Equivariant $\mathcal{O}_2$-absorption Theorem for Exact Groups

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Today's topics

① Non-commutative amenable action

- How to formulate? ('70s $\sim$ 2020)
- Examples? ('17 $\sim$)

② Classification

- equiv. version of $\mathcal{O}_2$-absorption theorems for EXACT groups

beyond amenable groups

Conventions

- C^*-algebras ; always unital

($\neq 0$) most case simple, separable, nuclear

- Groups Γ, G, ; always

countable, discrete, ex

① (NC) amenable actions.

Amenable actions (Zimmer)

Powerful concept to capture amenable sides of non-amenable groups.

Ex) $\mathbb{F}_d \curvearrowright \partial \mathbb{F}_d = \left\{ \begin{array}{c} \text{one sided} \\ \infty \text{ reduced} \\ \text{words} \end{array} \right\}$

$$\mu_n : \partial \mathbb{F}_d \longrightarrow \mathrm{Prob}\, \mathbb{F}_d$$
$$w \longmapsto \frac{1}{n} \sum_{k=1}^{n} w_k$$

$\leftarrow$ 1st kth words

$\exists$ Følner-type distributions

$$\left[\ \Gamma \overset{\text{discrete}}{<} G > \underset{\substack{\text{cocompact}\\ \text{amenable}}}{P} \implies \Gamma \curvearrowright \underset{\text{amenable.}}{G/P}\right.$$

Important applications

- Zimmer cocycle superrigidity
- Baum–Connes Conjecture
- von Neumann algebras
- orbit equivalences

G : exact group

$\overset{\text{Ozawa}}{\Longleftrightarrow}$ $\exists\, G \curvearrowright X$ amenable on cpct space

fairly mild (exterior) version of amenability

(a.k.a. boundary amenability)

Familiar grps are exact

- Linear groups $G < GL(n,k)$
- hyperbolic grps
- $MCG(\Sigma)$
- $Out(\mathbb{F}_d)$

⋮

NonCommutativity : (Sometimes)

Clearer, simpler,

more fruitful (e.g. classification)

NC amenable action ?

BAD NEWS : $\nexists$ Interesting example in W^*-setting...

Thm (Delaroche '79)

M : vN alg Then

$\Gamma \curvearrowright M$ amenable

$\Longleftrightarrow$ $\Gamma \curvearrowright Z(M)$ amenable

⚠ (factor) $\overline{\forall}$ $\binom{\text{non}}{\text{amen}}$ never amenable !

Similar defn/result expected in C^*-setting (Delaroche)

Problem ('02 Delaroche)

A : C^*-alg Γ : non-amenable

$A \rtimes \Gamma = A \rtimes_r \Gamma$

$\overset{??}{\Longrightarrow}$ $Z(A)$ non-trivial??

($\Gamma \curvearrowright \widehat{Z(A)}$ amenable??)

A. No!! (S'17)

Thm (S'17)

Γ ; exact Then

$\exists\, \Gamma \curvearrowright \mathcal{O}_2$

$$\mathcal{O}_2 \rtimes \Gamma = \mathcal{O}_2 \rtimes_r \Gamma$$

$$\simeq \underbrace{\mathcal{O}_2}_{\text{amenable}}$$

A correct formulation (S'20):

via $A_\omega = A^\omega \cap A'$

- useful/characterize in classification (Main Theorem)
- Equivalent to several known amenability-type defns ('20 Buss - Echterhoff - Willett)

② $\mathcal{O}_2$-absorption theorem

for exact group actions

(1st Classifiaction/Characterization
beyond amenable group)

- $\mathcal{O}_2$-absorption theorem

Thm (Kirchberg '94)

A : simple unital separable nuclear

Then $A \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$

$$\mathcal{O}_2 = C^*\left(S_1, S_2 \;\middle|\; \begin{array}{l} S_1^* S_1 = S_2^* S_2 \\ = S_1 S_1^* + S_2 S_2^* \end{array}\right)$$

Cuntz algebra

Cuntz algebra $\mathcal{O}_2$;

Most plain / homogeneous C^*-alg.

- $\forall$ Exact C^*-algs $\exists \hookrightarrow \mathcal{O}_2$
- $\mathrm{Hom}(\mathcal{O}_2, A)$ well-understood

Crucial ingredient in Phillips's classification theorem

Today's Goal (Main Thm)

Develop the equivariant version for exact groups

Background on NC dynamical systems

$\Gamma \curvearrowright A$

Motivations

- Important in Classification/Structure Theory of operator algs
- Powerful approach to provide new Examples & Presentations

Classification Problem

… In what sense … ?

Obvious Choice :

conjugacy $\left(\begin{matrix}\text{i.e.}\\ \text{equiv. isom}\end{matrix}\right)$ $(A, \alpha) \simeq (B, \beta)$

Usually TOO STRONG!!

Ex) $\alpha : \mathbb{Z} \curvearrowright A$

$u \in \mathcal{U}(A)$

$\longrightarrow \quad \alpha^{u} = \mathrm{Ad}\, u \circ \alpha$

[cocycle perturbation]

Typically $\alpha \not\simeq \alpha^{u}$

almost need to solve coboundary eq.

too algebraic... $\longrightarrow$ $u = v\,\alpha(v)^{*}$ in $\mathcal{U}(A)$

Cf. Thm (S '19)

Γ : infinite group Then

$\forall G$: AP group.

(e.g. $\mathbb{F}_d$, amenable grps)

$\exists \Gamma \curvearrowright \mathcal{O}_2$

$$\mathcal{O}_2^{\Gamma} \cong C^*_r G$$

⚠ Nakamura's thm : When $\Gamma = \mathbb{Z}$ they only differ up to cocycle perturbation.

Correct Problem : Classify

Up to Cocycle Conjugacy ($\underset{\text{c.c.}}{\sim}$)

Indeed natural :

- have the same crossed products
- Conjugate on $A_\omega = A^\omega \cap A'$

When classifiable?

Common Sense / Expectation

Iff Γ is amenable.

Indeed TRUE for von Neumann algebras

DICHOTOMY: $\Gamma \curvearrowright \mathcal{R}$: classifiable

(by many people) $\iff \Gamma$: amenable

C^*-setting : similar dichotomy expected

e.g. Conjectures by Izumi

- Many Successful Classification Theorems for amenable groups
- (Partial) non-classification for nonamenable groups

Main Theorem :

This is WRONG

in good way !

Main Thm (S'20) Γ : exact.

Then up to cocycle conjugacy

$\exists 1\ \delta : \Gamma \curvearrowright \mathcal{O}_2$

amenable, outer, $\mathcal{O}_2$-absorbing

action.

Consequences: (Absorption Thm)

$\delta : \Gamma \curvearrowright \mathcal{O}_2$ as above

A : simple unital sep. nuc C^*-alg.

Then:

- $\forall \alpha : \Gamma \curvearrowright A$.

$$\alpha \otimes \delta \underset{\text{c.c.}}{\simeq} \delta$$

- $\forall \alpha : \Gamma \curvearrowright A$ amenable outer action

$$\alpha \otimes 1_{\mathcal{O}_2} \underset{\text{c.c.}}{\simeq} \delta.$$

Motivating example/model of δ:

$\alpha_n : \Gamma \curvearrowright X_n$ amenable action

compact

Fix $C(X_n) \subset C(X_n) \rtimes \Gamma \overset{\iota_n}{\hookrightarrow} \mathcal{O}_2$

$\beta_n := \mathrm{ad}(\iota_n|_\Gamma) : \Gamma \curvearrowright \mathcal{O}_2$

$\delta := \bigotimes \beta_n : \Gamma \curvearrowright \left(\bigotimes \mathcal{O}_2\right) \simeq \mathcal{O}_2$

desired

(· amenable
· outer
· $\mathcal{O}_2$-absorbing)

Equivariant $\mathcal{O}_2$-absorption thm.

Previously known for

- $\mathbb{Z}$ Nakamura
- finite groups Izumi

$\vdots$

- amenable groups (Szabó)

How to prove MT?

follow Kirchberg (bury A in $(\mathcal{O}_2)_\omega$)

$A \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$

equivariant replacement:

$(\mathcal{O}_2)_\omega^\delta$ fixed point (= equivariant elements)

How to provide

$v \in (\mathcal{O}_2)_\omega^\delta$??

Ans. AVERAGING

amenable groups :

over Følner sets $F \subset \Gamma$

Our situation : no nice $F \subset \Gamma$.

But with coefficients

$\exists$ nice averaging !

$$\left(\ell^2(G) \rightsquigarrow \ell^2(G, (\mathcal{O}_2)_\omega) \right)$$

$(\mathcal{O}_2)_\omega^\delta$ rich enough

$\longrightarrow$ Connes's 2×2 Trick

$\longrightarrow$ Intertwining arguments

We also have

- Equiv. $\mathcal{O}_\infty$-absorption thm $(A \otimes \mathcal{O}_\infty \simeq A)$ for exact groups
- Equivariant analogue of $\mathcal{O}_2$-embedding thm

amenable action shares many outer actions