C^*-simplicity has no local obstraction

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1. Background & Setting

2. Previous Construction (S'16)

Solution of de la Harpe's problem

but NOT satisfactory ... why?

3. New Construction

→ The title follows

1. Background & setting of C^*-simplicity

G : locally compact group

admits natural unitary repn.

$$\lambda : G \curvearrowright L^2(G)$$

left translation

(called left regular repn)

Special among unitary repns

$\rightsquigarrow$ Fell's Absorption Principle

Linearize λ

$\rightsquigarrow C_c(G) \curvearrowright L^2(G)$

convolution action.

$$C^*_\lambda G := \overline{C_c(G)}^{\|\cdot\|} \subset B(L^2G)$$

Important (analytic / repn theoretic)

properties of G are reflected.

(One of) ultimate goals of OA:

Better Understanding of

G via $C^*_\lambda(G)$

Today's topic C^*-simplicity

<u>Def</u> G : C^*-simple

$\overset{\text{def}}{\Longleftrightarrow}$ $C^*_\lambda G$: simple

(i.e. no proper closed $*$-ideal)

$\Longleftrightarrow$ "minimality" of λ under weak containment

$\pi_1 \prec \pi_2$

matrix coefficients of π_1 simulated in π_2

Why important?

- Provide interesting examples of simple C^*-algs of natural background
- Better understanding of λ & related repns

$\rho \prec \lambda \Rightarrow$ same operator norms e.g. $H < G$ amenable

$$\Rightarrow \lambda_{G,H} : G \curvearrowright L^2(G/H)$$

$$\prec \lambda$$

- Technics extend to more general Op. Alg.

[e.g. free product,
crossed product]

Examples (discrete case)

- Free groups $\mathbb{F}_n$ (Powers '67)

$$\frac{1}{N}\sum_{i=1}^{N} \lambda_{s_i} x \lambda_{s_i}^{*} \to \tau(x)$$

averaging argument
based on combinatoric

actually works
to many interesting groups.

Characterization (discrete)

<u>Thm</u> (Kalantar – Kennedy '14)

Γ (discrete) C^*-simple

$\Longleftrightarrow$ $\Gamma \curvearrowright \partial_F \Gamma$ (topologically) free

geometric (non-O.A.) property

$\longrightarrow$ easily reprove previously known results.

–FIN–

How about non-discrete case?

Many new difficulties: e.g.

- $C^*_\lambda G$: non-unital
- "Canonical weight" unbounded
- $G \subset M(C^*_\lambda G)$ but $\not\subset C^*_\lambda G$

- $G \curvearrowright L^\infty G$ discontinuous

Continuous part: not injective.

⇝ proof for discrete groups:

not applicable

C^*-simplicity for non-discrete G:

still quite mysterious

Main Result: 1st examples of

TRULY NON-DISCRETE C^*-simple groups

(precisely, NON-ELEMENTARY)

[Wesolek]

2. Known examples

— why they are NOT satisfactory?

<u>Thm</u> (S'16)

G : lcg.

Assume :

$\exists$

$\cdots < K_n < K_{n-1} < \cdots < G_{n-1} < G_n < \cdots$

s.t.

① K_n cpct open in G

② $K_n \triangleleft G_n$

③ G_n / K_n C^*-simple

④ $K_n \searrow \{e\}$, $G_n \nearrow G$

Then G : C^*-simple

Ex: $\left(\bigoplus_{\mathbb{N}} \mathbb{F}_2\right) \rtimes \prod_{\mathbb{N}} \mathbb{Z}_2$

easy to produce by composing discrete groups

$\longrightarrow$ Solved de la Harpe's Problem

[$\exists$? non discrete C^*-simple] ('05 in fact 80s?)

Why not satistactory?

Because: Structure of G & $C^*_\lambda G$

: almost discrete.

(TOO ELEMENTARY)

- Averaging projection p_K :

For compact open $K < G$

$$p_K := \frac{1}{m(K)} \int_K \lambda_g \, dg$$

$$\in C_c(G) \subset C^*_\lambda G$$

: proj onto $\ell^2(K \backslash G)$.

$$p_K C_c(G) p_K = \mathbb{C}[K\backslash G/K]$$

Hecke algebra : Unital, canonical basis

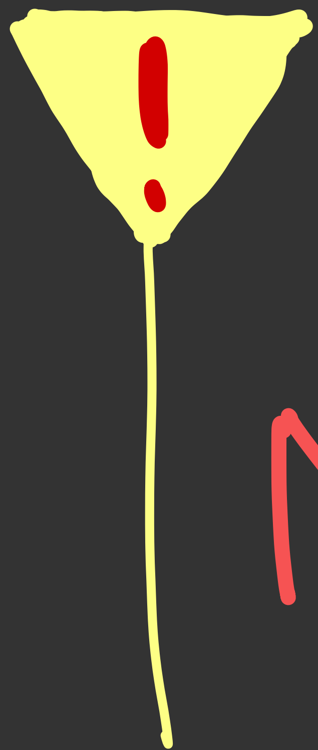
Pf of Thm S'16 :

$$
\begin{array}{c}
C^*_\lambda(G_n) \subset C^*_\lambda(G) \\
\cup \\
p_{K_n} C^*_\lambda(G_n) p_{K_n} \\
\cong \\
C^*_\lambda(G_n/K_n)
\end{array}
\longleftarrow \text{SIMPLE (assump.)}
$$

As $K_n \downarrow \{e\}$, $G_n \nearrow G$,

$$P_{K_n} C^*_\lambda(G_n) P_{K_n} \nearrow C^*_r(G)$$

$\square$



$C^*_\lambda G$: NOT REALLY NEW C^*...

Q. Is

non-discrete C^*-simplicity

REALY

interesting subject…?

A. (S'21) Yes.

3. Main Thm.

Thm (S'21)

$\forall$ Totally disconnected $G \xhookrightarrow{\text{open}} \exists \mathcal{G}$

C^*-simple

Many interesting t.d. G .

rejects open emb into groups from S'16

$SL_n(\mathbb{Q}_p)$

$Aut(T_d)$

etc

Non-elementary

$\cancel{\xhookrightarrow{\text{open}}}$ elementary (Wesolek)

Construction

G : f.d. grp

Ingredients : $E_n := \mathbb{Z}$ trivial G-action

$$\Upsilon_n := \bigoplus_{\substack{K<G \\ \text{cpct open}}} \bigoplus_{G/K} \mathbb{Z}_2$$

G-action : left shift $G \curvearrowright G/K$

Fact (van Dantzig '36)

G : t.d.

Then $\{K : \underset{\text{cpct open}}{K < G}\}$

forms a local basis at $e \in G$

Now set

$$\Gamma_1 = \Upsilon_1 \qquad \Lambda_1 = \Gamma_1 * \Xi_1$$

$$\Gamma_2 = \Lambda_1 \times \Upsilon_2 \qquad \Lambda_2 = \Gamma_2 * \Xi_2$$

$$\vdots \qquad\qquad \vdots$$

$$\Gamma_1 < \Lambda_1 < \Gamma_2 < \Lambda_2 < \cdots$$

(G-embeddings)

$\times\ \Upsilon_n$ $\rightsquigarrow$ "Central Free"

$*\ \Xi_n$ $\rightsquigarrow$ Simple

(Inspired by Vaes group)

$$\rightsquigarrow \quad \Lambda := \bigcup_n \Lambda_n$$

$$= \left(\left(\left(\left(\Gamma_1 * \Xi_1\right) \times \Gamma_2\right) * \Xi_2\right) \times \Gamma_3\right) * \cdots$$

$$G \curvearrowright \Lambda$$

Thm $\mathcal{G} := \Lambda \rtimes G$

is C^*-simple.

Lemma (replacement of canonical cond. exp.)

$K < G$ cpct open

$G \curvearrowright A$

Then

$\exists E_K : p_K (A \rtimes_r G) p_K \longrightarrow p_K A p_K$

faithful condi. exp.

$E_K(p_K a \lambda_s p_K) = \chi_K(s) p_K a p_K.$

Proof $0 \neq I \lhd C^*_r(\mathcal{G})$ given.

ISTS: $P_K \in I$

$\forall K < G$ cpct open

(van Dantzig)

$$E_K : P_K C^*_r(\mathcal{G}) P_K \longrightarrow C^*_r(\Lambda)^K P_K$$

WMA $P_k I P_k \neq 0$

(replace by smaller k)

Pick $a = P_k x P_k \overset{?}{\neq} 0$

$x \in I$.

$\rightarrow E_k(a) \neq 0$

WMA $\|E_k(a)\| = 1$

Pick $a_0 \in p_K C_c(\Lambda_n \rtimes G) p_K$ positive with

$$\begin{cases} \bullet\ \| a - a_0 \| < \frac{1}{2} \\ \bullet\ \| E_K(a_0) \| = 1 \end{cases}$$

$$a_0 = p_K a_0 p_K = \sum_{s \in K \backslash G / K} p_K x_s \lambda_s p_K$$

$x_s \in \mathbb{C}[\Lambda_n]$

x_e : K-inv

Point: Λ_n commute with

$$\bigoplus_{G/K} \mathbb{Z}_2 < \Upsilon_{n+1}$$

$$G \curvearrowright C^*_r\left(\bigoplus_{G/K} \mathbb{Z}_2\right)$$

$$\cong$$

$$C\left(\prod_{G/K} \{0,1\}\right)$$

$\rightsquigarrow \exists p \in C_r^*\left(\bigoplus_{G/K} \mathbb{Z}_2\right)^K$

nonzero p_j'.

$p a_0 p = \chi_e p \, p_K$

(Certain freeness of

$G \curvearrowright \prod_{G/K} \{0,1\}$)

Note:

$$\|x_e p\| = \|x_e\| \|p\| = 1$$

Observe:

$$x_e p \in C_r^*(\Gamma_{n+1})^k \subset C_r^*(\Gamma_{n+1})^k \underset{\tau}{*} C_r^*(E_{n+1})$$

- SIMPLE!

(Dykeme)

- K-invariant

$\rightsquigarrow \exists\, b_1, \dots, b_r \in B$

- $\sum b_i \, x p_e \, b_i^* = 1_B$

- $\left\| \sum b_i b_i^* \right\| \leq 2.$

$$\rightsquigarrow \sum b_i \, p \, a_0 \, p \, b_i^*$$

$$= \sum b_i \, p \, \lambda e \, b_i^* \, p_k$$

$$= p_k.$$

$$I \ni \sum b_i p a p b_i^* \underset{2 \times \frac{1}{2}}{\approx} p_k$$

Since I: ideal

$$p_k \in I \qquad \square$$

☆ Further results :

- Uniquess of KMS weight (w.r.t $\mathrm{Ad}(\Delta^{it})$)
- Factoriality of $L(\mathcal{G})$

Its Murray - vN - Connes type