(\alpha, u) : G \curvearrowright A$ twisted action.

Then the stabilization

$(\alpha \otimes 1, u \otimes 1) : G \curvearrowright A \otimes \mathbb{K}(L^2 G)$

can be untwisted.

Hence

$$A \rtimes_{r,\alpha,u} G \underset{\text{stably isom}}{\sim} (A \otimes \mathbb{K}(L^2 G)) \rtimes_{r,\beta} G$$

for a suitable genuine action β.

Return for untwist: $A \otimes \mathbb{K}(L^2 G)$ is non-unital.

Other motivations:

- non-discrete group C^*-algs $C^*_\lambda G$ are non-unital.
- Interesting phenomenon only happen in the non-unital case

e.g. unbounded traces & KMS weights.

Thus we want to extend BKKO theorems to non-unital case!

Main Theorems (S'21)

① Γ : C^*-simple.

Then simplicity of C^*-algs is STABLE under taking arbitrary twisted crossed product

$$\left(\text{i.e. } A : \text{simple} \Longrightarrow A \rtimes_{r,\alpha,u} \Gamma \ \text{simple} \right)$$

② Γ : UTP $\boxed{\Longleftrightarrow_{\text{BKKO}} \nexists 1 \neq N \triangleleft \Gamma \text{ amenable}}$

Then uniqueness of

(of cource up to scalar)

tracial weight is STABLE under taking twisted crossed product of trace preserving twisted actions.

$$\left(\begin{array}{l} \text{i.e. } \quad A : \text{unique trace } \tau \\ \qquad \alpha(G) \subset \mathrm{Aut}(A) \\ \qquad \text{preserves } \tau \\ \\ \Longrightarrow \quad A \rtimes_{r,\alpha,u} \Gamma \\ \qquad \text{has a unique trace.} \end{array} \right)$$

<u>Rem</u> The results are shown by Bryder-Kennedy ('18) in the UNITAL case

Again essential by the same reasons as BKKO

However, in the next

(Main) applications,

NON-UNITAL case

is substantial.

Cor (S'21)

① The class of (locally compact)

C^*-simple groups is

closed under discrete extension:

$$1 \to N \xrightarrow{\text{open}} G \to Q \to 1$$

N, Q: C^*-simple

$\Longrightarrow$ G : C^*-simple.

(2) Analogous result for UTP.

Pf) Use the fact

$$C^*_r G \simeq (C^*_r N) \rtimes_{r,\alpha,u} Q$$

and Main Theorems. $\square$