

Equivariant O_2 -absorption

Theorem for Exact Groups

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'20 9/7 @ Zoom (RIMS)

Today's topics

① Non-commutative amenable action

- How to formulate? ('70s ~ 2020)

- Examples? ('17 ~)

② Classification

- equiv. version of O_2 -absorption

theorems for EXACT groups
beyond amenable groups

Conventions

• C^* -algebras : always unital
($\neq 0$) most case simple, separable, nuclear

• Groups Γ, G : always
countable, discrete, ex

① (NC) amenable actions.

Amenable actions (Zimmer)

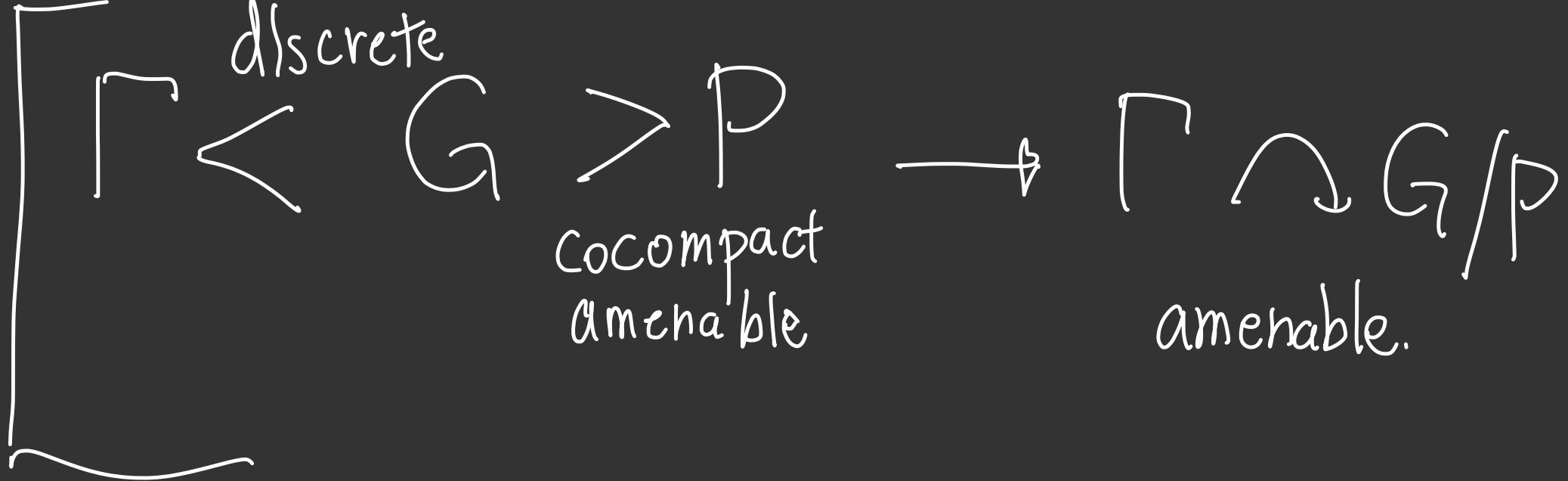
Powerful concept to
capture amenable sides
of non-amenable groups.

Ex) $\mathbb{F}_d \curvearrowright \partial \mathbb{F}_d = \left. \begin{array}{l} \text{one sided} \\ \infty \text{ reduced} \\ \text{words} \end{array} \right\}$

$\mu_n: \partial \mathbb{F}_d \rightarrow \text{Prob } \mathbb{F}_d$

$w \mapsto \frac{1}{n} \sum_{k=1}^n w_k$
1st kth words

\exists Følner-type distributions



Important applications

- Zimmer cocycle superrigidity
- Baum-Connes Conjecture
- von Neumann algebras
- orbit equivalences

G : exact group

Ozawa

$\iff \exists G \curvearrowright X$

amenable

on cpct space

fairly mild (exterior)

version of amenability

(a.k.a. boundary amenability)

Familiar grps
are exact

- Linear groups
 $G \subset GL(n, k)$
- hyperbolic
grps
- $MCG(\Sigma)$
- $Out(F_d)$

⋮
⋮

NonCommutativity : (Sometimes)

Clearer, simpler,

more fruitful (e.g. classification)

NC amenable action?


BAD NEWS: ~~is~~ Interesting example

Thm (Delaroché '79) in W^* -setting...

M : vN alg Then

$\Gamma \curvearrowright M$ amenable

$\iff \Gamma \curvearrowright \mathcal{Z}(M)$ amenable

 (factor) $\not\curvearrowright$ (non amenable) never amenable!

Similar defn/result expected
in C^* -setting (Delaroché)

Problem ('02 Delaroché)

$A : C^*$ -alg Γ : non-amenable

$$A \rtimes \Gamma = A \rtimes_r \Gamma$$

$\stackrel{??}{\implies} \mathcal{Z}(A)$ non-trivial??

($\Gamma \curvearrowright \mathcal{Z}(A)$ amenable??)

A. No!! (S'17)

Thm (S'17)

Γ : exact Then

$\exists \Gamma \curvearrowright \mathcal{O}_2$

$$\mathcal{O}_2 \rtimes \Gamma = \mathcal{O}_2 \rtimes_r \Gamma$$

$$\simeq \underline{\mathcal{O}_2}$$

amenable

A correct formulation ('20):

$$\text{via } A_\omega = A^\omega \cap A'$$

- useful / characterize
in classification (Main Theorem)

- Equivalent to

several known amenability-type defns

('20 Buss - Echterhoff - Willett)

② O_2 -absorption theorem

for exact group actions

(1st Classification/Characterization
beyond amenable group)

• \mathcal{O}_2 - absorption theorem

Thm (Kirchberg '94)

A : simple unital separable nuclear

Then $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$

$$\mathcal{O}_2 = C^*(s_1, s_2 \mid \begin{array}{l} s_1^* s_1 = s_2^* s_2 \\ s_1 s_1^* + s_2 s_2^* = 1 \end{array})$$

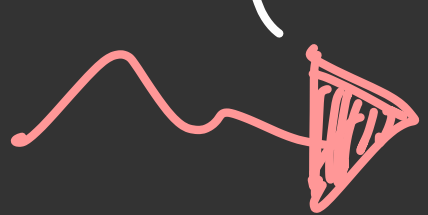
Cuntz algebra

Cuntz algebra \mathcal{O}_2 :

Most plain / homogeneous C^* -alg.

• \forall Exact C^* -algs $\exists \hookrightarrow \mathcal{O}_2$

• $\text{Hom}(\mathcal{O}_2, A)$ well-understood



Crucial ingredient in

Phillips's classification theorem

Today's Goal (Main Thm)

Develop the equivariant

version for exact groups

Background on NC dynamical systems

$\Gamma \curvearrowright A$

Motivations

- Important in Classification/Structure Theory of operator algs
- Powerful approach to provide new

Examples & Presentations

Classification Problem

... In what sense ... ?

Obvious Choice :
conjugacy (i.e. equiv. isom) $(A, \alpha) \simeq (B, \beta)$

Usually **TOO STRONG!!**

$$\text{Ex) } d: \mathbb{Z} \curvearrowright A$$

$$u \in \mathcal{U}(A)$$

$$\longrightarrow d^u = \text{ad } u \circ d$$

[cocycle perturbation]

$$\text{Typically } d \not\cong d^u$$

almost need to solve coboundary eq.

$$\text{too algebraic...} \rightarrow u = \mathcal{V} d(\mathcal{V})^* \text{ in } \mathcal{U}(A)$$

Cf. Thm (S '19)

Γ : infinite group Then

$\forall G$: AP group.

$\exists \Gamma \curvearrowright \mathcal{O}_2$ (e.g. \mathbb{F}_d , amenable grps)

$$\mathcal{O}_2^\Gamma \cong C_r^* G$$



Nakamura's thm : When $\Gamma = \mathbb{Z}$ they only differ up to cocycle perturbation.

Correct Problem : Classify

Up to Cocycle Conjugacy

Indeed natural : $(\underbrace{\quad}_{c.c.})$

- have the same
crossed products
- * Conjugate on $A_\omega = A^\omega \rtimes A'$

When classifiable?

Common Sense / Expectation

Iff Γ is amenable.

Indeed TRUE for
von Neumann algebras

DICHOTOMY: $\Gamma \curvearrowright \mathcal{R}$: classifiable
by many people) $\Leftrightarrow \Gamma$: amenable

C^* -setting : Similar dichotomy expected

e.g. Conjectures by Izumi

- Many Successful Classification Theorems for amenable groups
- (Partial) non-classification for nonamenable groups

Main Theorem :

This is WRONG

in good way !

Main Thm (S' 20) Γ : exact.

Then up to cocycle conjugacy

$\exists \delta : \Gamma \curvearrowright \mathbb{Q}_2$

amenable, outer, \mathbb{Q}_2 -absorbing

action.

Consequences: $\delta: \Gamma \curvearrowright \mathcal{O}_2$ as above

(Absorption Thm) A : simple unital sep. nuc C^* -alg.

Then: $\bullet \forall \alpha: \Gamma \curvearrowright A$.

$$\alpha \otimes \delta \underset{\text{c.c.}}{\simeq} \delta$$

$\bullet \forall \alpha: \Gamma \curvearrowright A$ amenable outer action

$$\alpha \otimes \text{id}_{\mathcal{O}_2} \underset{\text{c.c.}}{\simeq} \delta.$$

Motivating example / model of δ :

$\alpha_n: \Gamma \curvearrowright X_n$ amenable action
 \uparrow compact

Fix $C(X_n) \subset C(X_n) \rtimes \Gamma \xrightarrow{z_n} \mathcal{O}_2$

$\beta_n := \text{ad}(z_n|_{\Gamma}) : \Gamma \curvearrowright \mathcal{O}_2$

$\delta := \otimes \beta_n : \Gamma \curvearrowright (\otimes \mathcal{O}_2) \cong \mathcal{O}_2$ **desired**
(

- amenable
- outer
- \mathcal{O}_2 -absorbing

)

Equivariant O_2 -absorption thm.

Previously known for

- \mathbb{Z} Nakamura
- finite groups Izumi
- \vdots
- amenable groups (Szabó)

How to prove MT?

follow Kirchner (bury A in)

$$A \otimes \mathcal{O}_2 \simeq \mathcal{O}_2 \quad \left((\mathcal{O}_2)_\omega \right)$$

equivariant replacement :

$$(\mathcal{O}_2)_\omega^\delta$$

fixed point
(= equivariant)
elements

How to provide

$$\forall \epsilon \in (\mathbb{Q}_2)_\omega^\delta \quad ???$$

Ans. AVERAGING

amenable groups:

over Følner sets $F \subset \Gamma$

Our situation : no nice FCT.

But with coefficients

\exists nice averaging!

$$\left(\mathcal{L}^2(G) \rightsquigarrow \mathcal{L}^2(G, (\mathcal{O}_2)_\omega) \right)$$

$(\mathcal{O}_2)_\omega$ rich enough

→ Connes's 2×2 Trick

→ Intertwining arguments

We also have

- Equiv. O_∞ -absorption thm

$$(A \otimes O_\infty \simeq A)$$

for exact groups

amenable
action
shares
many outer
actions

- Equivariant analogue
of O_2 -embedding thm