Feb. 18, 2010 (北海道大学談話会)

# 超平面配置のトポロジー

## 極小セル分割とその周辺

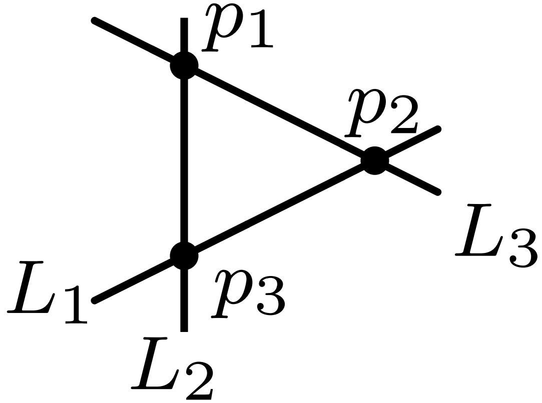
Masahiko Yoshinaga

Kyoto University

A *hyperplane arrangement* is a collection

$$\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$$

of affine hyperplanes $H_i \subset \mathbb{C}^\ell$ (or $H_i \subset \mathbb{P}^\ell$).

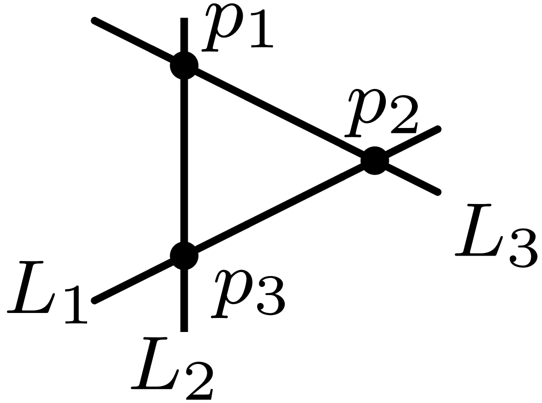



Combinatorics

A *hyperplane arrangement* is a collection

$$\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$$

of affine hyperplanes $H_i \subset \mathbb{C}^\ell$ (or $H_i \subset \mathbb{P}^\ell$).




Topology of the complement

$$M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup H_i$$

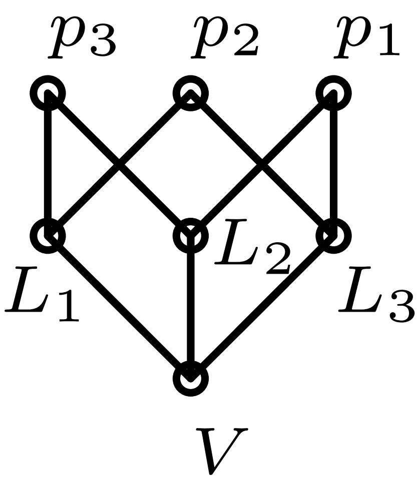
Combinatorics

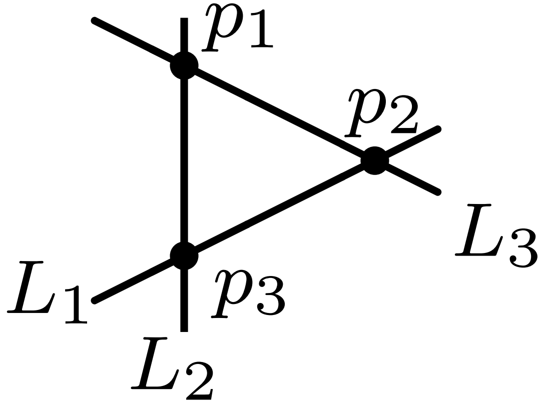
A *hyperplane arrangement* is a collection

$$\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$$

of affine hyperplanes $H_i \subset \mathbb{C}^\ell$ (or $H_i \subset \mathbb{P}^\ell$).

Combinatorial structures

$L(\mathcal{A}) =$ 


Topology of the complement

$$M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup H_i$$

Combinatorics

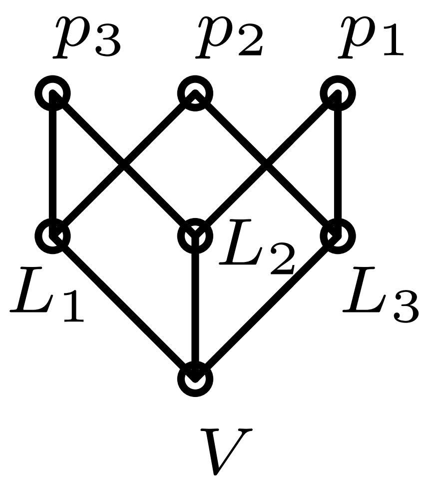
A *hyperplane arrangement* is a collection

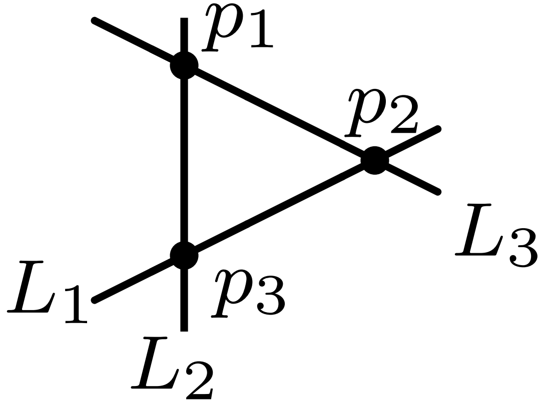
$$\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$$

of affine hyperplanes $H_i \subset \mathbb{C}^\ell$ (or $H_i \subset \mathbb{P}^\ell$).

Combinatorial structures

$L(\mathcal{A}) =$




Chambers

Topology of the complement

$$M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup H_i$$

Combinatorics controlls geometry via chambers.

# Homotopy type and cell decomposition

$$M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup_{H \in \mathcal{A}} H$$

Example,

# Homotopy type and cell decomposition

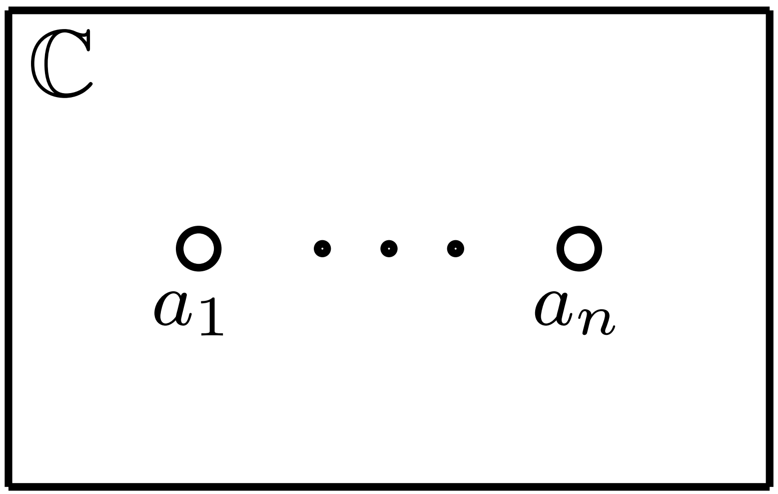
$$M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup_{H\in\mathcal{A}} H$$

Example, $\ell = 1$: $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{C}$.

# Homotopy type and cell decomposition

$$M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup_{H \in \mathcal{A}} H$$

Example, $\ell = 1$: $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{C}$.

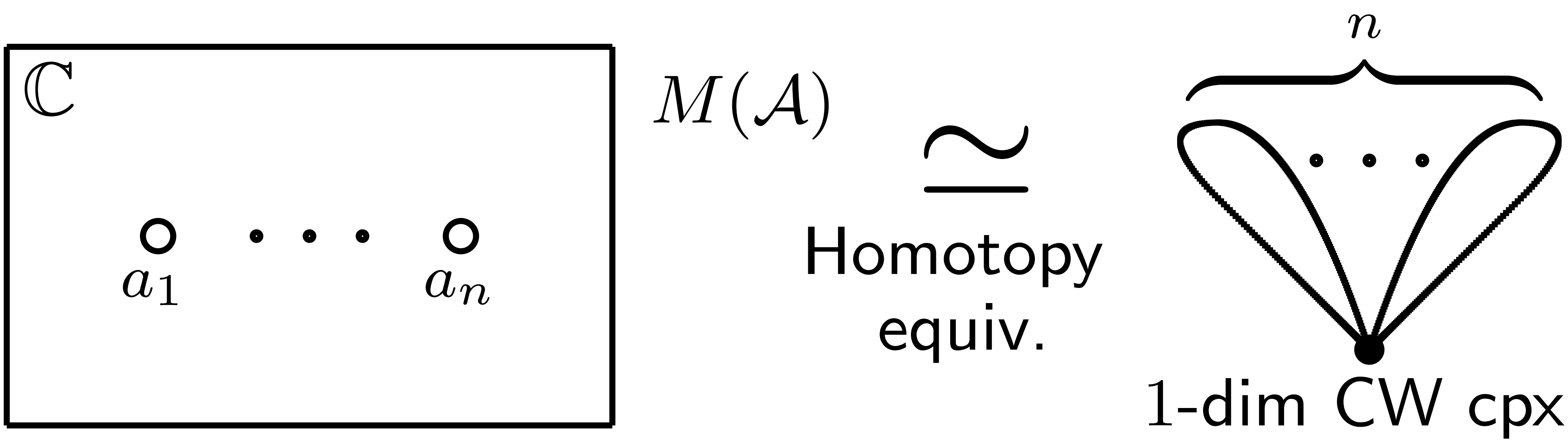



$M(\mathcal{A})$

# Homotopy type and cell decomposition

$$M(\mathcal{A}) = \mathbb{C}^{\ell} - \bigcup_{H \in \mathcal{A}} H$$

Example, $\ell = 1$: $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{C}$.



Example, $\ell = 2$, $\mathcal{A} = \{xy = 0\}$.

Example, $\ell = 2$, $\mathcal{A} = \{xy = 0\}$.

$$M(\mathcal{A}) = \{(x, y) \mid xy \neq 0\}$$

Example, $\ell = 2$, $\mathcal{A} = \{xy = 0\}$.

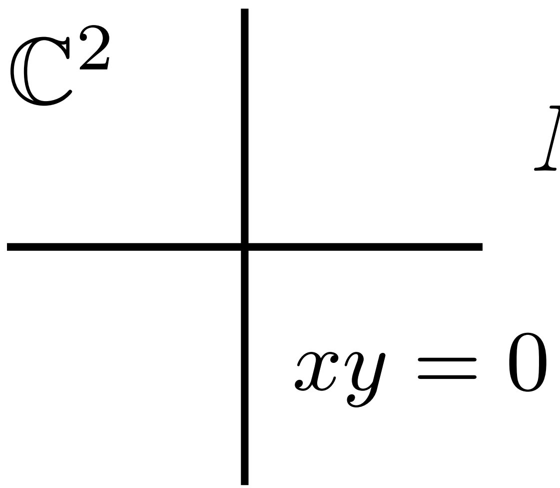
$$M(\mathcal{A}) = \{(x, y) \mid xy \neq 0\}$$
$$= \mathbb{C}^* \times \mathbb{C}^*$$

Example, $\ell = 2$, $\mathcal{A} = \{xy = 0\}$.

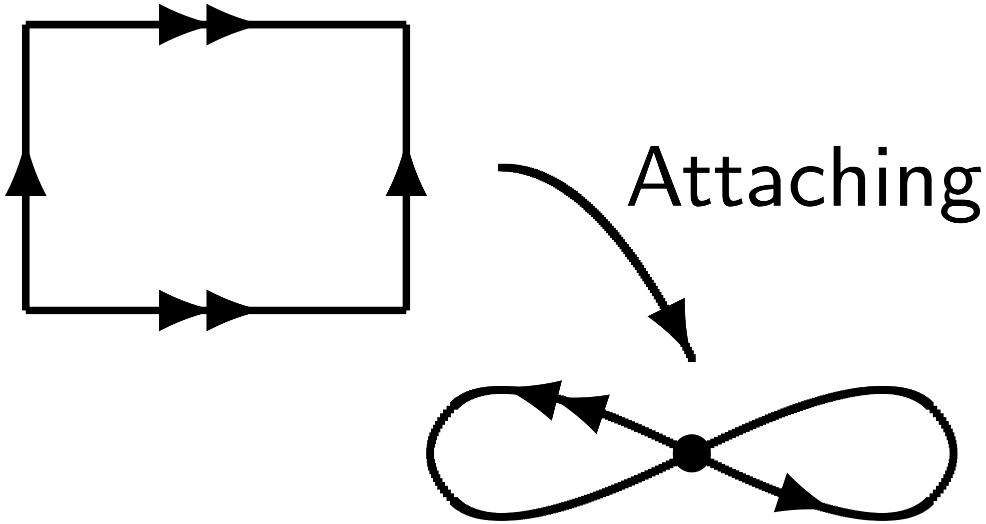
$$\begin{aligned} M(\mathcal{A}) &= \{(x, y) \mid xy \neq 0\} \\ &= \mathbb{C}^* \times \mathbb{C}^* \\ &\simeq S^1 \times S^1 \end{aligned}$$

# Example, $\ell = 2$, $\mathcal{A} = \{xy = 0\}$.

$$M(\mathcal{A}) = \{(x, y) \mid xy \neq 0\}$$
$$= \mathbb{C}^* \times \mathbb{C}^*$$
$$\simeq S^1 \times S^1$$




$M(\mathcal{A}) \simeq$



# 0 Contents

# 0 Contents

# 0 Contents

1. Aomoto's observation.
2. Minimality of $M(\mathcal{A})$.
   (Dimca, Papadima, Suciu, Randell)

# 0 Contents

1. Aomoto's observation.
2. Minimality of $M(\mathcal{A})$.
   (Dimca, Papadima, Suciu, Randell)
3. Real cases.

# 0 Contents

1. Aomoto's observation.
2. Minimality of $M(\mathcal{A})$.
   (Dimca, Papadima, Suciu, Randell)
3. Real cases.
4. Application to local systems.

# 0 Contents

1. Aomoto's observation.
2. Minimality of $M(\mathcal{A})$.
   (Dimca, Papadima, Suciu, Randell)
3. Real cases.
4. Application to local systems.
   (arXiv:1002.2038)

# 1 Aomoto's observation

# 1 Aomoto's observation

concerning dimensions of local system homology groups

$$\dim H_k(M(\mathcal{A}), \mathcal{L})$$

for rank one local system $\mathcal{L}$ on $M(\mathcal{A})$.

## 1.1 Local system homology groups

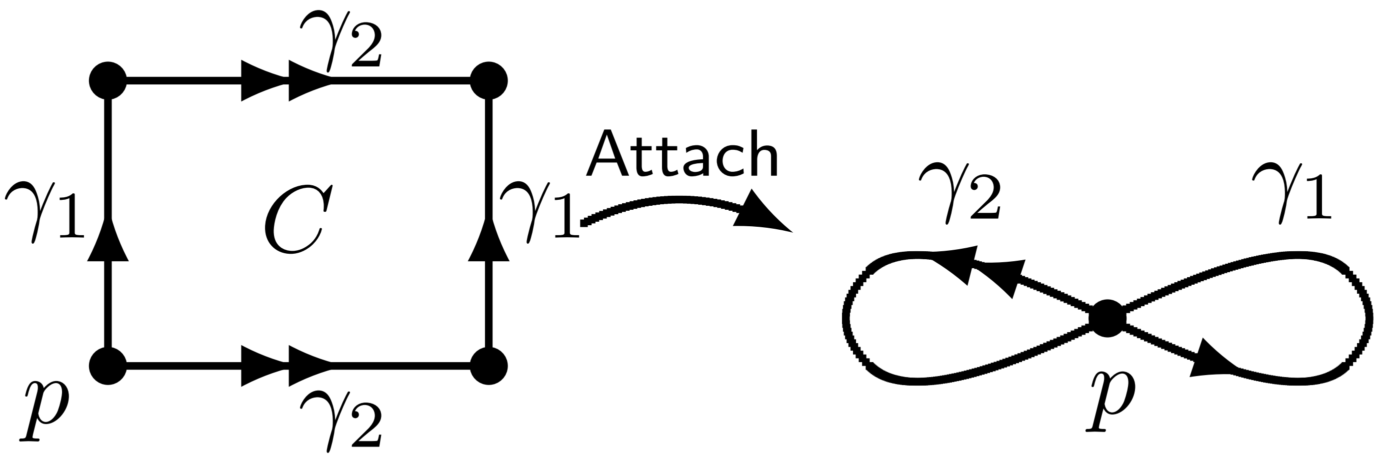
## 1.1 Local system homology groups

$\ell = 2$, $\mathcal{A} = \{xy = 0\}$.

## 1.1 Local system homology groups

$\ell = 2$, $\mathcal{A} = \{xy = 0\}$.

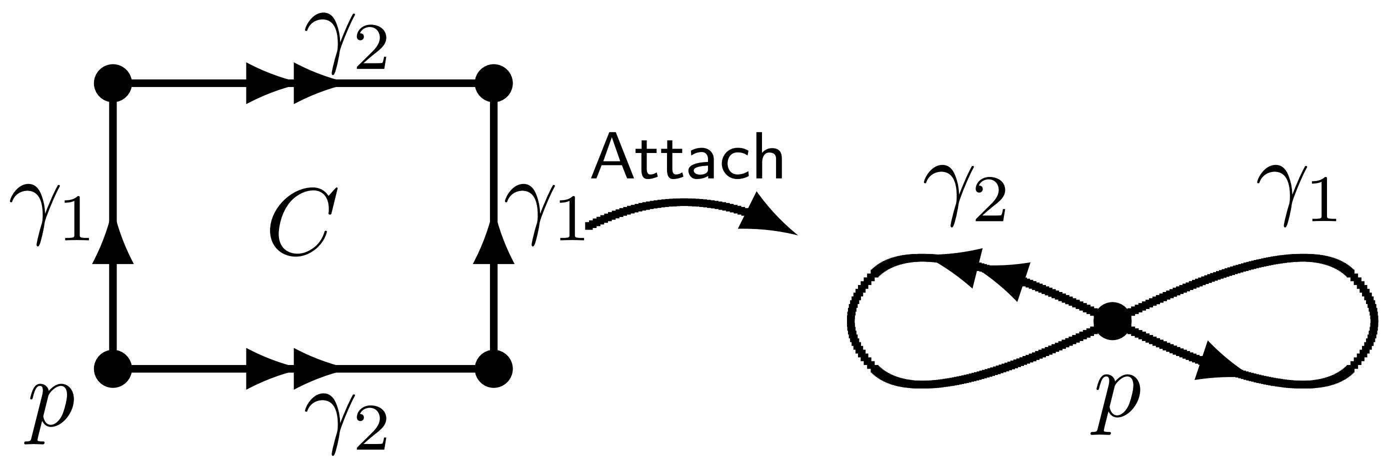
$$M(\mathcal{A}) \simeq$$



## 1.1 Local system homology groups

$\ell = 2$, $\mathcal{A} = \{xy = 0\}$.

$M(\mathcal{A}) \simeq$

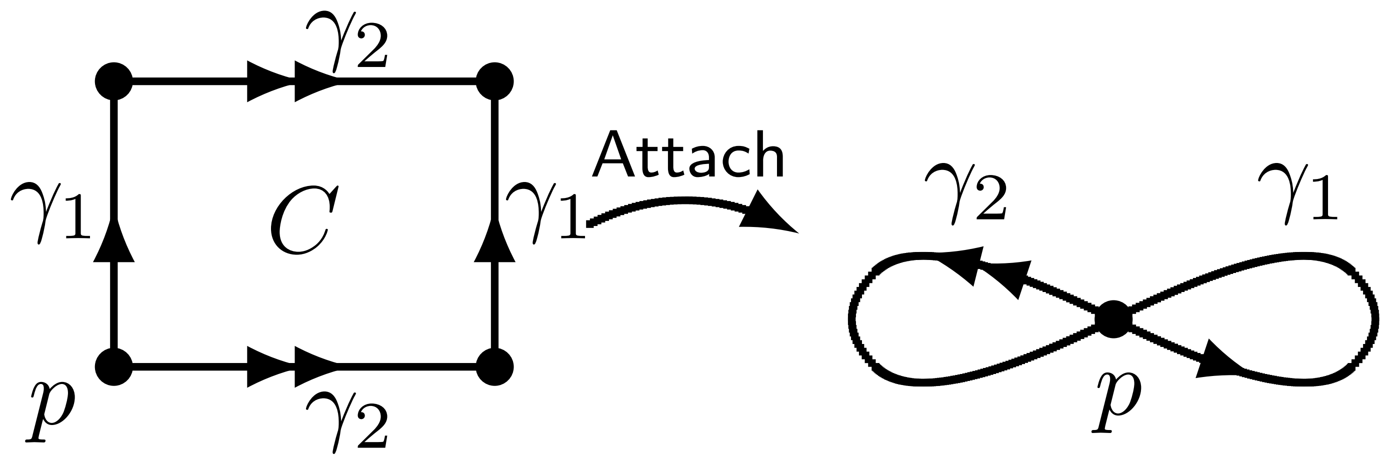



A local system $\mathcal{L}$ is determined by

$$\rho : \pi_1(M(\mathcal{A})) \longrightarrow \mathbb{C}^*,$$

i.e. by $\rho([\gamma_1]) = t_1, \rho([\gamma_2]) = t_2 \in \mathbb{C}^*$.

## 1.1 Local system homology groups

$M \simeq$ 


Chain complex

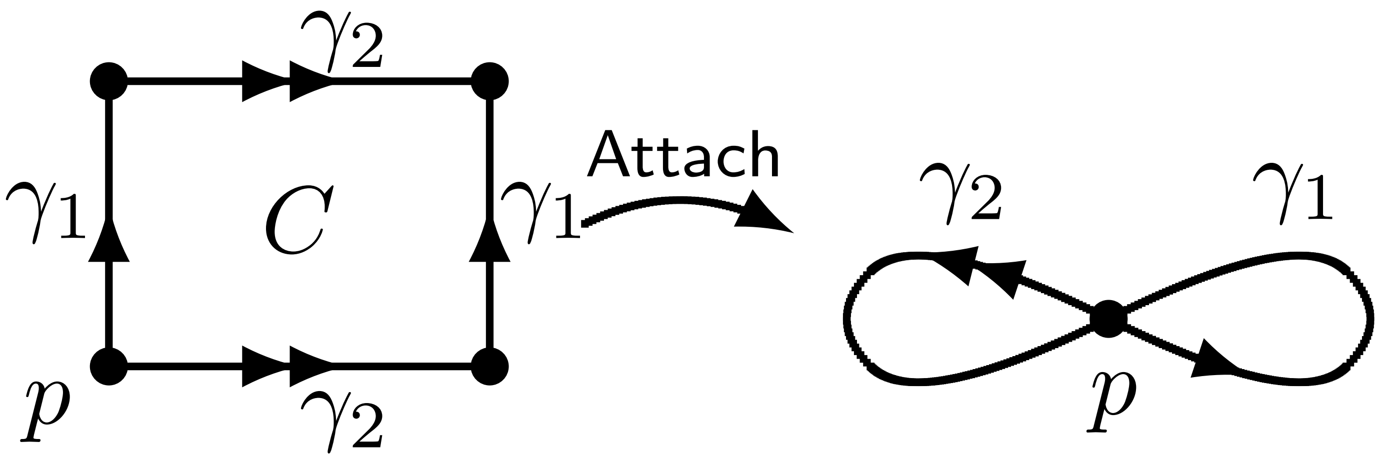
$$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

$$[C] \longmapsto [\gamma_2] + [\gamma_1] - [\gamma_2] - [\gamma_1]$$

$$[\gamma_1] \longmapsto [p] - [p]$$

$$[\gamma_2] \longmapsto [p] - [p]$$

## 1.1 Local system homology groups

$M \simeq$ 


Chain complex (Twisted by $\mathcal{L}$):

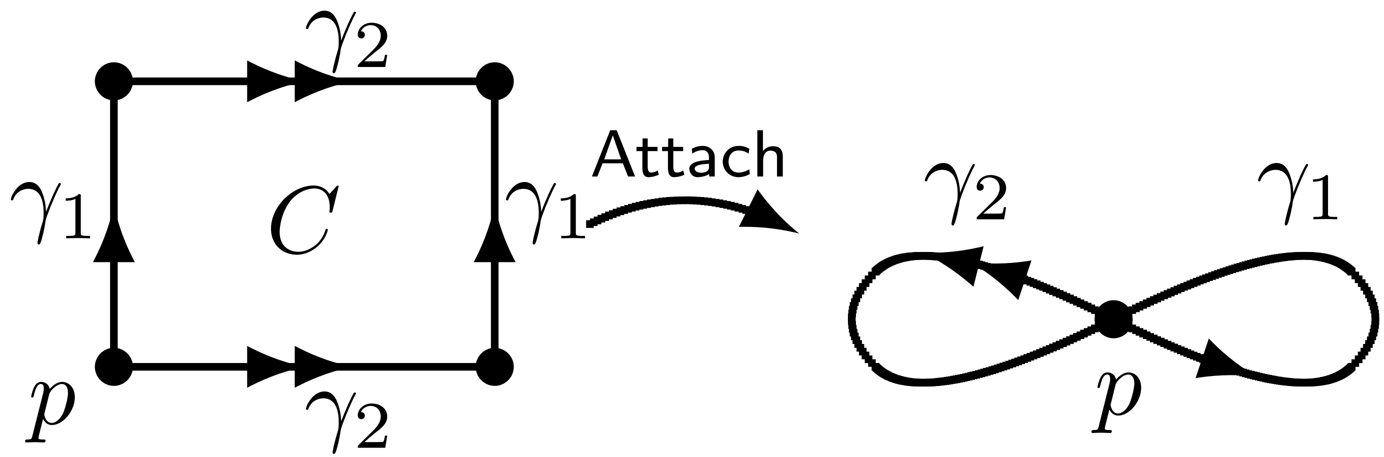
$$C_2 \xrightarrow{\partial_{\mathcal{L}}} C_1 \xrightarrow{\partial_{\mathcal{L}}} C_0$$

$$[C] \longmapsto [\gamma_2] + t_2[\gamma_1] - t_1[\gamma_2] - [\gamma_1]$$

$$[\gamma_1] \longmapsto t_1[p] - [p]$$

$$[\gamma_2] \longmapsto t_2[p] - [p]$$

## 1.1 Local system homology groups

$M \simeq$ 


Chain complex (Twisted by $\mathcal{L}$):

$$C_2 \xrightarrow{\partial_{\mathcal{L}}} C_1 \xrightarrow{\partial_{\mathcal{L}}} C_0$$

$$[C] \longmapsto (t_2 - 1)[\gamma_1] - (t_1 - 1)[\gamma_2]$$

$$[\gamma_1] \longmapsto (t_1 - 1)[p]$$

$$[\gamma_2] \longmapsto (t_2 - 1)[p]$$

## 1.1 Local system homology groups

$$C_2 \xrightarrow{\partial_{\mathcal{L}}} C_1 \xrightarrow{\partial_{\mathcal{L}}} C_0$$

$$[C] \longmapsto (t_2 - 1)[\gamma_1] - (t_1 - 1)[\gamma_2]$$

$$[\gamma_1] \longmapsto (t_1 - 1)[p]$$

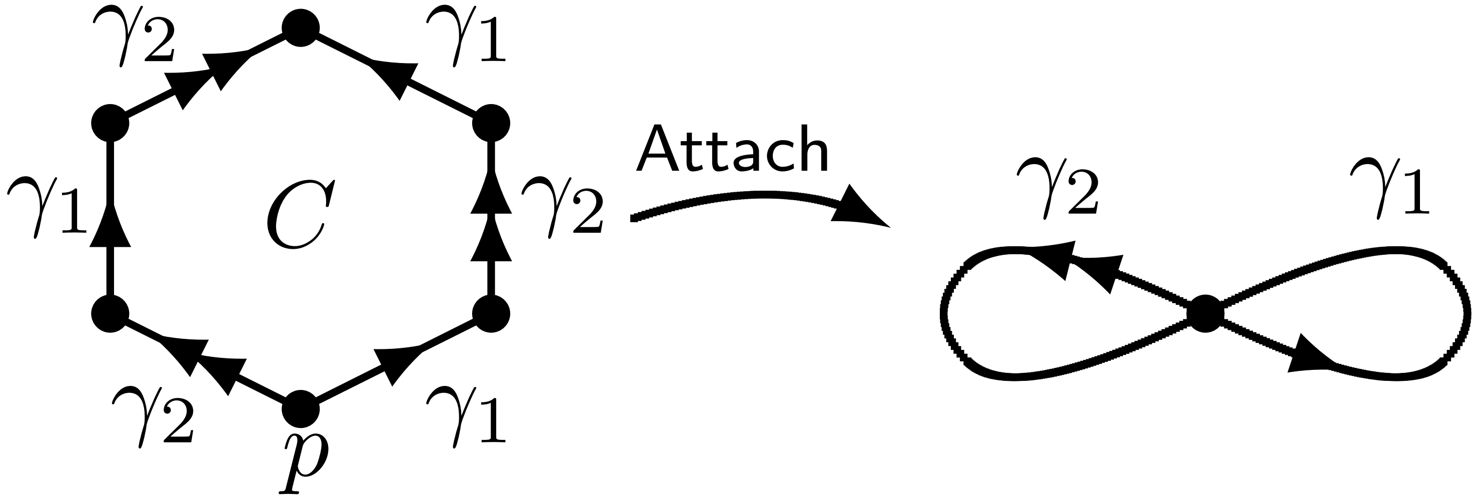
$$[\gamma_2] \longmapsto (t_2 - 1)[p]$$

| $\mathcal{L}$ | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| Trivial $(t_i = 1)$ | $\mathbb{C}$ | $\mathbb{C}^2$ | $\mathbb{C}$ |
| not trivial | $0$ | $0$ | $0$ |

# 1.1 Local system homology groups

Example.

$X =$

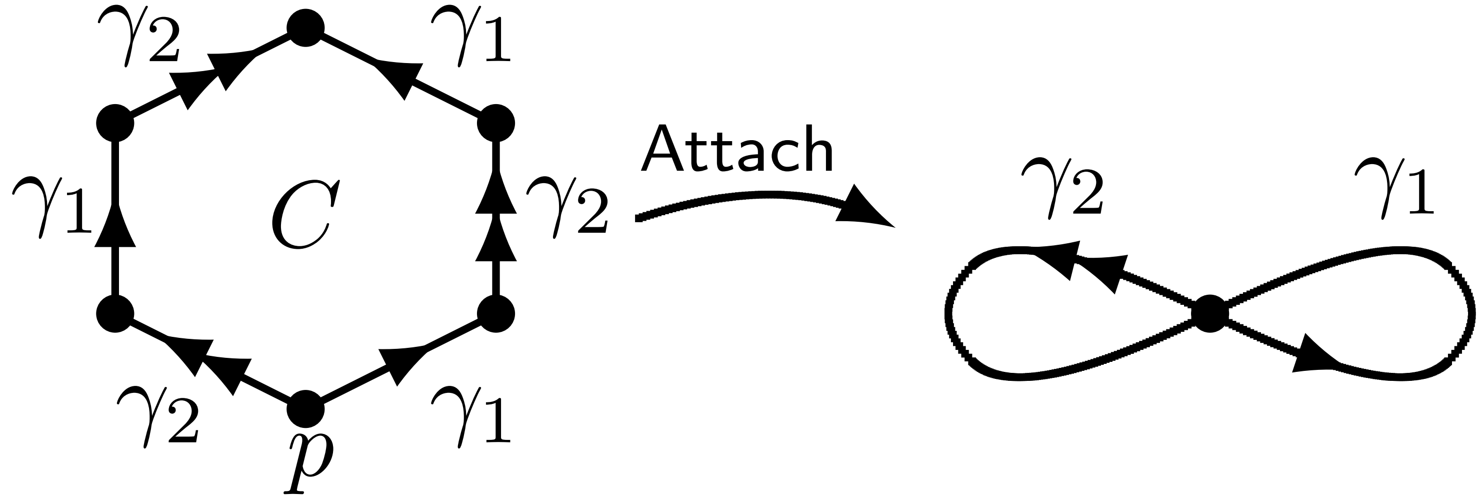



Remark:

# 1.1 Local system homology groups

Example.

$$X =$$

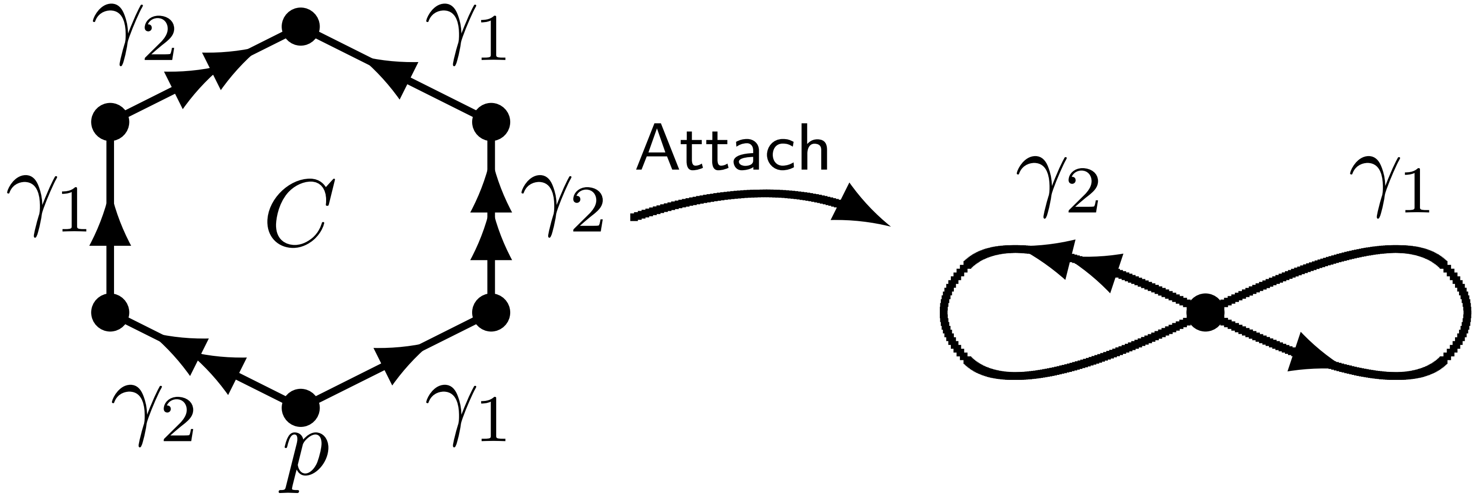



Remark: $X \simeq \{x^2 - y^3 \neq 0\}$.

## 1.1 Local system homology groups

Example.

$$X =$$



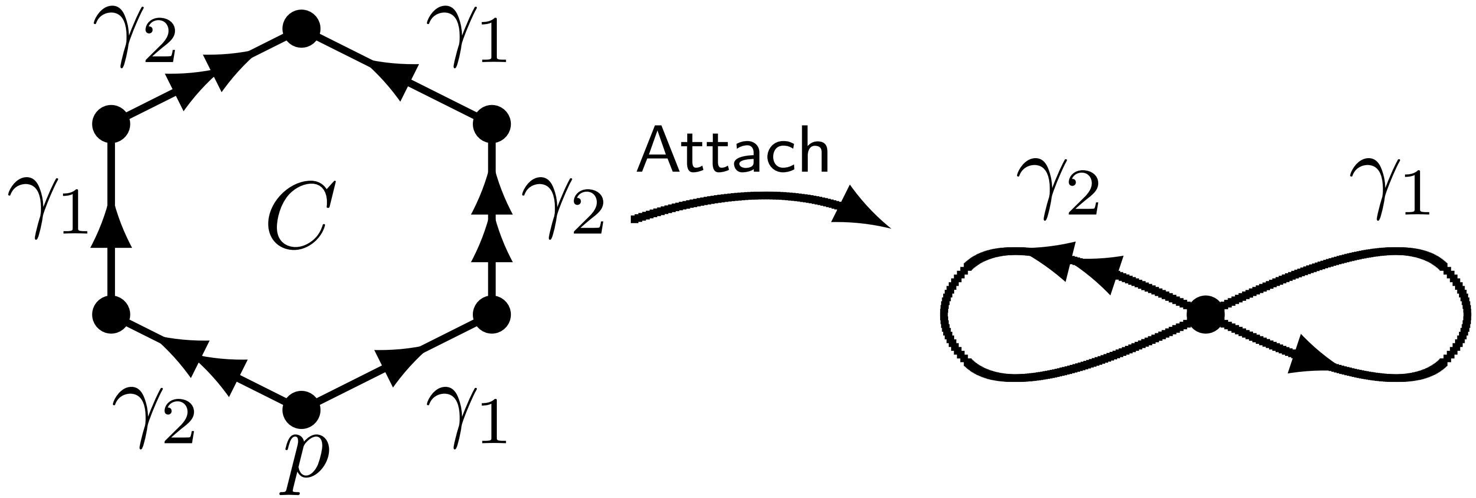

Remark: $X \simeq \{x^2 - y^3 \neq 0\}$.

Since $\partial([C]) = [\gamma_1] - [\gamma_2]$, a local system $\mathcal{L}_t$ is determined by

$$\rho([\gamma_1]) = \rho([\gamma_2]) =: t \in \mathbb{C}^*.$$

## 1.1 Local system homology groups

$X =$




Chain complex with $\mathcal{L}_t$-coefficients:

$$C_2 \xrightarrow{\partial_{\mathcal{L}_t}} C_1 \xrightarrow{\partial_{\mathcal{L}_t}} C_0$$

$$[C] \longmapsto (1-t+t^2)[\gamma_1] - (1-t+t^2)[\gamma_2]$$

$$[\gamma_1] \longmapsto (t-1)[p]$$

$$[\gamma_2] \longmapsto (t-1)[p]$$

## 1.1 Local system homology groups

$$C_2 \xrightarrow{\partial_{\mathcal{L}_t}} C_1 \xrightarrow{\partial_{\mathcal{L}_t}} C_0$$

$$[C] \longmapsto (1 - t + t^2)([\gamma_1] - [\gamma_2])$$

$$[\gamma_1] \longmapsto (t-1)[p]$$

$$[\gamma_2] \longmapsto (t-1)[p]$$

| $\mathcal{L}_t$ | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| Trivial ($t = 1$) | $\mathbb{C}$ | $\mathbb{C}$ | $0$ |
| $t = e^{\pm \pi i/3}$ | $0$ | $\mathbb{C}$ | $\mathbb{C}$ |
| others | $0$ | $0$ | $0$ |

## 1.2 Aomoto's observation

## 1.2 Aomoto's observation

| $\{xy \neq 0\}$ | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $\mathcal{L}$ : trivial | $\mathbb{C}$ | $\mathbb{C}^2$ | $\mathbb{C}$ |
| not trivial | $0$ | $0$ | $0$ |

| $\{x^2 \neq y^3\}$ | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $\mathcal{L}_t$ : Trivial | $\mathbb{C}$ | $\mathbb{C}$ | $0$ |
| $t = e^{\pm \frac{\pi i}{3}}$ | $0$ | $\mathbb{C}$ | $\mathbb{C}$ |
| others | $0$ | $0$ | $0$ |

## 1.2 Aomoto's observation

| $\{xy \neq 0\}$ | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $\mathcal{L}$ : trivial | $\mathbb{C}$ | $\mathbb{C}^2$ | $\mathbb{C}$ |
| not trivial | $0$ | $0$ | $0$ |

| $\{x^2 \neq y^3\}$ | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $\mathcal{L}_t$ : Trivial | $\mathbb{C}$ | $\mathbb{C}$ | $0$ |
| $t = e^{\pm \frac{\pi i}{3}}$ | $0$ | $\mathbb{C}$ | $\mathbb{C}$ |
| others | $0$ | $0$ | $0$ |

$\mathcal{A}$: a hyperplane arrangement, $\mathcal{L}$: a rank one local system on the complement $M(\mathcal{A})$. Aomoto conjectured:

## 1.2 Aomoto's observation

| $\{xy \neq 0\}$ | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $\mathcal{L}$ : trivial | $\mathbb{C}$ | $\mathbb{C}^2$ | $\mathbb{C}$ |
| not trivial | $0$ | $0$ | $0$ |

| $\{x^2 \neq y^3\}$ | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $\mathcal{L}_t$ : Trivial | $\mathbb{C}$ | $\mathbb{C}$ | $0$ |
| $t = e^{\pm \frac{\pi i}{3}}$ | $0$ | $\mathbb{C}$ | $\mathbb{C}$ |
| others | $0$ | $0$ | $0$ |

$\mathcal{A}$: a hyperplane arrangement, $\mathcal{L}$: a rank one local system on the complement $M(\mathcal{A})$. Aomoto conjectured:

$$\dim H_i(M(\mathcal{A}), \mathcal{L}) \leq b_i(M(\mathcal{A})).$$

## 1.2 Aomoto's observation

Aomoto's conjecture:

$$\dim H_i(M(\mathcal{A}), \mathcal{L}) \leq b_i(M(\mathcal{A}))$$

## 1.2 Aomoto's observation

Aomoto's conjecture:

$$\dim H_i(M(\mathcal{A}), \mathcal{L}) \leq b_i(M(\mathcal{A}))$$

was proved by D. Cohen.

## 1.2 Aomoto's observation

Aomoto's conjecture:

$$\dim H_i(M(\mathcal{A}), \mathcal{L}) \le b_i(M(\mathcal{A}))$$

was proved by D. Cohen.

$\Longleftarrow$ (Stronger result):

## 1.2 Aomoto's observation

Aomoto's conjecture:

$$\dim H_i(M(\mathcal{A}), \mathcal{L}) \leq b_i(M(\mathcal{A}))$$

was proved by D. Cohen.

$\Longleftarrow$ (Stronger result):

"Minimality of $M(\mathcal{A})$"

# 2 Minimality of $M(\mathcal{A})$

# 2 Minimality of $M(\mathcal{A})$

## 2.1 Minimal CW-complex

# 2 Minimality of $M(\mathcal{A})$

## 2.1 Minimal CW-complex

Def. A finite CW-cpx $X$ is *minimal* if

$$\sharp(k\text{-cells}) = b_k(X), \text{ for } k \geq 0.$$

# 2 Minimality of $M(\mathcal{A})$

## 2.1 Minimal CW-complex

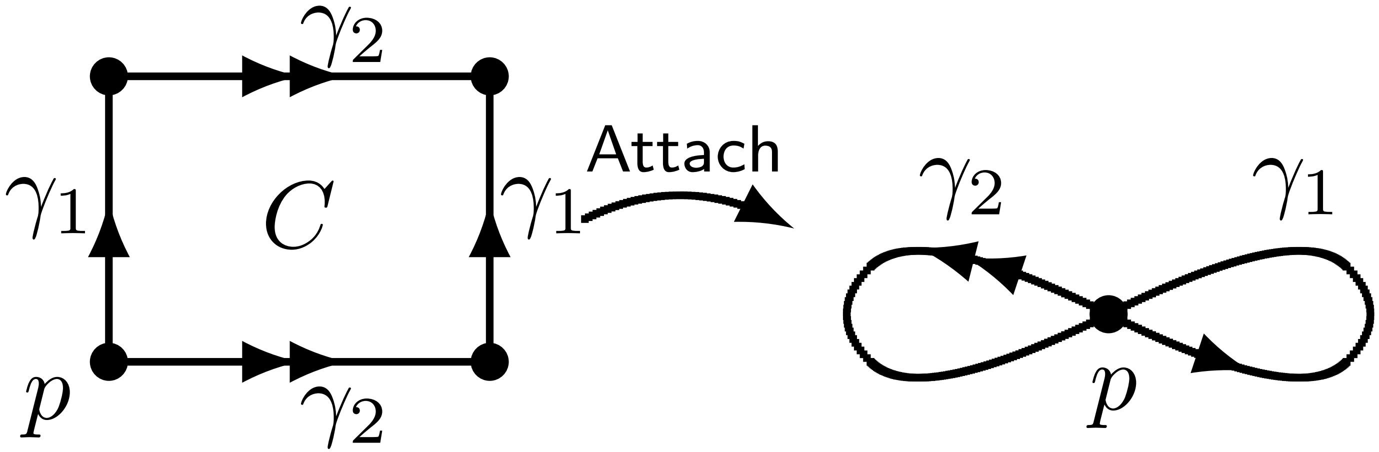
Def. A finite CW-cpx $X$ is *minimal* if

$$\sharp(k\text{-cells}) = b_k(X), \text{ for } k \geq 0.$$

Rem. In general,

$$\sharp(k\text{-cells}) \geq b_k(X).$$

## 2.1 Minimal CW-complex

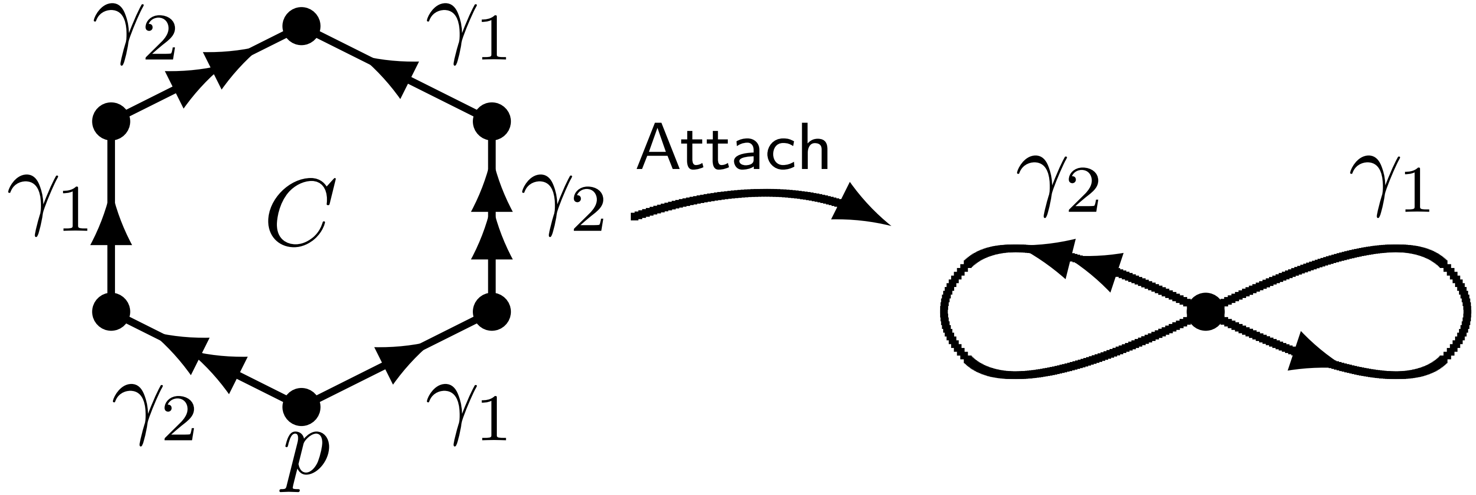
$M \simeq$ 


$M$ is minimal. Indeed

| $k$ | 0 | 1 | 2 |
|---|---|---|---|
| $b_k(M)$ | 1 | 2 | 1 |
| $\sharp$ of $k$-cells | 1 | 2 | 1 |

## 2.1 Minimal CW-complex

$X =$




$X$ is not minimal.

| $k$ | 0 | 1 | 2 |
|---|---|---|---|
| $b_k(M)$ | 1 | 1 | 0 |
| $\sharp$ of $k$-cells | 1 | 2 | 1 |

## 2.1 Minimal CW-complex

Prop.

## 2.1 Minimal CW-complex

Prop. $X$: a minimal CW-cpx. Then Aomoto's conj holds, i.e.,

$$\dim H_i(X, \mathcal{L}) \leq b_i(X).$$

## 2.1 Minimal CW-complex

Prop. $X$: a minimal CW-cpx. Then Aomoto's conj holds, i.e.,

$$\dim H_i(X, \mathcal{L}) \leq b_i(X).$$

$\because$) $H_i(X, \mathcal{L}) = H_i(C_\bullet(X, \mathcal{L}), \partial_{\mathcal{L}})$, and $\dim C_i(X, \mathcal{L}) = b_i(X)$. $\square$

## 2.2 Minimality of $M(\mathcal{A})$

## 2.2 Minimality of $M(\mathcal{A})$

$\mathcal{A}$ is arrangement in $\mathbb{C}^\ell$.

## 2.2 Minimality of $M(\mathcal{A})$

$\mathcal{A}$ is arrangement in $\mathbb{C}^\ell$.

Thm. (DPSR) $M(\mathcal{A})$ has the homotopy type of a $\ell$-dim minimal CW-cpx. i.e., there is an $\ell$-dim minimal CW-cpx $X$ such that

$$M(\mathcal{A}) \simeq X.$$

## 2.3 Proof of minimality

## 2.3 Proof of minimality

Proof is based on two results:

## 2.3 Proof of minimality

Proof is based on two results:

– Lefschetz Theorem on hyperplane section.

## 2.3 Proof of minimality

Proof is based on two results:

– Lefschetz Theorem on hyperplane section.

– Combinatorial description of cohomology ring $H^\bullet(M(\mathcal{A}), \mathbb{Z})$ (Orlik-Solomon).

## 2.3 Proof of minimality

$M = M(\mathcal{A})$,

$F \subset \mathbb{C}^\ell$: a generic hyperplane.

## 2.3 Proof of minimality

$M = M(\mathcal{A})$,

$F \subset \mathbb{C}^\ell$: a generic hyperplane.

Thm.(Lefschetz)

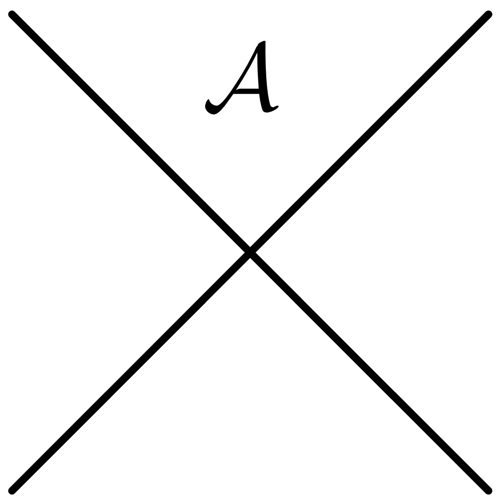
$$M \simeq (M \cap F) \underbrace{\cup_\varphi \bigcup_{i=1}^{b} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$

## 2.3 Proof of minimality

$$M \simeq (M \cap F) \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell}}_{\text{attach } \ell\text{-dim cells}}$$

## 2.3 Proof of minimality

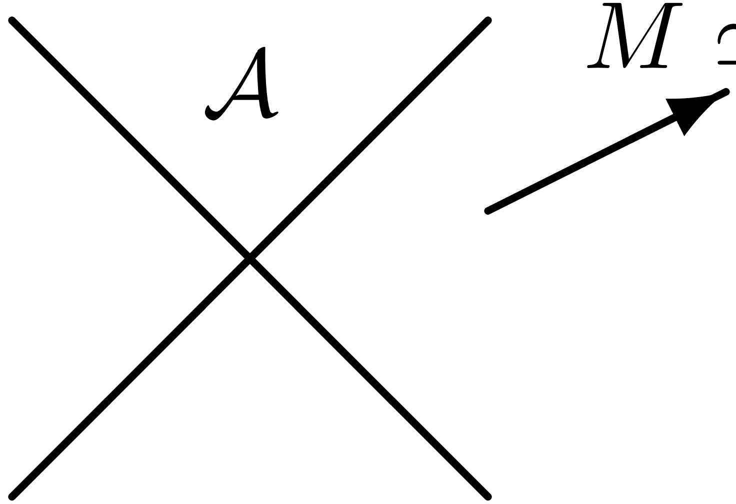
$$M \simeq (M \cap F) \ \cup_{\varphi} \underbrace{\bigcup_{i=1}^{b} D^{\ell}}_{\text{attach } \ell\text{-dim cells}}$$



# 2.3 Proof of minimality

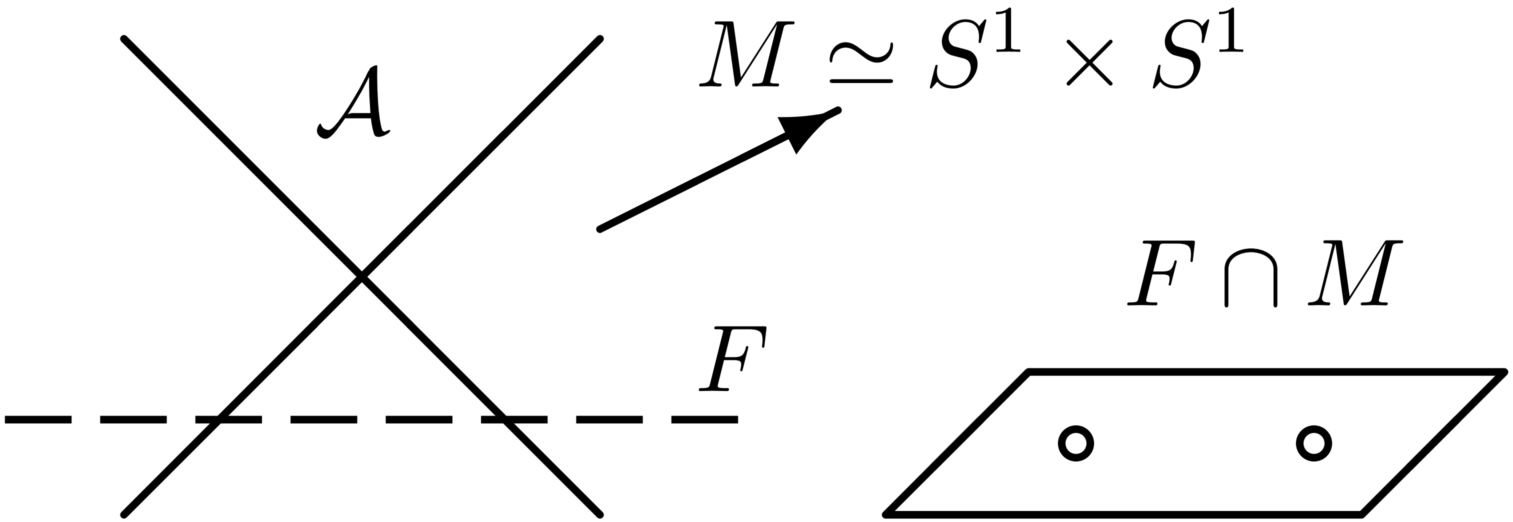
$$M \simeq (M \cap F) \underbrace{\cup_\varphi \bigcup_{i=1}^{b} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$

$$M \simeq S^1 \times S^1$$



# 2.3 Proof of minimality

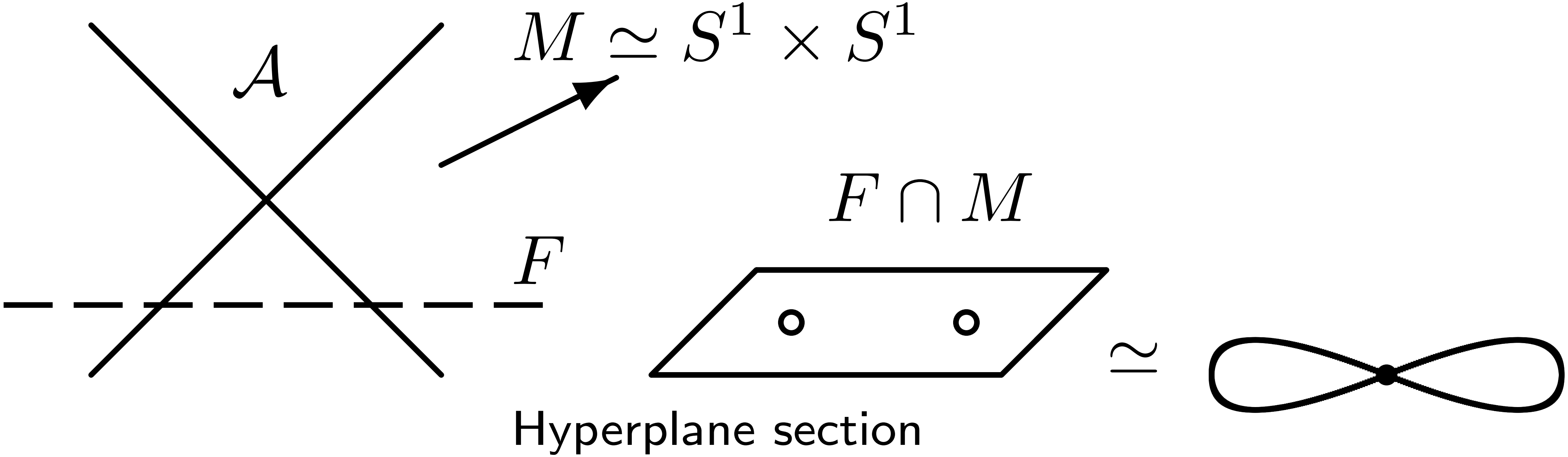
$$M \simeq (M \cap F) \ \cup_\varphi \underbrace{\bigcup_{i=1}^{b} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$




Hyperplane section

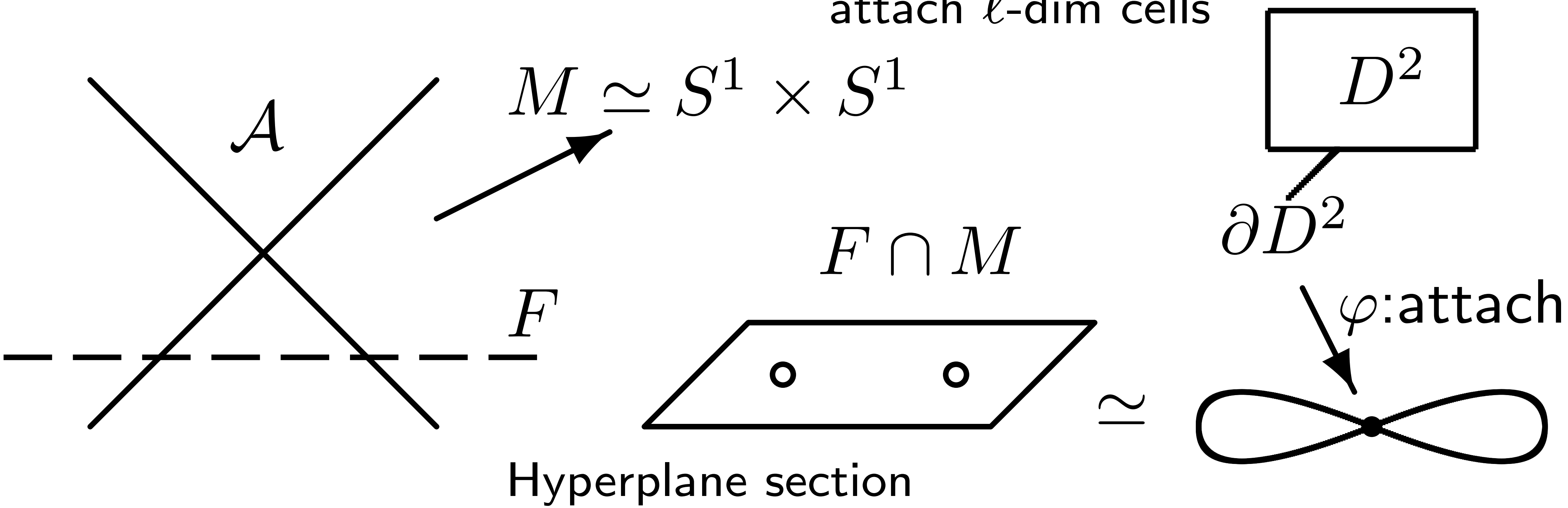
## 2.3 Proof of minimality

$$M \simeq (M \cap F) \ \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell}}_{\text{attach } \ell\text{-dim cells}}$$



## 2.3 Proof of minimality

$$M \simeq (M \cap F) \; \underbrace{\cup_\varphi \bigcup_{i=1}^{b} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$



## 2.3 Proof of minimality

$$M \simeq (M \cap F) \ \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell}}_{\text{attach } \ell\text{-dim cells}}$$

## 2.3 Proof of minimality

$$M \simeq (M \cap F) \underbrace{\cup_\varphi \bigcup_{i=1}^{b} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$

How many $\ell$-dim cells to attach?

## 2.3 Proof of minimality

$$M \simeq (M \cap F) \underbrace{\cup_\varphi \bigcup_{i=1}^{b} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$

How many $\ell$-dim cells to attach?

$$\Longrightarrow b = \dim H_\ell(M, M \cap F; \mathbb{C}).$$

## 2.3 Proof of minimality

$$M \simeq (M \cap F) \underbrace{\cup_\varphi \bigcup_{i=1}^{b} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$

$b = \dim H_\ell(M, M \cap F; \mathbb{C})$.

Fact. (Orlik-Solomon)

$$H_\ell(M) \xrightarrow{\cong} H_\ell(M, M \cap F).$$

## 2.3 Proof of minimality

$$M \simeq (M \cap F) \underbrace{\cup_\varphi \bigcup_{i=1}^{b} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$

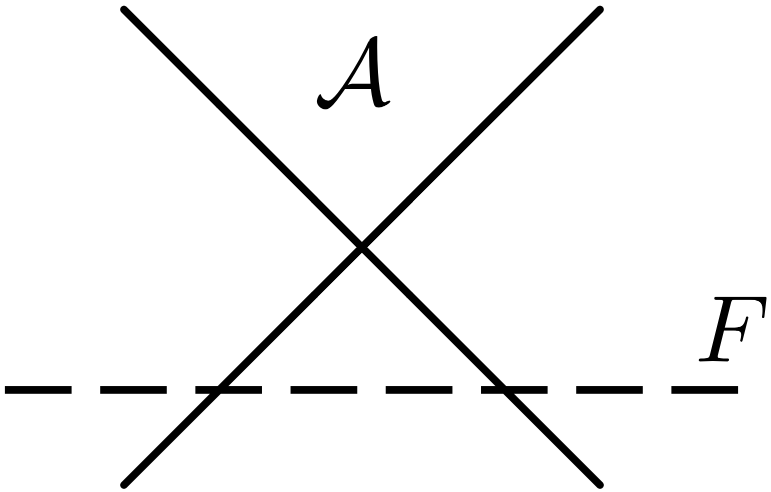
$b = b_\ell(M)$, by induction

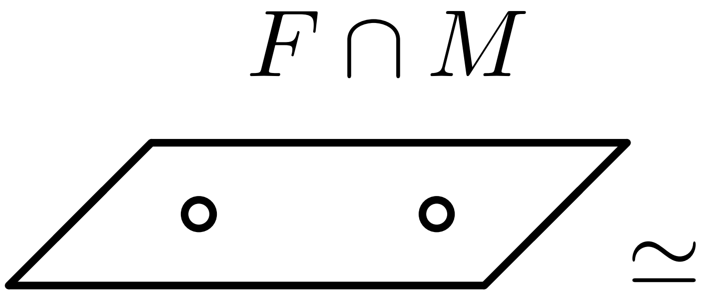
$\longrightarrow$ minimality of $M$. $\square$

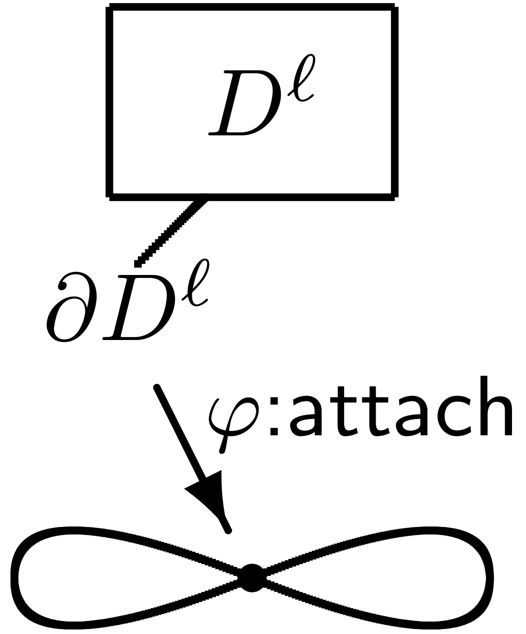
## 2.4 Problems

# 2.4 Problems

$$M \simeq (M \cap F) \quad \cup_{\varphi} \underbrace{\bigcup_{i=1}^{b_\ell} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$



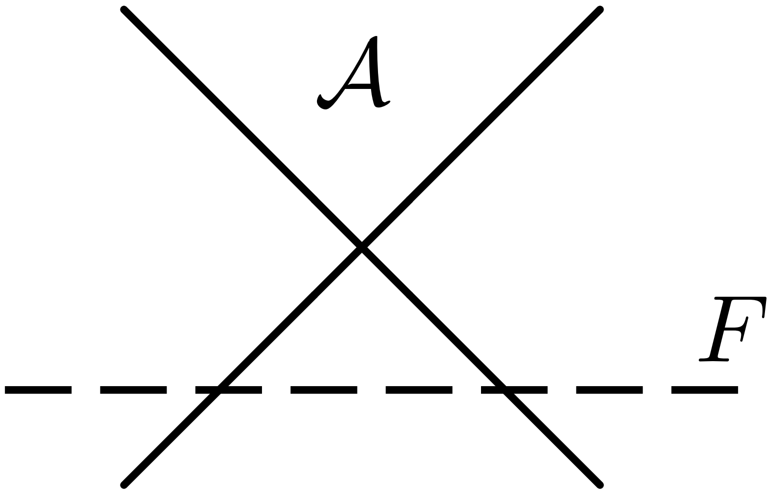



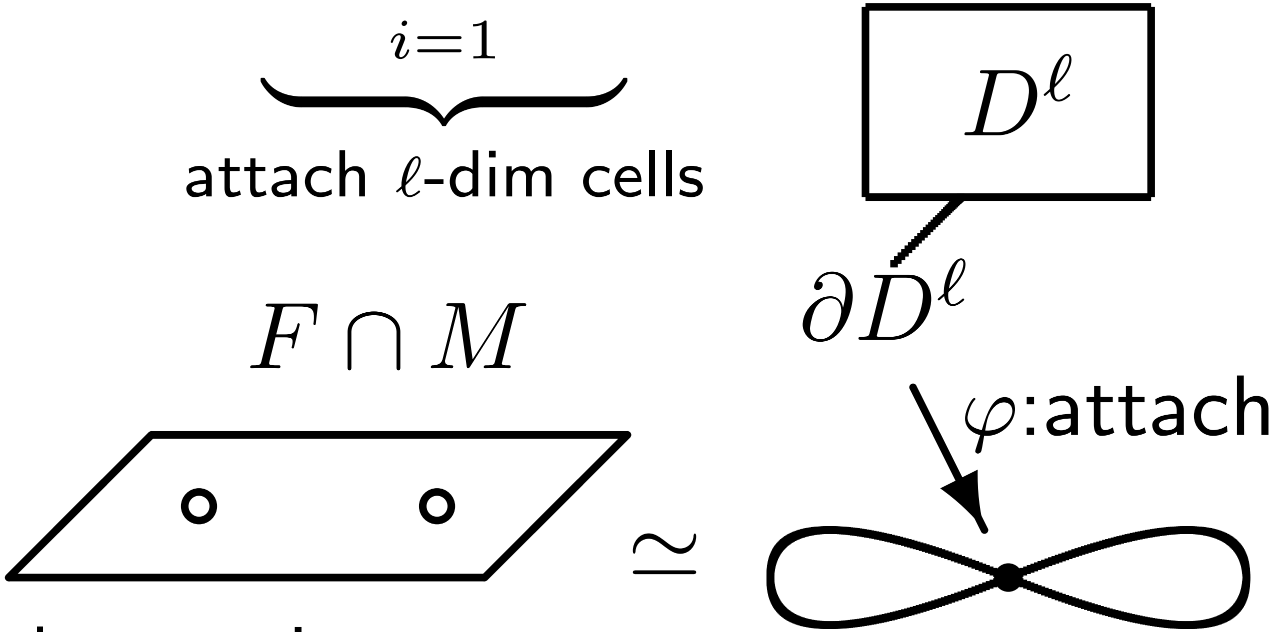




Hyperplane section

## 2.4 Problems

$$M \simeq (M \cap F) \quad \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b_\ell} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$



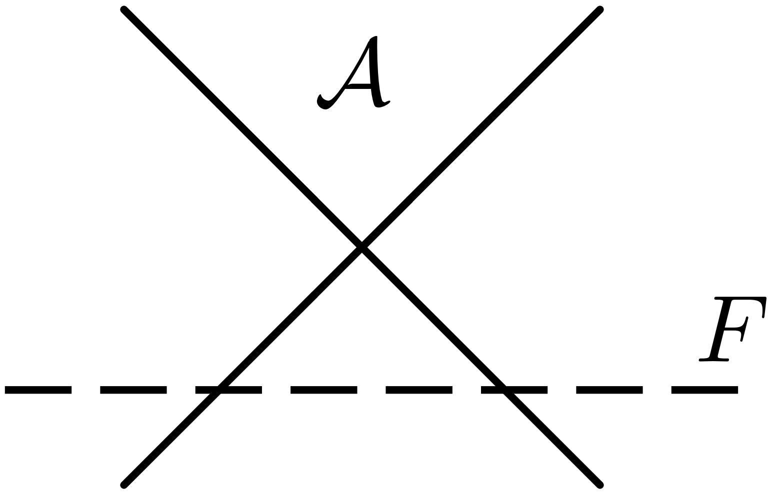



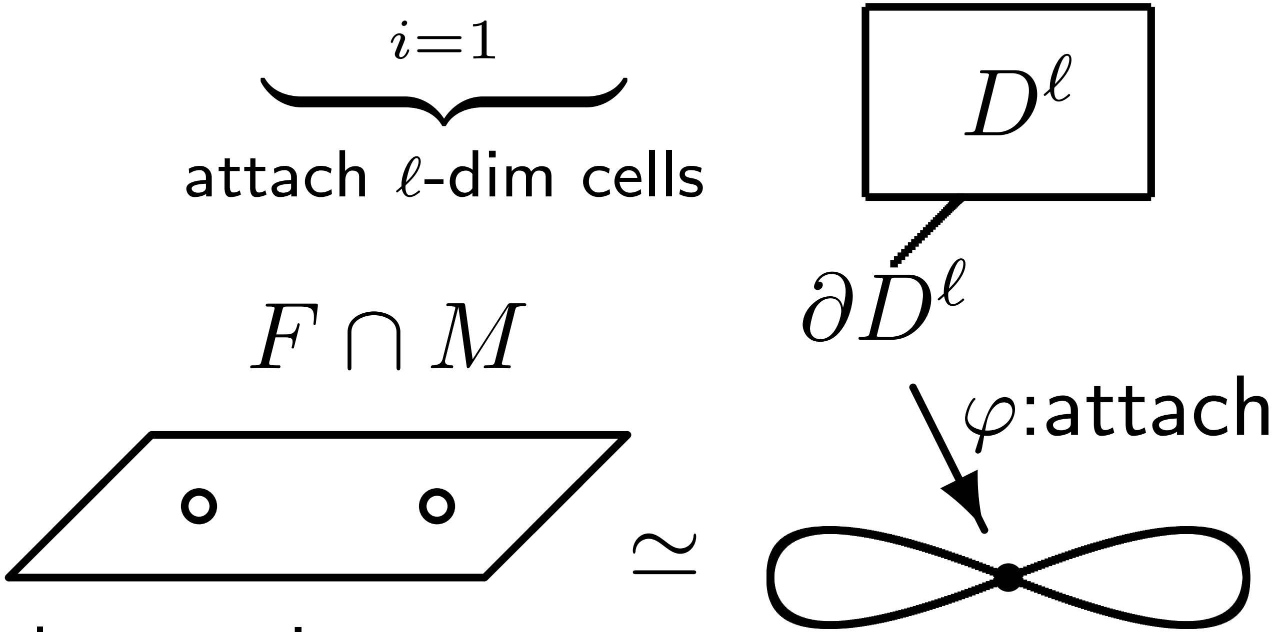

Hyperplane section

How cells attach?

## 2.4 Problems

$$M \simeq (M \cap F) \quad \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b_\ell} D^\ell}_{\text{attach } \ell\text{-dim cells}}$$




Hyperplane section

How cells attach?

How cells are labeled?

# 3 Real cases

# 3 Real cases

From now, every $H_i \in \mathcal{A}$ is defined $/\mathbb{R}$.

# 3 Real cases

From now, every $H_i \in \mathcal{A}$ is defined $/\mathbb{R}$. Connected comp. of $M(\mathcal{A}) \cap \mathbb{R}^\ell$ is called a *chamber*.

## 3 Real cases

From now, every $H_i \in \mathcal{A}$ is defined $/\mathbb{R}$.

Connected comp. of $M(\mathcal{A}) \cap \mathbb{R}^\ell$ is called a *chamber*.

$\mathrm{ch}(\mathcal{A})$: set of all chambers.

$\mathrm{bch}(\mathcal{A})$: set of all bounded chambers.

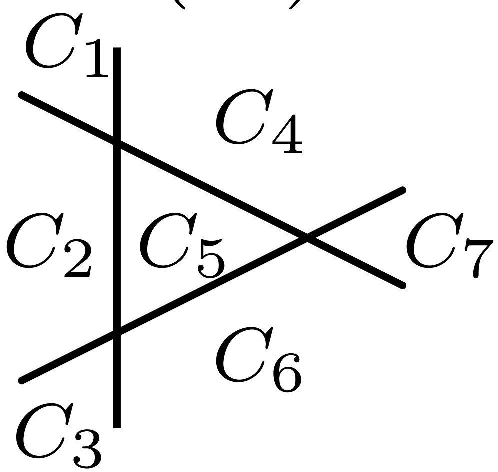
# 3 Real cases

From now, every $H_i \in \mathcal{A}$ is defined $/\mathbb{R}$.

Connected comp. of $M(\mathcal{A}) \cap \mathbb{R}^\ell$ is called a *chamber*.

$\mathrm{ch}(\mathcal{A})$: set of all chambers.

$\mathrm{bch}(\mathcal{A})$: set of all bounded chambers.




$$\mathrm{ch}(\mathcal{A}) = \{C_1, C_2, \ldots, C_7\}$$

$$\mathrm{bch}(\mathcal{A}) = \{C_5\}$$

# 3 Real cases

$\mathrm{(b)ch}(\mathcal{A})$ has information about $M(\mathcal{A})$.

<u>Thm.</u> (Zaslawski)

(i) $$\sum_{i=0}^{\ell} b_i(M(\mathcal{A})) = \sharp\,\mathrm{ch}(\mathcal{A}).$$

(ii) $$\left|\sum_{i=0}^{\ell} (-1)^i b_i(M(\mathcal{A}))\right| = \sharp\,\mathrm{bch}(\mathcal{A}).$$

# 3 Real cases

Let $F \subset \mathbb{C}^\ell$ be a generic hyperplane defined $/\mathbb{R}$.

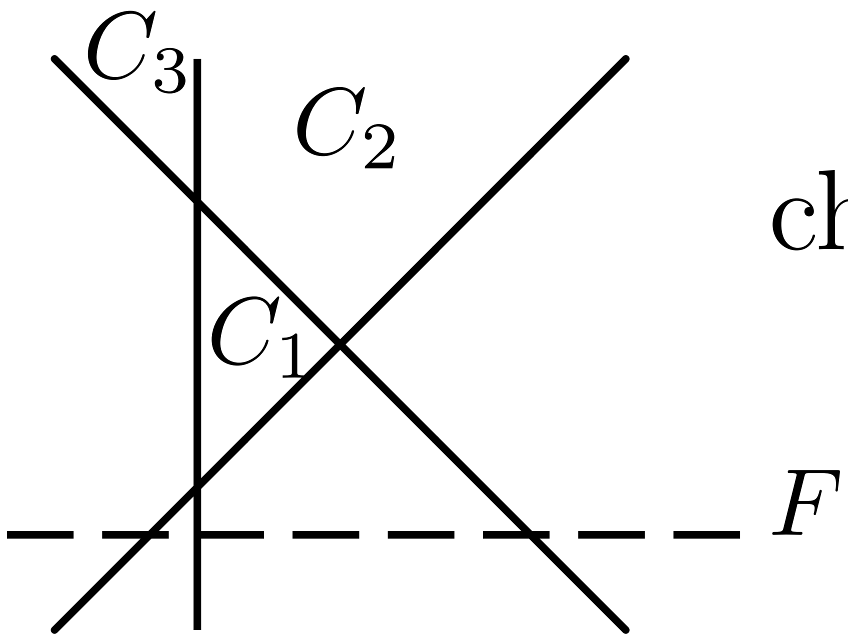
## 3 Real cases

Let $F \subset \mathbb{C}^\ell$ be a generic hyperplane defined $/\mathbb{R}$. Define

# 3 Real cases

Let $F \subset \mathbb{C}^\ell$ be a generic hyperplane defined $/\mathbb{R}$. Define

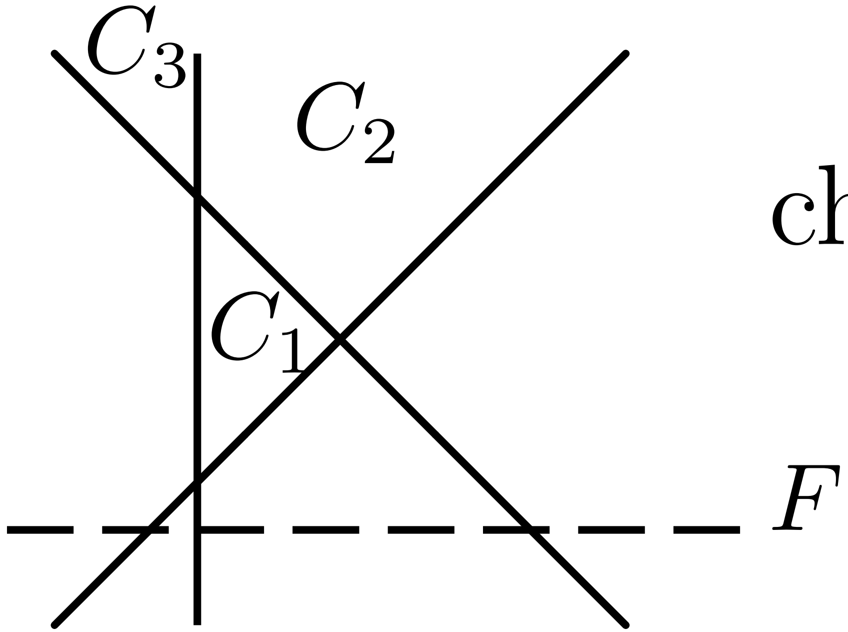
$$\mathrm{ch}_F(\mathcal{A}) := \{C \in \mathrm{ch}(\mathcal{A}) \mid F \cap C = \emptyset\}$$




$$\mathrm{ch}_F(\mathcal{A}) := \{C_1, C_2, C_3\}$$

# 3 Real cases

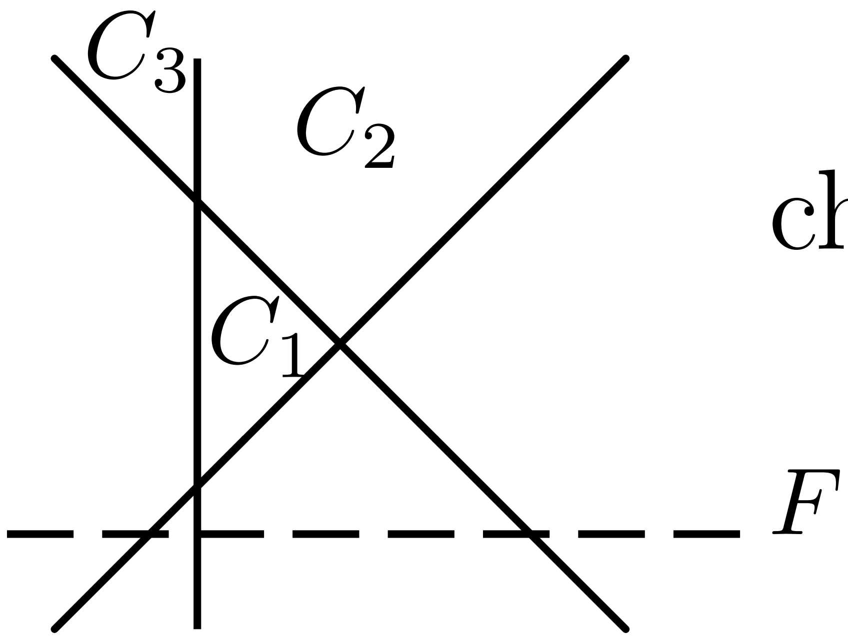
$$\mathrm{ch}_F(\mathcal{A}) := \{C \in \mathrm{ch}(\mathcal{A}) \mid F \cap C = \emptyset\}$$




$$\mathrm{ch}_F(\mathcal{A}) := \{C_1, C_2, C_3\}$$

# 3 Real cases

$$\mathrm{ch}_F(\mathcal{A}) := \{C \in \mathrm{ch}(\mathcal{A}) \mid F \cap C = \emptyset\}$$

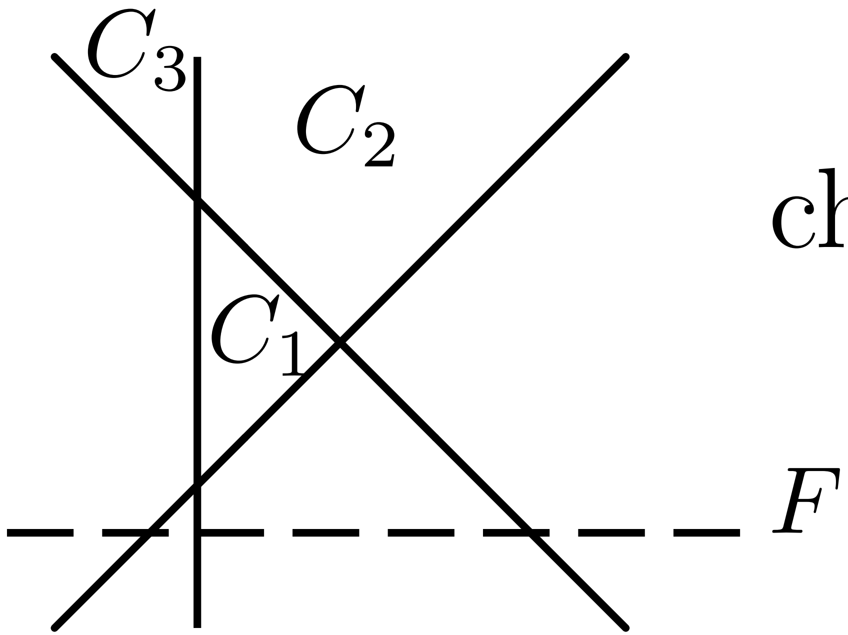



$$\mathrm{ch}_F(\mathcal{A}) := \{C_1, C_2, C_3\}$$

Prop. $\sharp\,\mathrm{ch}_F(\mathcal{A}) = b_\ell(M(\mathcal{A}))$.

# 3 Real cases

$$\mathrm{ch}_F(\mathcal{A}) := \{C \in \mathrm{ch}(\mathcal{A}) \mid F \cap C = \emptyset\}$$




$$\mathrm{ch}_F(\mathcal{A}) := \{C_1, C_2, C_3\}$$

Prop. $\sharp\,\mathrm{ch}_F(\mathcal{A}) = b_\ell(M(\mathcal{A}))$.

$\Longrightarrow \mathrm{ch}_F(\mathcal{A})$ labeling $\ell$-dim cells.

## 3.1 What Morse theory tells us

## 3.1 What Morse theory tells us

$\mathcal{A} = \{H_1, \ldots, H_n\}$. Set $H_i = \alpha_i^{-1}(0)$.
$Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$: the defining equation of $\mathcal{A}$.

## 3.1 What Morse theory tells us

$\mathcal{A} = \{H_1, \ldots, H_n\}$. Set $H_i = \alpha_i^{-1}(0)$.

$Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$: the defining equation of $\mathcal{A}$.

$F = \{f = 0\}$: a generic hyperplane.

## 3.1 What Morse theory tells us

$\mathcal{A} = \{H_1, \ldots, H_n\}$. Set $H_i = \alpha_i^{-1}(0)$.

$Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$: the defining equation of $\mathcal{A}$.

$F = \{f = 0\}$: a generic hyperplane.

Consider a Morse function

$$\varphi := \left| \frac{f^{n+1}}{Q} \right| : M(\mathcal{A}) \longrightarrow \mathbb{R}_{\geq 0}$$

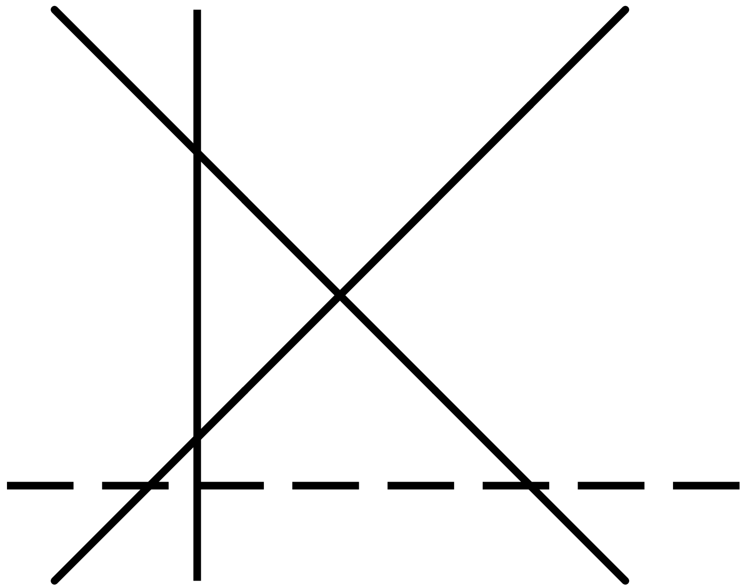
## 3.1 What Morse theory tells us

$$\varphi := \left| \frac{f^{n+1}}{Q} \right| : M(\mathcal{A}) \longrightarrow \mathbb{R}_{\geq 0}$$

## 3.1 What Morse theory tells us

$$\varphi := \left| \frac{f^{n+1}}{Q} \right| : M(\mathcal{A}) \longrightarrow \mathbb{R}_{\geq 0}$$

$$\mathrm{Cr}(\varphi) := \{p : \text{critical pt}, \ \varphi(p) \neq 0\}$$



$F = \{f = 0\}$

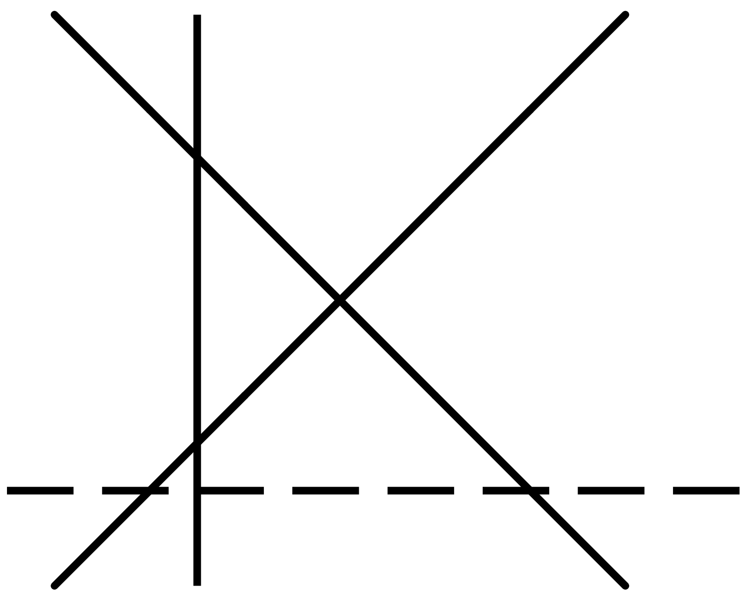
## 3.1 What Morse theory tells us

$$\varphi := \left| \frac{f^{n+1}}{Q} \right| : M(\mathcal{A}) \longrightarrow \mathbb{R}_{\geq 0}$$

$$\mathrm{Cr}(\varphi) := \{p : \text{critical pt, } \varphi(p) \neq 0\}$$

$\mathcal{S}(p)$: stable mfd ($p \in \mathrm{Cr}(\varphi)$)

$\mathcal{U}(p)$: unstable mfd

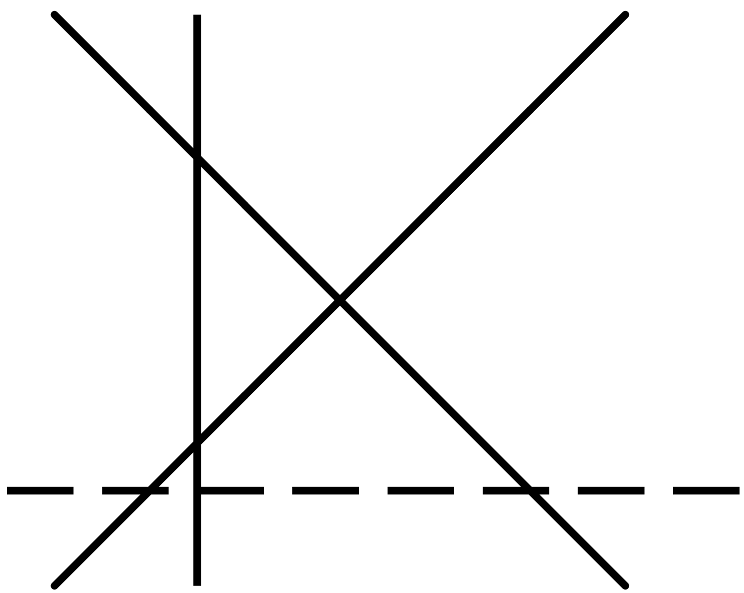


$F = \{f = 0\}$

## 3.1 What Morse theory tells us

$$\varphi := \left| \frac{f^{n+1}}{Q} \right| : M(\mathcal{A}) \longrightarrow \mathbb{R}_{\geq 0}$$

$$\mathrm{Cr}(\varphi) := \{p : \text{critical pt, } \varphi(p) \neq 0\}$$



$\mathcal{S}(p)$: stable mfd ($p \in \mathrm{Cr}(\varphi)$)

$\mathcal{U}(p)$: unstable mfd

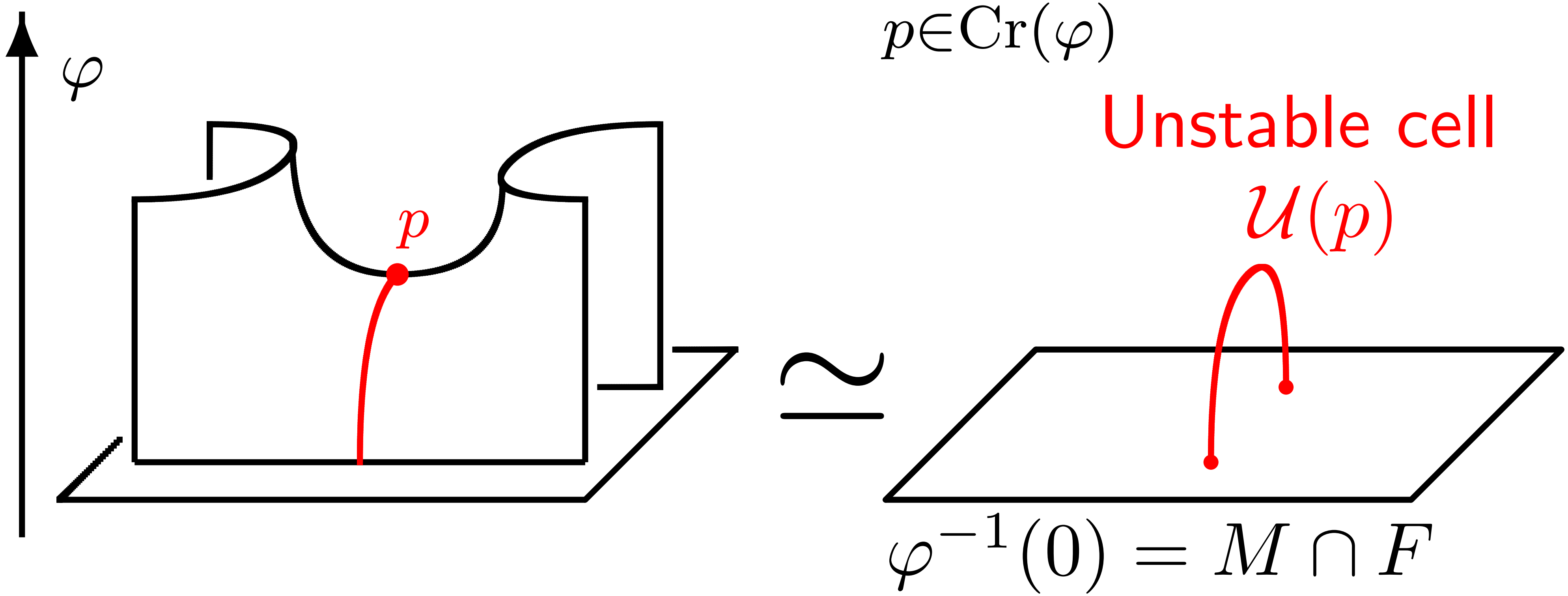
$$\varphi^{-1}(0) = M(\mathcal{A}) \cap F$$

$F = \{f = 0\}$

## 3.1 What Morse theory tells us

$\mathcal{U}(p)$: unstable manifold of $p \in \mathrm{Cr}(\varphi)$.

$$M \simeq (M \cap F) \cup \bigcup_{p \in \mathrm{Cr}(\varphi)} \mathcal{U}(p).$$



## 3.1 What Morse theory tells us

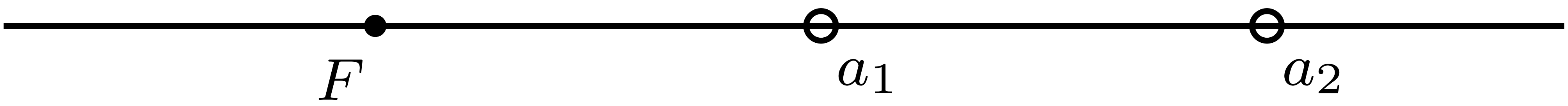
Example. $\ell = 1$, $\mathcal{A} = \{a_1, a_2\}$ and $F = \{b\}$.

$$\varphi(z) = \left| \frac{(z-b)^3}{(z-a_1)(z-a_2)} \right|$$

## 3.1 What Morse theory tells us

Example. $\ell = 1$, $\mathcal{A} = \{a_1, a_2\}$ and $F = \{b\}$.

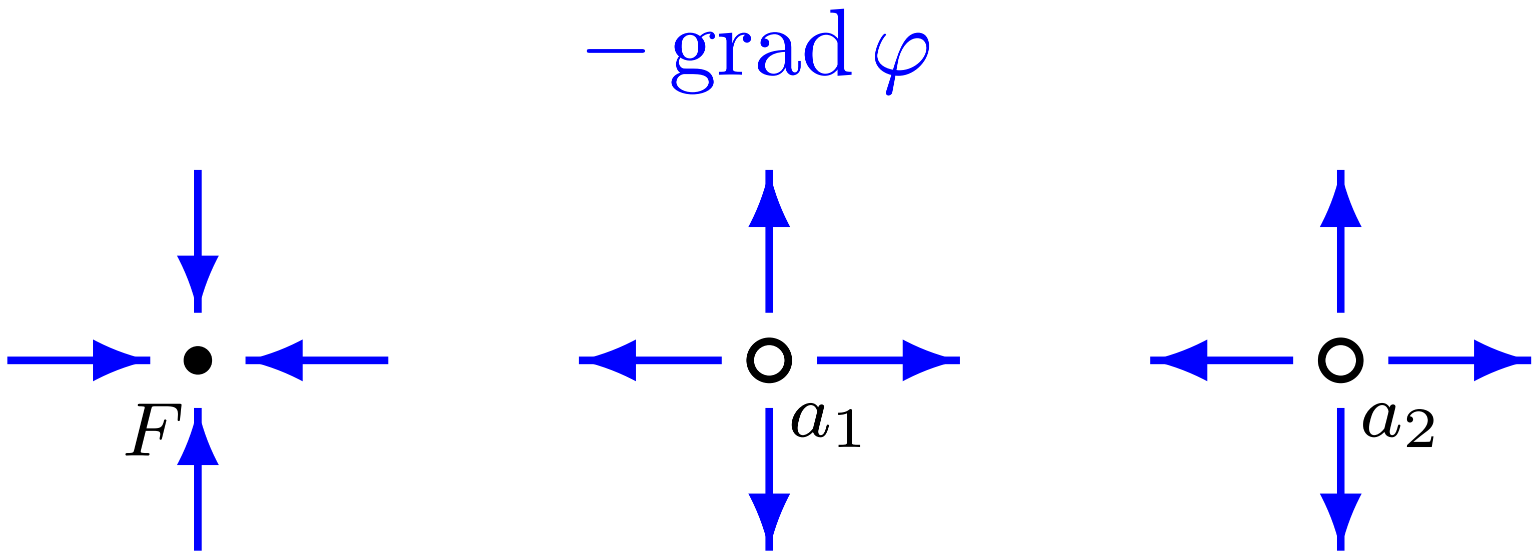
$$\varphi(z) = \left| \frac{(z-b)^3}{(z-a_1)(z-a_2)} \right|$$



## 3.1 What Morse theory tells us

Example. $\ell = 1$, $\mathcal{A} = \{a_1, a_2\}$ and $F = \{b\}$.

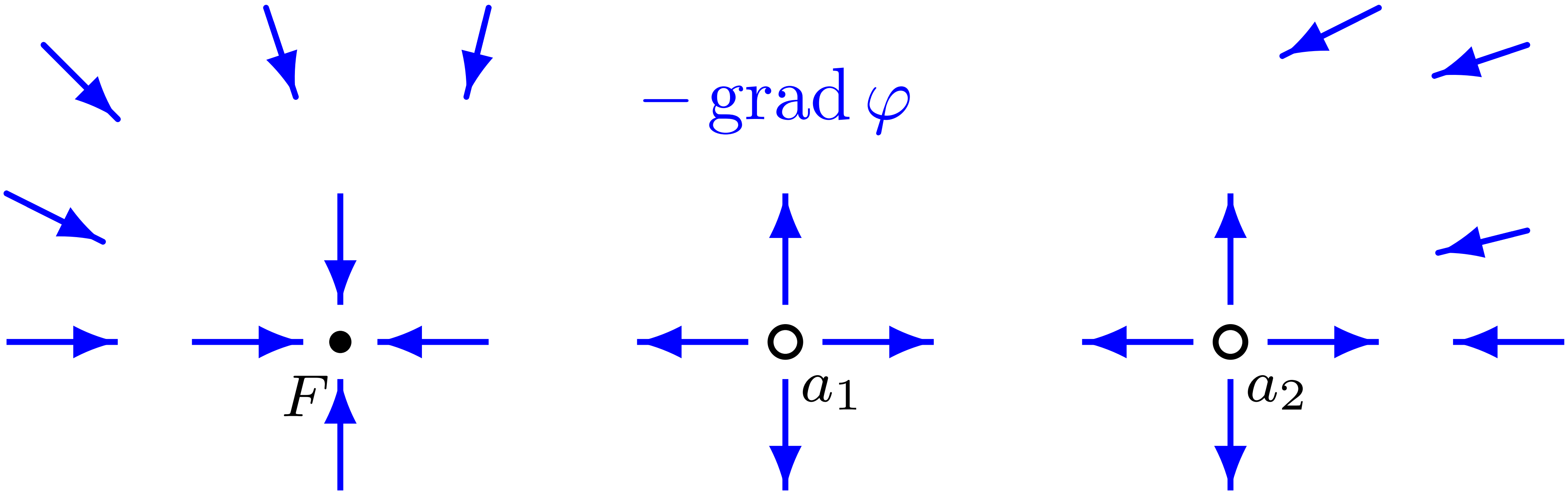
$$\varphi(z) = \left| \frac{(z-b)^3}{(z-a_1)(z-a_2)} \right|$$



## 3.1 What Morse theory tells us

<u>Example.</u> $\ell = 1$, $\mathcal{A} = \{a_1, a_2\}$ and $F = \{b\}$.

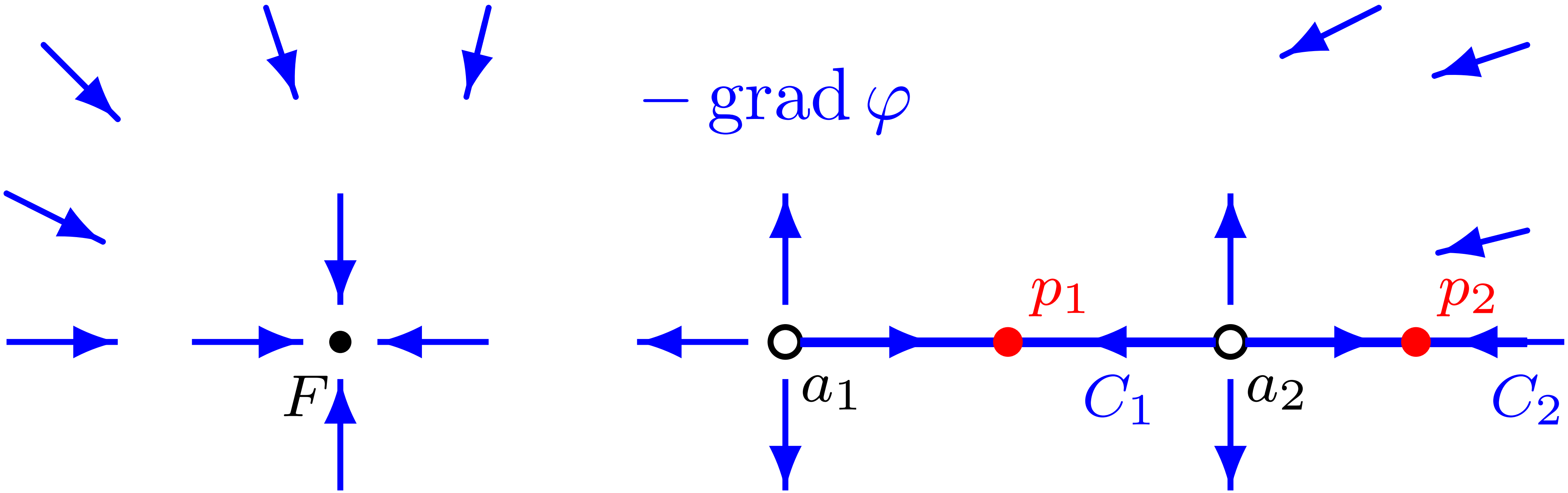
$$\varphi(z) = \left| \frac{(z-b)^3}{(z-a_1)(z-a_2)} \right|$$



## 3.1 What Morse theory tells us

Example. $\ell = 1$, $\mathcal{A} = \{a_1, a_2\}$ and $F = \{b\}$.

$$\varphi(z) = \left| \frac{(z-b)^3}{(z-a_1)(z-a_2)} \right|$$

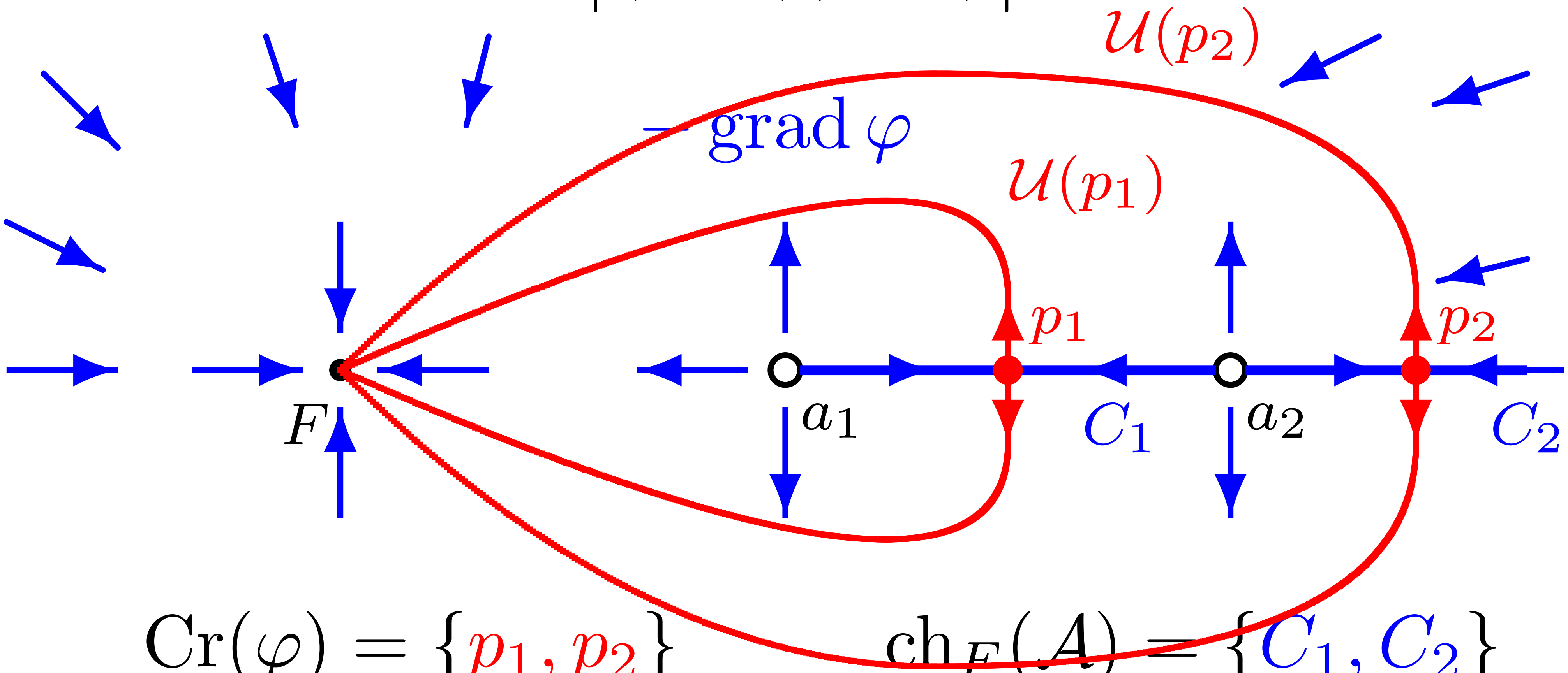



$\mathrm{Cr}(\varphi) = \{p_1, p_2\}$ $\quad\quad$ $\mathrm{ch}_F(\mathcal{A}) = \{C_1, C_2\}$

## 3.1 What Morse theory tells us

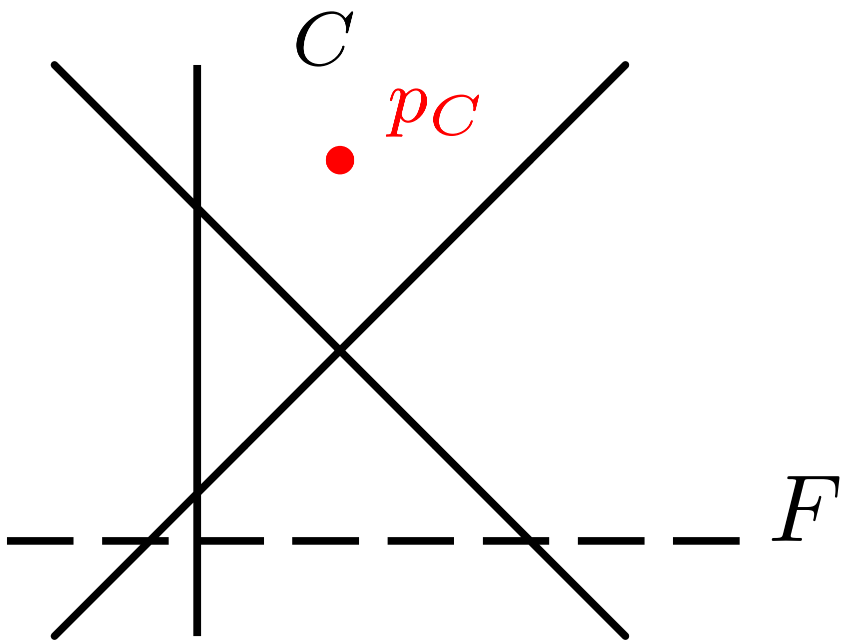
Example. $\ell = 1$, $\mathcal{A} = \{a_1, a_2\}$ and $F = \{b\}$.

$$\varphi(z) = \left| \frac{(z-b)^3}{(z-a_1)(z-a_2)} \right|$$




$\mathrm{Cr}(\varphi) = \{p_1, p_2\}$ $\qquad$ $\mathrm{ch}_F(\mathcal{A}) = \{C_1, C_2\}$

## 3.1 What Morse theory tells us



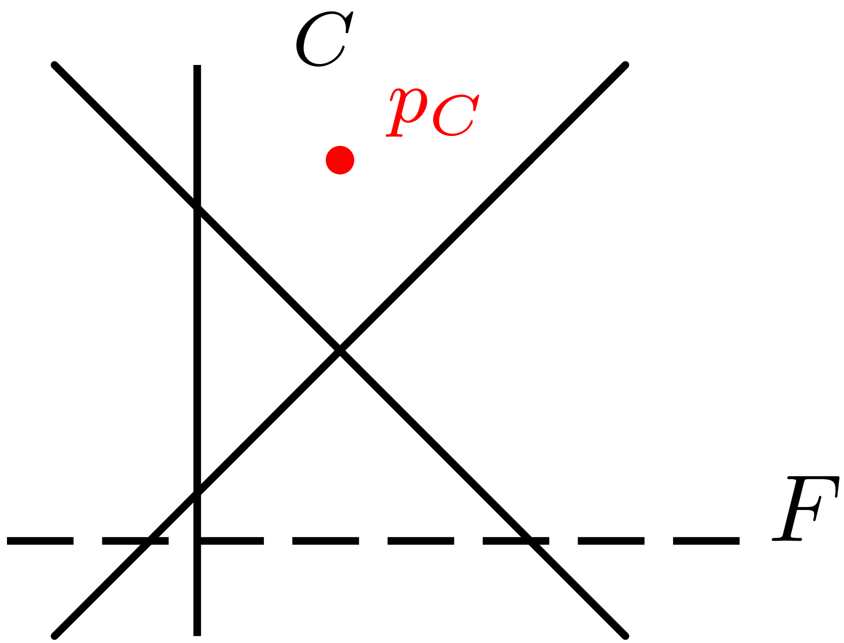

**Thm.**

(1) $\mathrm{ch}_F(\mathcal{A}) \overset{1:1}{\longleftrightarrow} \mathrm{Cr}(\varphi)$

$C \longleftrightarrow p_C$

(2) $\mathcal{S}(p_C) = C$

## 3.1 What Morse theory tells us




Thm.

(1) $\mathrm{ch}_F(\mathcal{A}) \overset{1:1}{\longleftrightarrow} \mathrm{Cr}(\varphi)$

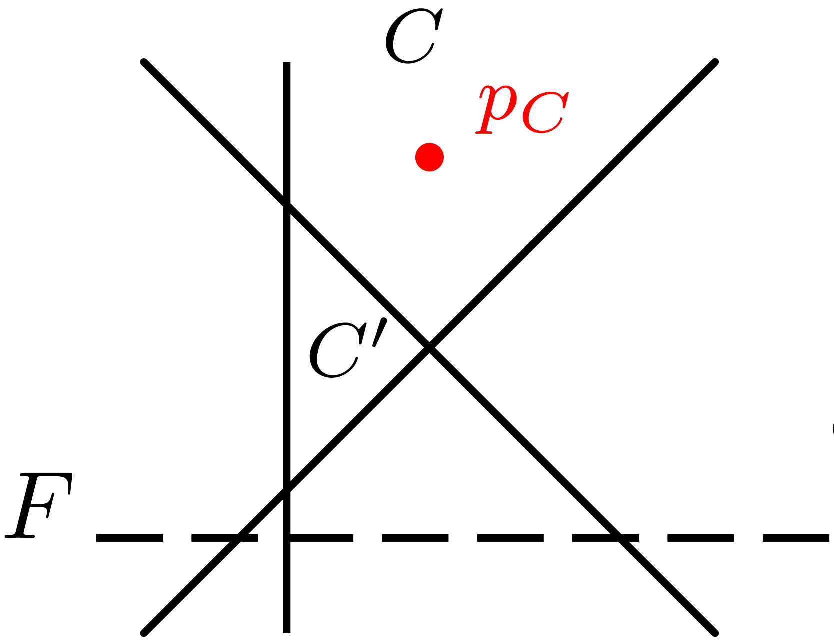
$C \longleftrightarrow p_C$

(2) $\mathcal{S}(p_C) = C$

How about unstable cell $\mathcal{U}(p_C)$?

## 3.2 Unstable cells

## 3.2 Unstable cells

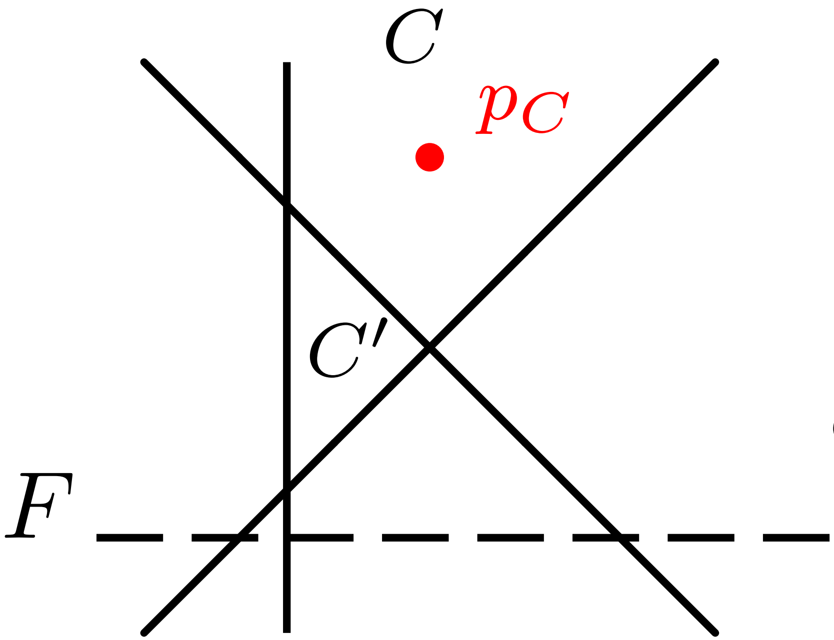



Unstable cell can be considered as a map

$$\sigma_C : (D^\ell, \partial D^\ell) \to (M, M \cap F)$$

such that

## 3.2 Unstable cells




Unstable cell can be considered as a map

$$\sigma_C : (D^\ell, \partial D^\ell) \to (M, M \cap F)$$

such that

(i) $\sigma_C(D^\ell) \pitchfork C = \{p_C\}$.

(ii) $\sigma_C(D^\ell) \cap C' = \emptyset$ for $C' \in \mathrm{ch}_F \setminus \{C\}$.

## 3.2 Unstable cells

(i) and (ii) above characterize the homotopyt type of $\mathcal{U}(p_C)$.

## 3.2 Unstable cells

(i) and (ii) above characterize the homotopyt type of $\mathcal{U}(p_C)$.

<u>Thm.</u> Suppose

$\sigma'_C : (D^\ell, \partial D^\ell) \longrightarrow (M, M \cap F)$ satisfies

(i) $\sigma'_C(D^\ell) \pitchfork C = \{p_C\}$.

(ii) $\sigma'_C(D^\ell) \cap C' = \emptyset$ for $C' \in \mathrm{ch}_F \setminus \{C\}$.

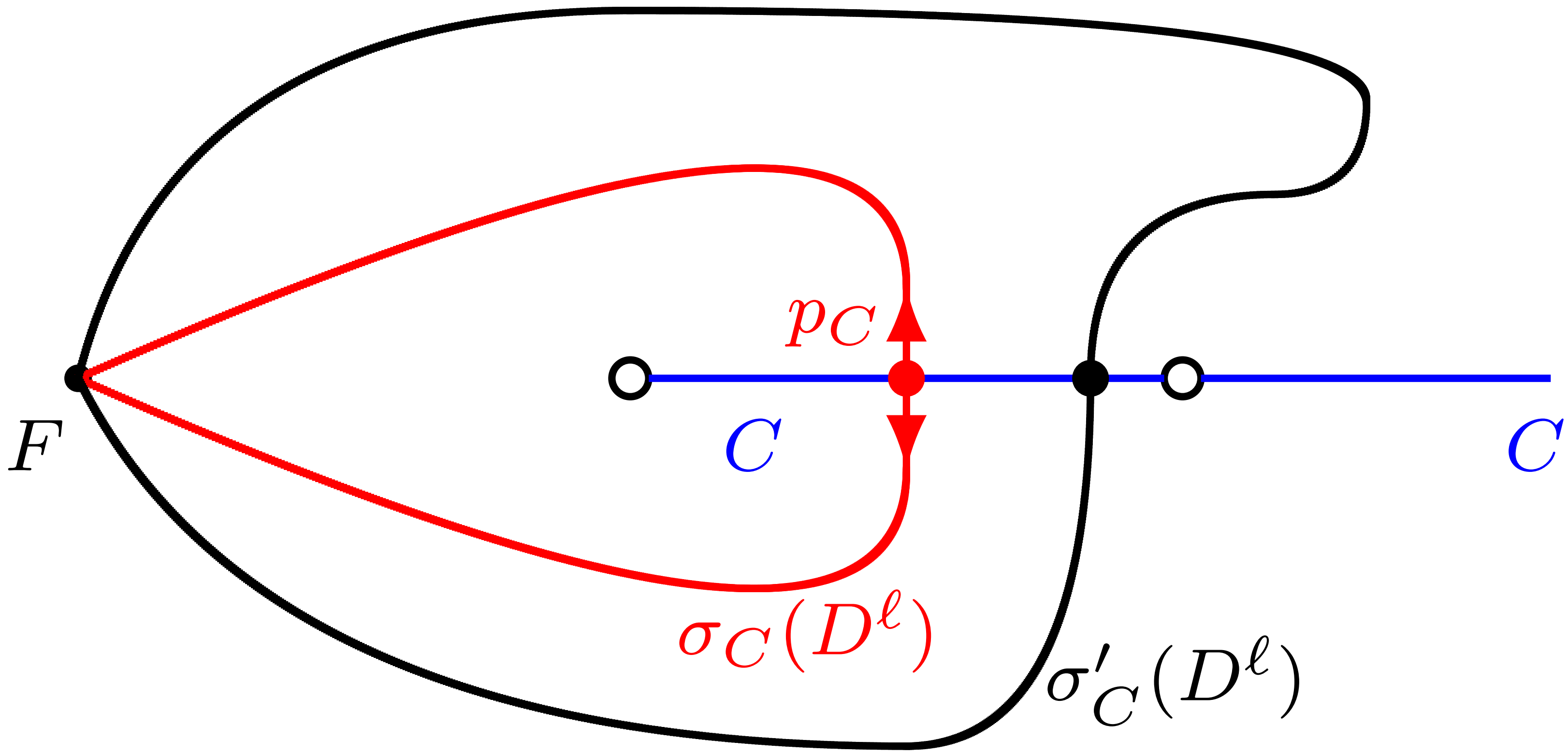
Then $\sigma'_C \simeq \sigma_C$.

## 3.2 Unstable cells

Sketch of the proof:

## 3.2 Unstable cells

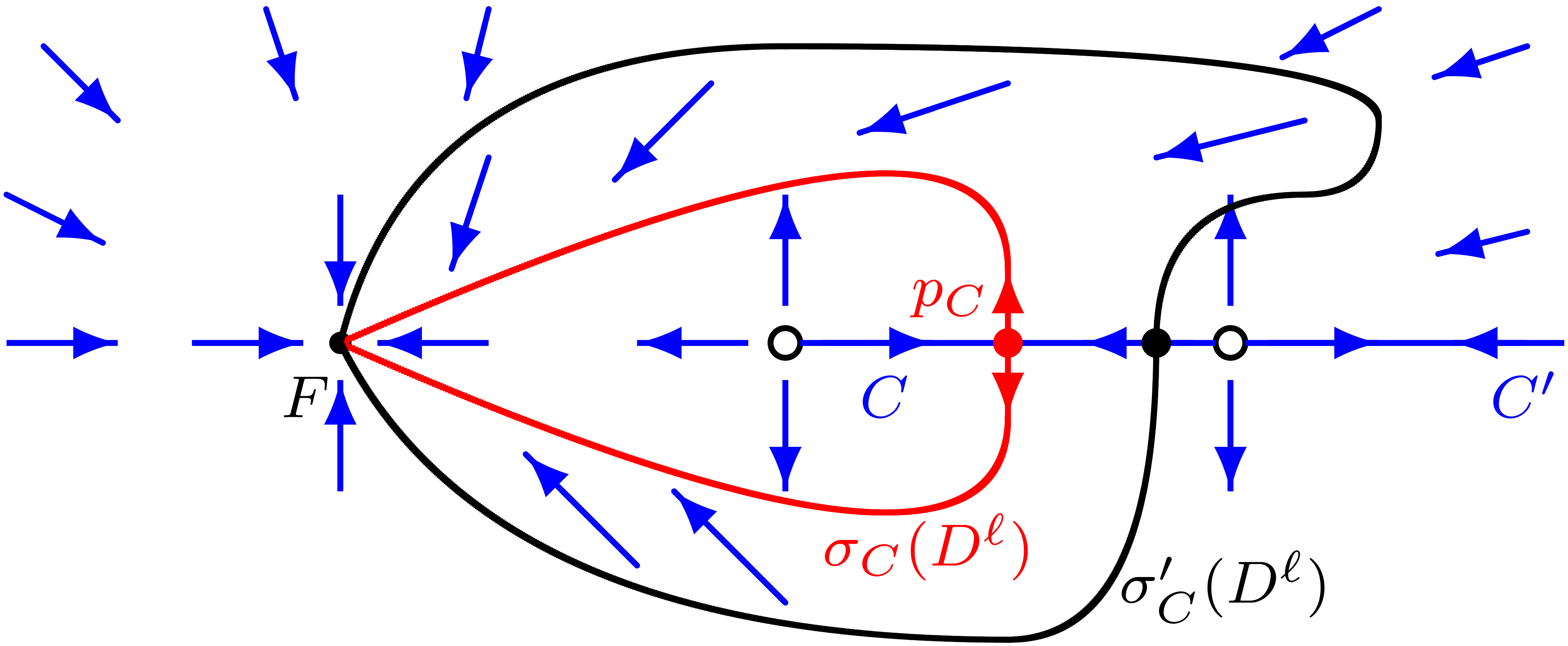
Sketch of the proof:



## 3.2 Unstable cells

Sketch of the proof:

$\phi_t$:1-parameter diffeo generated by $-\operatorname{grad}\varphi$

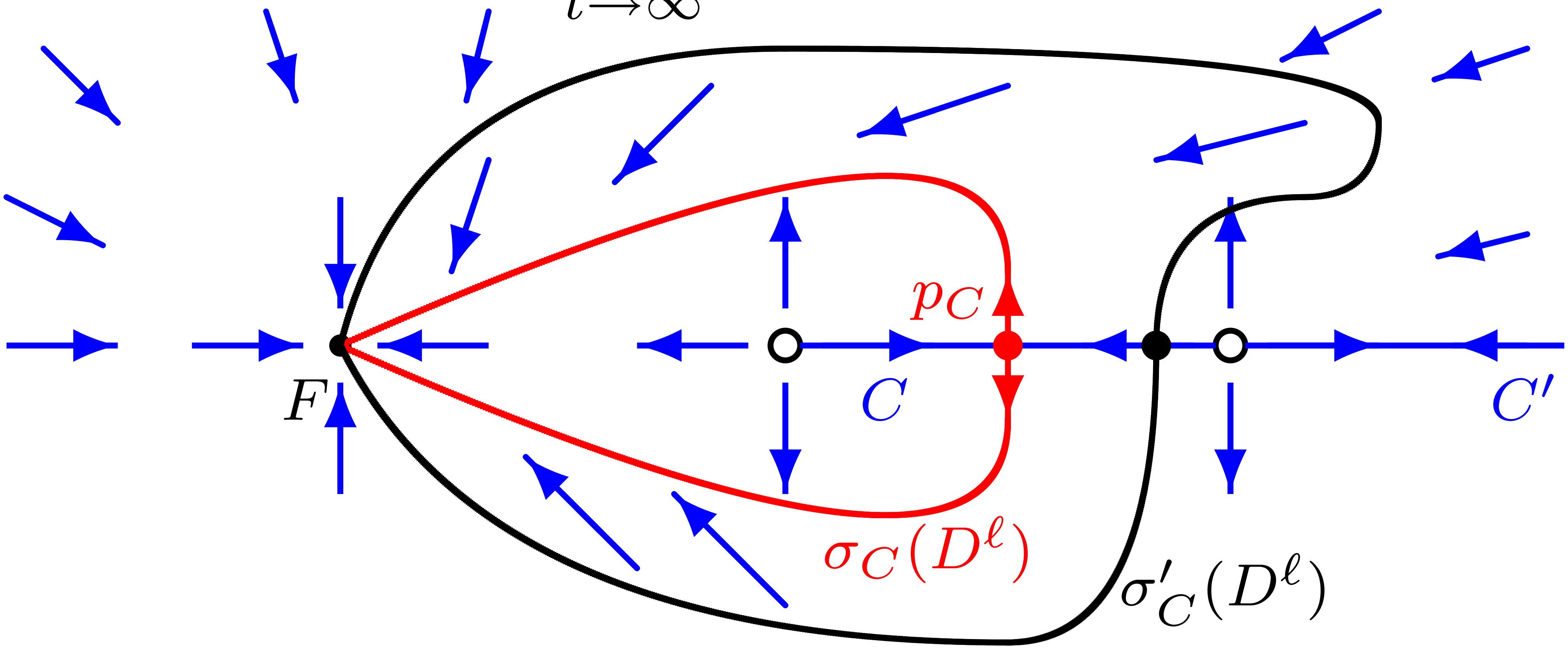


## 3.2 Unstable cells

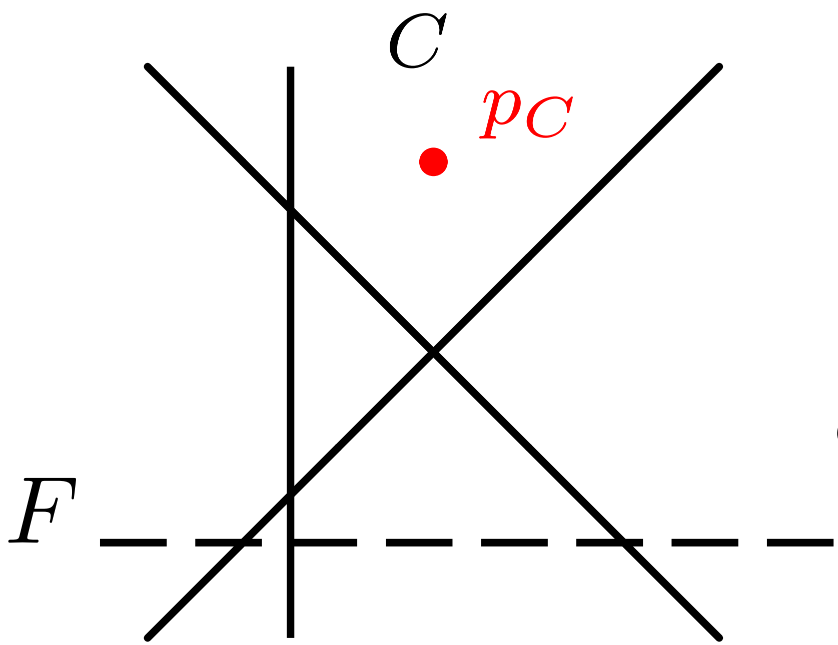
Sketch of the proof:

$\phi_t$:1-parameter diffeo generated by $-\operatorname{grad}\varphi$

$$\lim_{t\to\infty} \phi_t \circ \sigma'_C = \sigma_C$$



## 3.2 Unstable cells




**Thm.**

We can construct a concrete map

$$\sigma'_C : (D^\ell, \partial D^\ell) \to (M, M \cap F)$$

such that

(i) $\sigma'_C(D^\ell) \pitchfork C = \{p_C\}$.

(ii) $\sigma'_C(D^\ell) \cap C' = \emptyset$ for $C' \in \mathrm{ch}_F \setminus \{C\}$.

## 3.3 Recent works

Salvetti-Settepanella, Delucchi: “Discrete Morse theory on Salvetti complex”.

$\Longrightarrow$ Another description of attaching maps.

# 4 An application

Topological proof of vanishing theorem on $H_k(M(\mathcal{A}), \mathcal{L})$ and a refinement.

# 4 An application

Vanishing Thm. (Aomoto, Kita-Noumi, Kohno, ...)

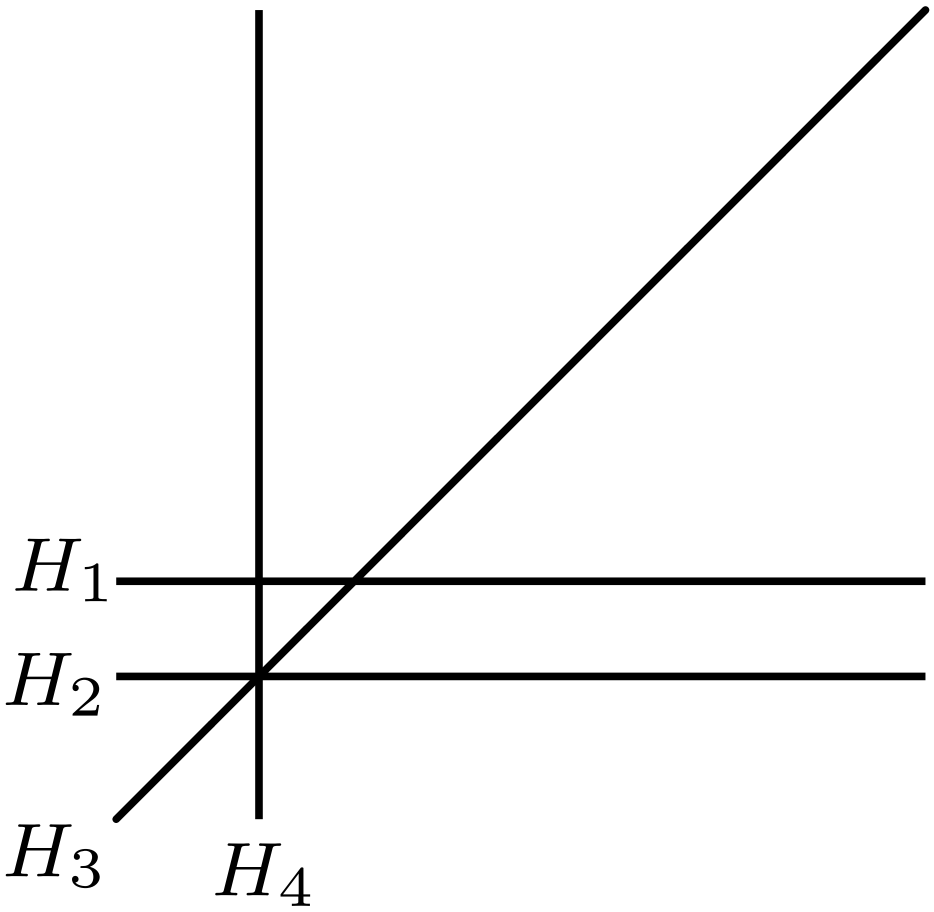
Suppose $\mathcal{L}$ is "generic". Then

$$H^k(M, \mathcal{L}) = \begin{cases} 0 & k \neq \ell, \\ \bigoplus_{C \in \mathrm{bch}} [C] & k = \ell. \end{cases}$$

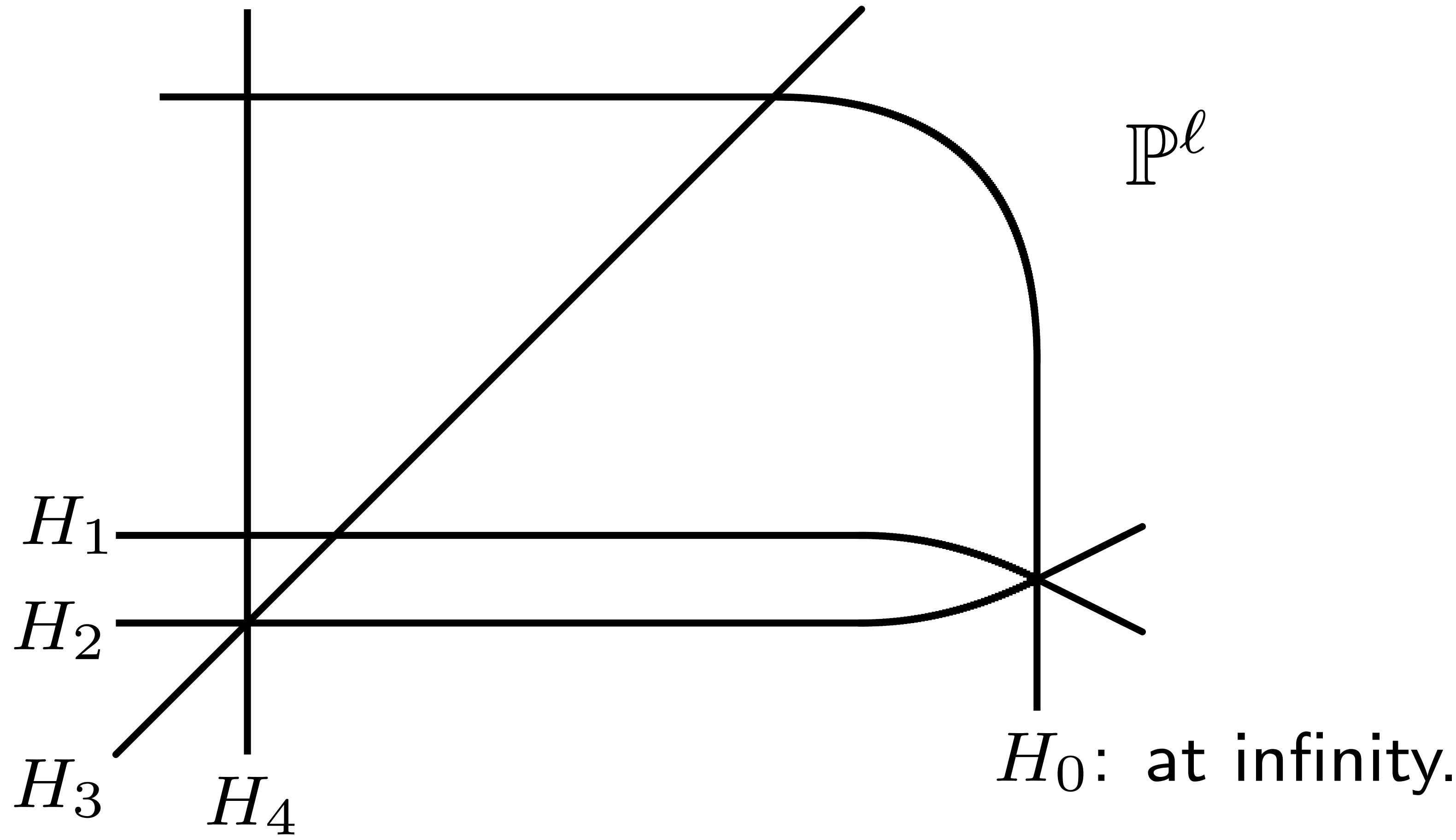
## 4.0 Notation

A local system $\mathcal{L}$ is determined by $(t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$. Consider $t_i$ is the local monodromy around $H_i$.

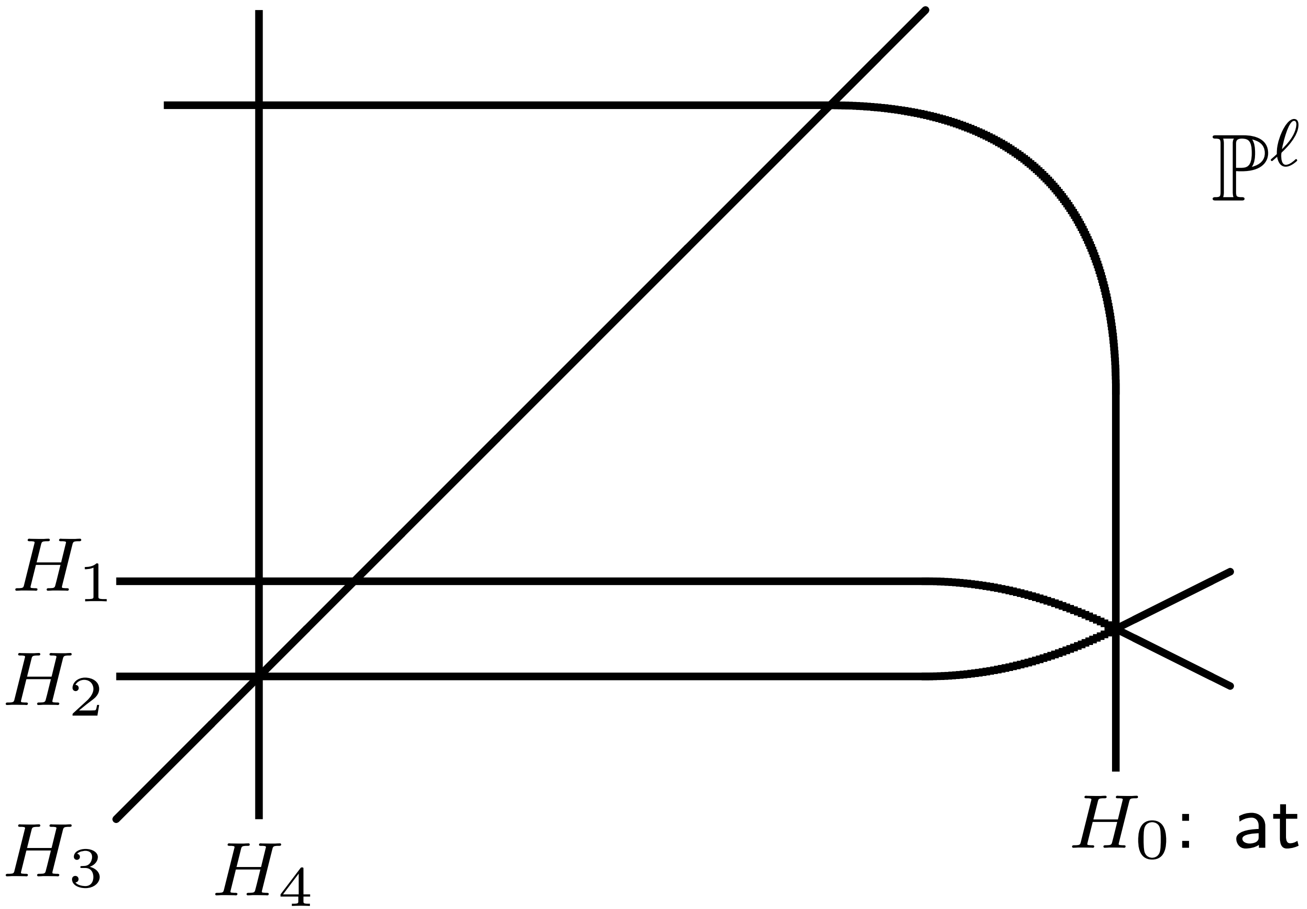
# 4.0 Notation



# 4.0 Notation

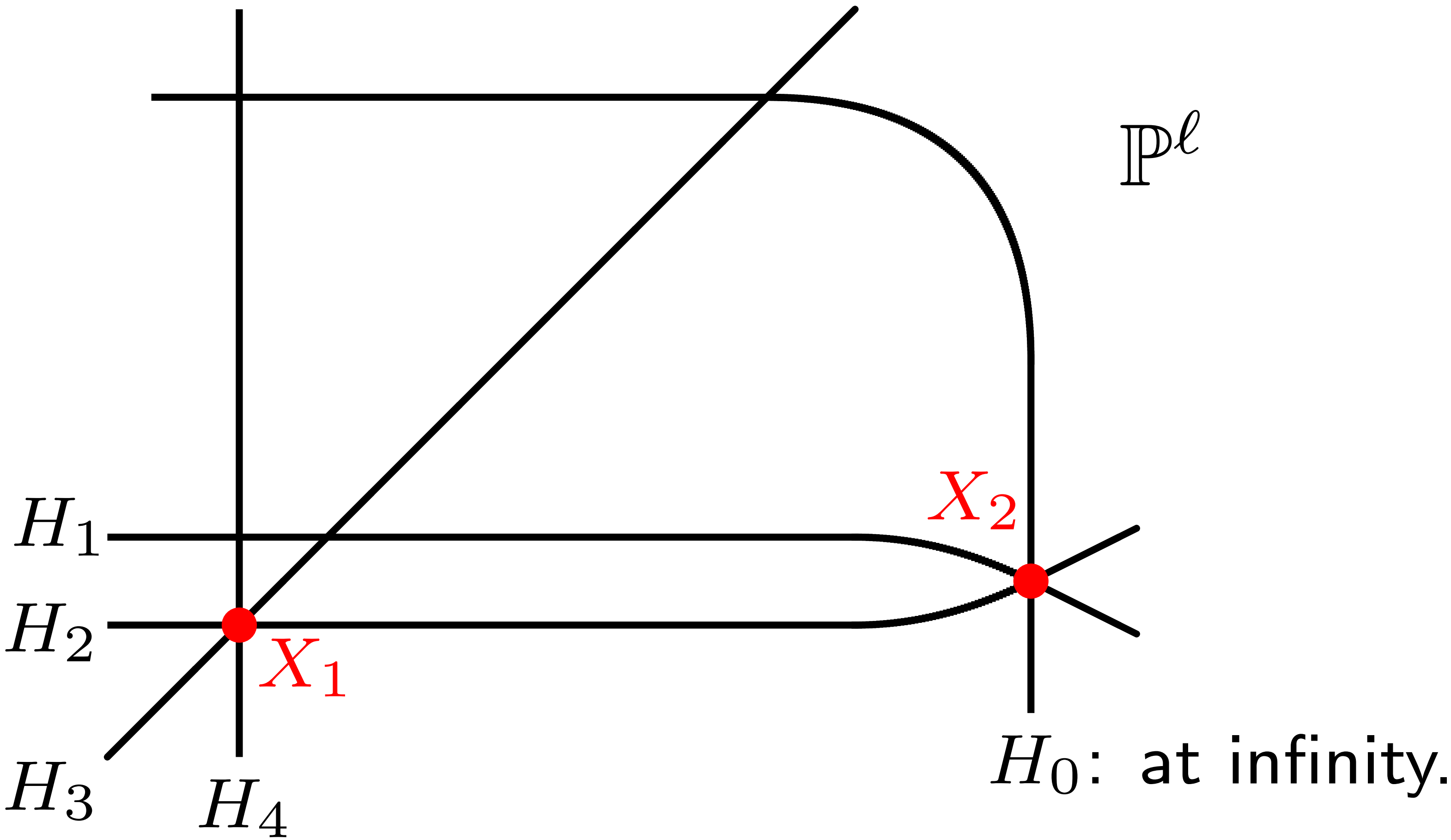


# 4.0 Notation




$$t_0 = (t_1 t_2 \cdots t_n)^{-1}$$

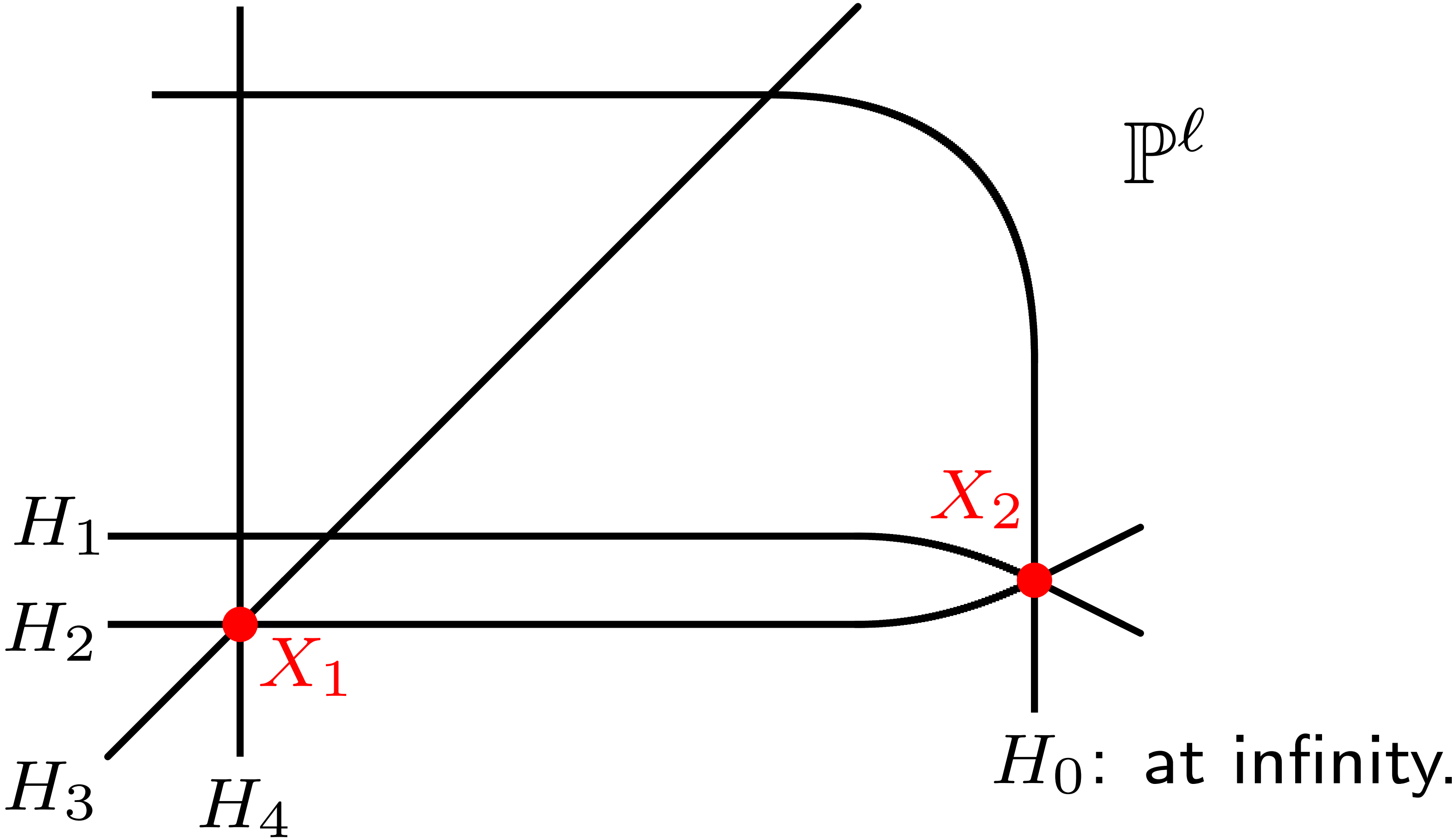
# 4.0 Notation




$$t_0 = (t_1 t_2 \cdots t_n)^{-1}$$

$X_1, X_2$: dense edge

## 4.0 Notation




$t_{X_1} := t_2 t_3 t_4$

$t_0 = (t_1 t_2 \cdots t_n)^{-1}$

$X_1, X_2$: dense edge

# 4 An application

Thm. (Cohen-Dimca-Orlik)

Suppose $\mathcal{L}$ satisfies the condition:

(*) $t_i \neq 1$ $(i = 0, 1, \ldots, n)$ and $t_X \neq 1$ for any dense edge $X \subset H_0$. Then

$$H^k(M, \mathcal{L}) = \begin{cases} 0 & k \neq \ell, \\ \bigoplus_{C \in \mathrm{bch}} [C] & k = \ell. \end{cases}$$

# 4 An application

Concrete attaching maps of the previous section enable us to have a purely topological proof to the vanishing result. Moreover, also a converse:

# 4 An application

Thm. $\ell = 2$, $\mathcal{A}$: indecomposable. Then TFAE.

(1) $\mathcal{L}$ satisfies (*).

(2)

$$H^k(M, \mathcal{L}) = \begin{cases} 0 & k \neq \ell, \\ \bigoplus_{C \in \mathrm{bch}} [C] & k = \ell. \end{cases}$$

(3) $\{[C]\}_{C \in \mathrm{bch}}$ generate $H^\ell(M, \mathcal{L})$.

# 5 References

A. Dimca, S Papadima, Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements. Ann. of Math (2) 158 (2003).

R. Randell, Morse theory, Milnor fibers and minimality of hyperplane arrangements. Proc. A. M. S. 130 (2002).

M. Salvetti, S. Settepanella, Combinatorial Morse theory and minimality of hyperplane arrangements. Geometry and Topology, 11 (2007)

M. Yoshinaga, Hyperplane arrangements and Lefschetz's hyperplane section theorem. Kodai Math. J. 30 (2007)