Equivariant $\mathcal{O}_2$-absorption theorem for exact groups

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Throughout the talk,

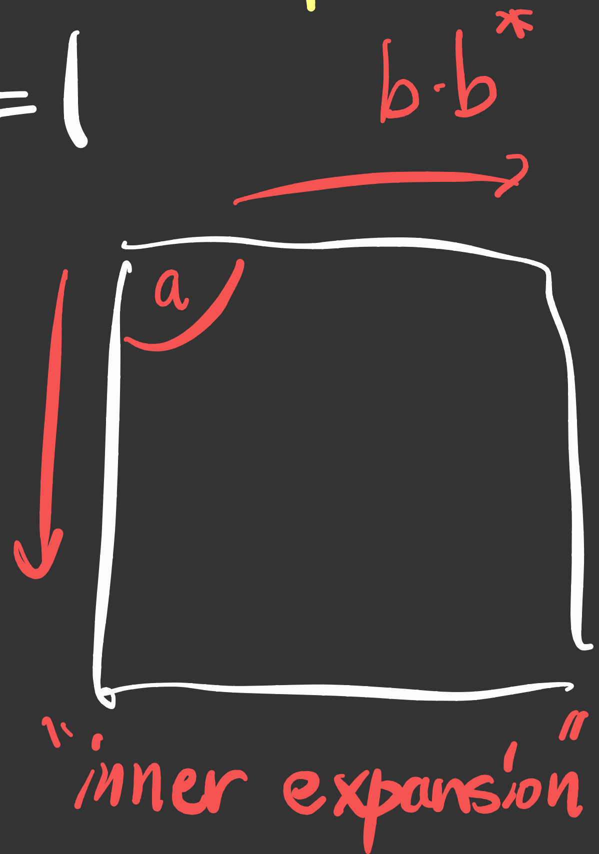
ALWAYS ASSUME:

- C^*-algebras: Unital (nonzero)
  (mostly separable & simple & nuclear)
- G: Countable discrete group

Preliminaries & Backgrounds

A (unital) C^*-alg $\mathcal{A}$ is purely infinite simple

if $\forall a \in A_+ \setminus \{0\}$ $\exists b \in A$ $bab^* = 1$



Cuntz

K-theory rigidity for purely infinite simple A.

$$K_0(A) = \left\{ \begin{matrix} \text{nonzero} \\ \text{proj} \end{matrix} \right\} \Big/ _{M \sim N}, \quad K_1(A) = \mathcal{U}(A) / \mathcal{U}_0(A)$$

Examples ① Cuntz algebras $2 \leq n \leq \infty$

$$\mathcal{O}_n := C^*\left((S_i)_{i=1}^n \;\middle|\; \begin{array}{l} S_i^* S_j = \delta_{ij} \\ \& \ \sum S_i S_i^* = 1 \\ \text{when } n < \infty \end{array} \right)$$

"Smallest" properly infinite C^*-alg

with $(n-1)[1]_0 = 0$

Examples ② G ; nonamenable group

Many minimal action $G \curvearrowright X$

$\rightsquigarrow$ $C(X) \rtimes_r G$ purely ∞ simple

(Laca-Spielberg, Delaroche, ...)

③ Preserved under

$\rtimes_r G$ for outer actions

(Kishimoto)

etc...

Kirchberg–Phillips classification theorem

Recall: Kirchberg algebra

$\overset{\text{def}}{\Longleftrightarrow}$ simple separable nuclear

+ p.i.s.

Thm (Kirchberg, Phillips)

A, B Kirchberg algs.

Then $A \underset{\text{stab}}{\sim} B$

iff $A \underset{KK}{\sim} B$

+ UCT

then iff

$K_*(A) \cong K_*(B)$

Exms of Kirchberg algs (with UCT)

- Cuntz algs $\mathcal{O}_n$
- $C(X) \rtimes_r G$ for many amenable minimal $G \curvearrowright X$

etc

Today's Main Thm : first step to develop its G-equiv version for exact groups (e.g. $\mathbb{F}_n$, $SL(n, \mathbb{Z})$, ...)

Phillips's approach:

Realize $KK(A, -)$

as (asymp) homs $/\sim$

need to · make it a group.

· nice functor

$\longrightarrow$ Elliott intertwining argument

Three key ingredients on $\mathcal{O}_n$
(Kirchberg '94)

① $C \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$ $\forall C$ unital simple separable nuclear

→ Neutralize C $\left(\begin{array}{l} \mathrm{Hom}(\mathcal{O}_2, -) \\ \text{well-understood} \end{array} \right)$

② $\mathcal{O}_\infty \otimes A \simeq A$ $\forall A$ Kirchberg

→ Accept to treat as "$\mathcal{O}_\infty \subset Z(A)$"

Note: $\boxed{\mathcal{O}_\infty \simeq \bigotimes_{1}^{\infty} \mathcal{O}_\infty}$

③ $\exists D \hookrightarrow \mathcal{O}_2$ $\forall D$ sep exact.

$\to$ $\mathrm{Hom}(A, B)$ non-trivial

Def

$D \otimes -$ preserves short exact sequences

Much weaker version of nuclearity

$D \otimes - = D \otimes_{\max} -$

<u>Today's goal</u>: Give G-equiv version of these three for exact countable group G

Exact groups (a.k.a. boundary amenability)

Recall (Kirchberg - Wassermann)

G : exact $\Longleftrightarrow$ $- \rtimes_r G$ preserves G-short exact sequences

($\Longleftrightarrow$ $C^*_r G$: exact)

Thm (Ozawa) G : discrete TFAE

① G : exact

② G has Yu's property A (metric space amenability)

③ $\exists G \curvearrowright X$ amenable action on cpct sp.

<u>Examples</u>

- Amenable groups (trivial)
- linear groups $G < GL(n, K)$
  ($n \in \mathbb{N}$ K: field)
- hyperbolic groups
  ($G \curvearrowright \partial G$ amenable)
- $MCG(\Sigma_g)$
- $Out(\mathbb{F}_n)$ ... etc

Preserved by extensions, increasing unions, amalgamated free products.

Backgrounds on

Noncommutative dynamical systems

1. Classification of

$$G \curvearrowright A$$

2. Amenability of $G \curvearrowright A$

1 $G \curvearrowright A$ & $\mathrm{Aut}(A)$ (for A simple)

Naturally appear in

- Structure/Classification theory
- construction of new examples / presentations

Classification of $G \curvearrowright A$: important

In what sense ?

Obvious choice : Conjugacy (i.e. equivariant isomorphism)

But Conjugacy : too strong !

Ex) $\alpha : \mathbb{Z} \curvearrowright A$

$u \in \mathcal{U}(A)$

$\longrightarrow \alpha^u := \mathrm{ad}\, u \circ \alpha$

Typically $\alpha \not\simeq \alpha^u$

☆ need to solve coboundary equation

$$u = v\,\alpha(v)^*$$

in $\mathcal{U}(A)$

too algebraic ...

Cf. **Thm (S'19)**

G : infinite countable group

Γ : countable group with AP

(e.g. amenable group, $\mathbb{F}_n$, hyperbolic groups, $\mathbb{Z}^2 \rtimes SL_2\mathbb{Z}$, ...)

Then $\exists\, \alpha_\Gamma : G \curvearrowright \mathcal{O}_2$ pointwise outer

s.t. $\mathcal{O}_2^{\alpha_\Gamma} \cong C^*_r\Gamma$.

Nakamura : when $G = \mathbb{Z}$, they only differ as before.

Fortunatery, α and α^u share important things.

- $A \rtimes_\alpha \mathbb{Z} \simeq A \rtimes_{\alpha^u} \mathbb{Z}$
- Induce the same action on $A_\omega = A^\omega \cap A'$

Right identification:

Cocycle conjugacy

<u>Common Sense</u> (Believed for long time)

Classification of $G \curvearrowright A$ requires

amenability of G.

Indeed true for vNalgs

Classification theorem established for amenable G

(Connes, Jones, Ocneanu, Takesaki, Sutherland, Katayama, Kawahigashi, ...) (A: AFD factor)

Non-classification results for non-amenable G

(Jones, M. Choda, Popa, ..., Brothier-Vaes) (on $\mathcal{R}$)

C^*-alg case : Similar dichotomy expected

(e.g. Izumi Conjectures)

Some deep classification thm : by many hands

(Jones, Ocneanu, Kishimoto, Nakamura,
Izumi, Matui , Sato , Szabó , ...)

Today : We will see

this "Common Sense"

is WRONG ($G \curvearrowright A$ classifiable $\Leftrightarrow$ G amenable)

for C^*-algs in GOOD way!

Main Thm (S'20)

G: exact

Then up to cocycle conjugacy,

$\exists ! \, \delta : G \curvearrowright \mathcal{O}_2$

pointwise outer, QAP,

equiv $\mathcal{O}_2$-absorbing action.

amenability type condition

The first classification/uniqueness result beyond amenable G

$\rightarrow$ G-equiv $\mathcal{O}_2$-absorption :

- $\delta \otimes \alpha \underset{\text{c.c.}}{\sim} \delta \quad \forall \alpha : G \curvearrowright A$

(A : simple unital sep. nuc.)

- $\beta \otimes 1_{\mathcal{O}_2} \underset{\text{c.c.}}{\sim} \delta \quad \forall \beta : G \curvearrowright A$ QAP, outer

Note Any condition in MT doesn't follow from the other two

$\rightarrow$ The equiv $\mathcal{O}_2$-thm for exact groups

Motivating results & examples & questions.

Thm (S'17) G : exact

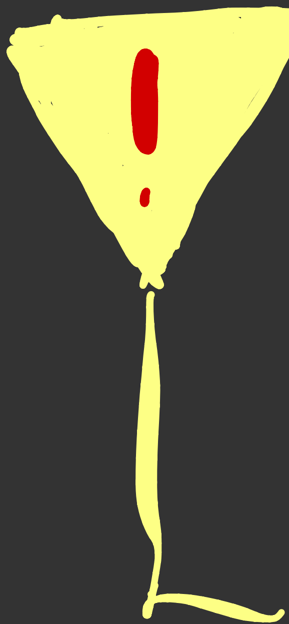
$\exists\, G \curvearrowright \mathcal{O}_2$

$\mathcal{O}_2 \rtimes G = \mathcal{O}_2 \rtimes_r G$

$\simeq \mathcal{O}_2$

nuclear

amenability-like conditions!



$\nexists$ such actions on a factor (unless G amenable) (Delaroche '79)

$\longrightarrow$ Really C^*-algebraic new phenomena.

Ex Take $G \curvearrowright X_n$ amenable $\mathrm{ad}(2^{n}|G)$

$$C(X_n) \to C(X_n) \rtimes G \overset{2^n}{\hookrightarrow} \mathcal{O}_2 \curvearrowleft G$$

equiv embedding.

$$\rightsquigarrow \delta : G \curvearrowright^{\bigotimes_{n=1}^{\infty} \mathrm{ad}(2^n)} \bigotimes_{n=1}^{\infty} \mathcal{O}_2 \quad (\cong \mathcal{O}_2)$$

pointwise outer, QAP, $\mathcal{O}_2$-abs.

How to witness amenability of $G \overset{\delta}{\curvearrowright} A$?

Good Case : $X_n = X$

for ∞ many n

$\longrightarrow$ $C(X) \underset{\text{equiv}}{\hookrightarrow} \big((\mathcal{O}_2)_\omega , \delta_\omega \big)$

General choice : not obvious

(In fact, the same is TRUE by MT!)

Some abstract formulation :

QAP (Quasi-central Approximation Property
Buss-Echterhoff-Willett '19)

$\exists$ Square root of the Følner-type partition of unity

$$\subset \text{“}\ell^2(A)_\omega\text{”}$$

! Need a slight reformulation via $\ell^2(A_\omega)$

Equivariant $\mathcal{O}_2$-absorption thm

- In fact previously known

for
- $\mathbb{Z}$ (Nakamura)
- finite groups (Izumi)

⋮

- amenable groups (Szabó)

Proof: Follow Kirchberg, (Bury C inside $(\mathcal{O}_2)_\omega$)

but need to use

$\tilde{v} \in (\mathcal{O}_2)^\beta_\omega$ ⟶ Equivariant intertwining argument works!

nice v: Constructed by averaging over Følner set

Our situation : $\nexists$ Følner sets

But with coefficients,

we have nice averagings!

$\longrightarrow$ Some fixed point algebras are purely infinite simple

$\xrightarrow{\text{Cuntz}}$ Connes's 2×2 trick!!

We also get

- $\mathcal{O}_\infty$-abs results for QAP actions on Kirchberg algebras

share many outer $G \curvearrowright \mathcal{O}_\infty$ as $\otimes$-component

- Appropriate analogue of $\mathcal{O}_2$-emb thm

Thank You for your attension!