Euler characteristic reciprocity for chromatic and order polynomials.

(j.w. T. Hasebe)

Masahiko Yoshinaga (Hokkaido U.)

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Contents

Reference: T. Hasebe, M. Yoshinaga, arxiv: 1601.00254

1. Combinatorial Reciprocity

1. Combinatorial Reciprocity

An Example from high-school math.

$p, n > 0$, $[n] = \{1, 2, 3, \cdots, n\}$.

$$\mathcal{O}_p^{<}(n) := \#\{(x_1, \cdots, x_p) \in [n]^p \mid x_1 < x_2 < \cdots < x_p\}$$

$$\mathcal{O}_p^{\leq}(n) := \#\{(x_1, \cdots, x_p) \in [n]^p \mid x_1 \leq x_2 \leq \cdots \leq x_p\}$$

The following formulae are well-known.

$$\mathcal{O}_p^{<}(n) = \binom{n}{p} = \frac{n(n-1)\cdots(n-p+1)}{p!} \in \mathbb{Q}[n]$$

$$\mathcal{O}_p^{\leq}(n) = \binom{n+p-1}{p} = \frac{(n+p-1)\cdots(n+1)n}{p!} \in \mathbb{Q}[n]$$

1. Combinatorial Reciprocity

$$\#\{(x_1,\cdots,x_p)\in[n]^p \mid x_1<x_2<\cdots<x_p\} = O_p^{<}(n) = \frac{n(n-1)\cdots(n-p+1)}{p!}$$

$$\#\{(x_1,\cdots,x_p)\in[n]^p \mid x_1\le x_2\le\cdots\le x_p\} = O_p^{\le}(n) = \frac{(n+p-1)\cdots(n+1)n}{p!}$$

Consider these as polynomials on n.

These two polynomials are related by "reciprocity"

$$O_p^{<}(n) = (-1)^p\cdot O_p^{\le}(-n).$$

1. Combinatorial Reciprocity

"Combinatorial reciprocity" (after R. Stanley):

An enumerative problem

$\downarrow$

The counting function

$F_1(n)$

Another enumerative problem

$\downarrow$

The counting function

$F_2(n)$

$$F_1(n) = \pm F_2(-n)$$

Different enumerative problems are sometimes related by "combinatorial reciprocity".

1. Combinatorial Reciprocity

"Combinatorial reciprocity" (after R. Stanley) :

An enumerative problem
$\downarrow$
The counting function
$F_1(n)$

Another enumerative problem
$\downarrow$
The counting function
$F_2(n)$

$$F_1(n) = \pm F_2(-n)$$

<u>Examples</u> (Stanley)

① P : a finite poset. $\#\mathrm{Hom}^{<}(P,[n]) \longleftrightarrow \#\mathrm{Hom}^{\leq}(P,[n])$.

(A generalization of "high-school reciprocity". disscussed in detail later.)

② G : a finite simple graph.

Regular n-coloring $\longleftrightarrow$ Acyclic orientation with compatible maps to $[n]$.

(NOT discussed today)

1. Combinatorial Reciprocity

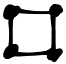
Example

③ Ehrhart reciprocity: Let $P \subset \mathbb{R}^d$ be a lattice polytope.

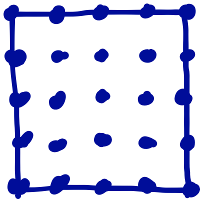
$$L_P(n) := \#(\mathbb{Z}^d \cap n \cdot P)$$

$$L_P^{\circ}(n) := \#(\mathbb{Z}^d \cap n \cdot P^{\circ})$$ The interior of P

$$\Longrightarrow \quad L_P^{\circ}(n) = (-1)^d \cdot L_P(-n).$$

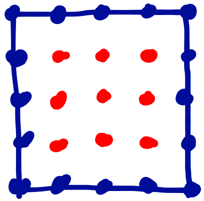
$P =$ 

The unit square $\Longrightarrow$



$(n \leftrightarrow -n)$

$\longleftrightarrow$



$L_P(n) = (n+1)^2$ $\qquad$ $L_P^{\circ}(n) = (n-1)^2$

1. Combinatorial Reciprocity

Back to "high school" again. $p>0$.

$\#\{(x_1,\cdots,x_p)\in[n]^p \mid x_1<x_2<\cdots<x_p\}$

$= O_p^{<}(n)$

$$= \frac{n(n-1)\cdots(n-p+1)}{p!}$$

$\#\{(x_1,\cdots,x_p)\in[n]^p \mid x_1\le x_2\le\cdots\le x_p\}$

$= O_p^{\le}(n)$

$$= \frac{(n+p-1)\cdots(n+1)n}{p!}$$

These polynomials count the # of maps of posets.

$$O_p^{<}(n) = \#\{f:[p]\longrightarrow[n] \mid i<j \Rightarrow f(i)<f(j)\}$$

$$O_p^{\le}(n) = \#\{f:[p]\longrightarrow[n] \mid i<j \Rightarrow f(i)\le f(j)\}$$

1. Combinatorial Reciprocity

These polynomials count the # of maps of posets.

$$\Omega_p^{<}(n) = \#\{f : [p] \to [n] \mid i < j \Rightarrow f(i) < f(j)\}$$

$$\Omega_p^{\leq}(n) = \#\{f : [p] \to [n] \mid i < j \Rightarrow f(i) \leq f(j)\}$$

Abstraction (after Stanley) : Replace $[p], [n]$ by posets.

Def. Let P, Q be poset (Partially Ordered Set).

Define the set of strictly / weakly increasing maps (poset hom) by

$$\mathrm{Hom}^{\leq}(P, Q) := \left\{ f : P \to Q \;\middle|\; \begin{array}{l} x_1, x_2 \in P,\ x_1 < x_2 \\ \Rightarrow f(x_1) \leq f(x_2) \end{array} \right\}$$

1. Combinatorial Reciprocity

<u>Def.</u> Let P, Q be poset (Partially Ordered Set).

$$\mathrm{Hom}^{\le}(P,Q) := \left\{ f : P \to Q \;\middle|\; \begin{array}{l} x_1, x_2 \in P,\ x_1 < x_2 \\ \quad \Rightarrow f(x_1) \le f(x_2) \end{array} \right\}$$

Let $Q = [n] = \{1, 2, \cdots, n\}$. Then,

$$\mathrm{Hom}^{\le}(P,[n]) = \{ f : P \to [n] \mid x_1 < x_2 \Longrightarrow f(x_1) \le f(x_2) \}.$$

<u>Thm.</u> (Stanley) Let P be a finite poset.

(i) $\exists\, \mathcal{O}_P^{<}(t),\ \mathcal{O}_P^{\le}(t) \in \mathbb{Q}[t]$ s.t. $\mathcal{O}_P^{\le}(n) = \#\mathrm{Hom}^{\le}(P,[n])$.

(ii) (Reciprocity) $\mathcal{O}_P^{<}(t) = (-1)^{\#P} \cdot \mathcal{O}_P^{\le}(-t)$.

1. Combinatorial Reciprocity

$\mathrm{Hom}^{\lneq}(P,[n]) := \{ f : P \to [n] \mid x_1 < x_2 \Rightarrow f(x_1) \lneq f(x_2) \}$.

Thm. (Stanley) Let P be a finite poset.

(i) $\exists\, \mathcal{O}_P^{<}(t), \mathcal{O}_P^{\leq}(t) \in \mathbb{Q}[t]$ s.t. $\mathcal{O}_P^{\lneq}(n) = \#\mathrm{Hom}^{\lneq}(P,[n]\}$.

(ii) (Reciprocity) $\mathcal{O}_P^{<}(t) = (-1)^{\#P} \cdot \mathcal{O}_P^{\leq}(-t)$.

A Question. Can we generalize Stanley's result $[n] \rightsquigarrow Q$? (a poset)

Very informaly, Stanley's reciprocity can be stated as

"$\#\mathrm{Hom}^{<}(P,[n]) = (-1)^{\#P} \cdot \mathrm{Hom}^{\leq}(P, [-n])$"

Can we make rigorous "$\#\mathrm{Hom}^{<}(P,Q) = (-1)^{\#P} \cdot \#\mathrm{Hom}^{\leq}(P, -Q)$"?

$\rightsquigarrow$ Yes!

2. Euler characteristics

2. Euler characteristics

Starting observation:

the Euler characteristics of semi-algebraic sets satisfy "reciprocity-like" formula.

Recall, $X \subseteq \mathbb{R}^N$ is a semi-algebraic set iff it is a Boolean connection $(\cup, \cap, -^c)$ finitely many sets of the form $\{x \in \mathbb{R}^N \mid f(x) > 0\}$, for $f \in \mathbb{R}[x_1, \cdots, x_N]$

2. Euler characteristics

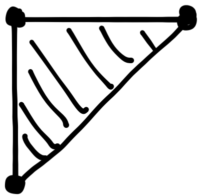
Recall, $X \subseteq \mathbb{R}^N$ is a semi-algebraic set iff it is a Boolean connection $(\cup, \cap, -^c)$ finitely many sets of the form $\{x \in \mathbb{R}^N \mid f(x) > 0\}$, for $f \in \mathbb{R}[x_1, \cdots, x_N]$

<u>Example</u> Closed / Open simplex

$$\sigma_d := \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_d \leq 1\}$$

$$\sigma_d^\circ := \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < x_2 < \cdots < x_d < 1\}$$

σ_2 :



σ_2° :

2. Euler characteristics

$$\sigma_d := \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_d \leq 1\}$$

$$\sigma_d^\circ := \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < x_2 < \cdots < x_d < 1\}$$

Every semi-algebraic set $X \subseteq \mathbb{R}^N$ has a stratification

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

s.t.

$$\# \Lambda < \infty, \quad X_\lambda \cong \sigma_{d_\lambda}^\circ \text{ for some } d_\lambda \geq 0.$$

$(\sigma_0 = \sigma_0^\circ = \{pt\}$ by convention$)$

<u>Def-Thm</u> $e(X) := \sum_{\lambda \in \Lambda} (-1)^{d_\lambda} \in \mathbb{Z}$ is independent of the stratification. (the Euler characteristic)

2. Euler characteristics

$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$, $X_\lambda \cong \sigma^\circ_{d_\lambda}$ for some $d_\lambda \geq 0$.

<u>Def-Thm</u> $e(X) := \sum_{\lambda \in \Lambda} (-1)^{d_\lambda} \in \mathbb{Z}$ is independent of the stratification. (the Euler characteristic)

<u>Facts</u> ① $X \simeq Y$ (homeomorphic) $\Rightarrow e(X) = e(Y)$

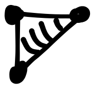
② $e(X \sqcup Y) = e(X) + e(Y)$

③ $e(X \times Y) = e(X) \cdot e(Y)$.

<u>Remark</u> $e(X)$ is not a homotopy invariant

<u>Example</u>

$e(\sigma^\circ_d) = (-1)^d$, $\sigma(\sigma_d) = 1$

$\Bigg(e\Big($  $\Big) = e(\sigma^\circ_2) + 3 \cdot e(\sigma^\circ_1) + 3 \cdot e(\sigma^\circ_0) = 1 \Bigg)$

2. Euler characteristics

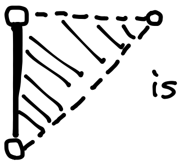
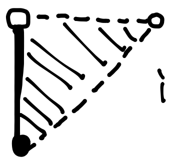
$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$, $X_\lambda \cong \sigma^\circ_{d_\lambda}$ for some $d_\lambda \geq 0$.

<u>Def-Thm</u> $e(X) := \sum_{\lambda \in \Lambda} (-1)^{d_\lambda} \in \mathbb{Z}$ is independent of the stratification. (the Euler characteristic)

<u>Remark</u> If X is a locally compact semi-algebraic set, $e(X)$ is the Euler characteristic of the Borel-Moore homology $H^{BM}_*(X, \mathbb{Z})$, which implies the well-definedness of $e(X)$.

It should be noted that even X is not loc. cpt, $e(X)$ is well-defined.

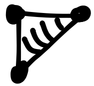
<u>Example</u>

 is locally compact,  is not locally compact.

2. Euler characteristics

Again back to ....

<u>Excample</u>

$e(\sigma_d^\circ) = (-1)^d$, $\sigma(\sigma_d) = 1$ $\Bigg($ $e\Big($  $\Big) = e(\sigma_2^\circ) + 3 \cdot e(\sigma_1^\circ) + 3 \cdot e(\sigma_0^\circ)$ $= 1$ $\Bigg)$

A weaker result may be

$$e(\sigma_d^\circ) = (-1)^d \cdot e(\sigma_d).$$

This is

- looking alike "reciprocity"
- related to the poset structure of the interval

$$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}.$$

2. Euler characteristics

$$\#\{(x_1,\cdots,x_p)\in[n]^p \mid x_1<x_2<\cdots<x_p\}$$

$$= \mathcal{O}_p^{<}(n) = \frac{n(n-1)\cdots(n-p+1)}{p!}$$

$$\#\{(x_1,\cdots,x_p)\in[n]^p \mid x_1\le x_2\le\cdots\le x_p\}$$

$$= \mathcal{O}_p^{\le}(n) = \frac{(n+p-1)\cdots(n+1)n}{p!}$$

$$\mathcal{O}_p^{<}(n) = (-1)^p\cdot \mathcal{O}_p^{\le}(-n)$$

open simplex

$$\sigma_d^\circ = \{(x_1,\cdots,x_d)\in\mathbb{R}^d \mid 0<x_1<\cdots<x_d<1\}$$

$$e(\sigma_d^\circ) = (-1)^d$$

closed simplex

$$\sigma_d = \{(x_1,\cdots,x_d)\in\mathbb{R}^d \mid 0\le x_1\le\cdots\le x_d\le 1\}$$

$$e(\sigma_d) = 1$$

$$e(\sigma_d^\circ) = (-1)^d\cdot e(\sigma_d)$$

These examples should be unified...

2. Euler characteristics

Another story on the Euler characteristic:

The quest for "Negative sets".

Schanuel (1991)

$$\begin{tikzcd}
(\text{Finite sets}) \arrow[r, hook] \arrow[d, "\#"'] & ?? \arrow[d] \\
\mathbb{N}=\{0,1,2,\cdots\} \arrow[r, hook] & \mathbb{Z}
\end{tikzcd}$$

2. Euler characteristics

Another story on the Euler characteristic:

The quest for "Negative sets".

Schanuel (1991)

$$\begin{array}{ccc} (\text{Finite sets}) & \hookrightarrow & (\text{Semi-algebraic Sets}) \\ \# \downarrow & & \downarrow e : \text{the Euler char.} \\ \mathbb{N} = \{0,1,2,\cdots\} & \hookrightarrow & \mathbb{Z} \end{array}$$

Other possibilities: Semi-linear (polyhedral) sets, o-minimal ...

3. Euler characteristic reciprocity

3. Euler characteristic reciprocity

Problems so far: We would like to ...

- generalize Stanley's reciprocity by making rigorous a formula of the form

$$\text{"}\#\mathrm{Hom}^{<}(P, Q) = (-1)^{\#P} \cdot \#\mathrm{Hom}^{\leq}(P, -Q)\text{"}$$

- include $e(\sigma_d^\circ) = (-1)^d \cdot e(\sigma_d)$ as an example of reciprocity.

Schanuel's View suggests to consider semi-algebraic sets and Euler characteristic.

3. Euler characteristic reciprocity

Joint Work with Takahiro Hasebe (Hokkaido U.)

<u>Definition</u> $(P, \leq)$ is a semi-algebraic poset

$\underset{\text{def}}{\iff}$ • $(P, \leq)$ is a poset, and

• P admits a structure of semi-algebraic set such that

$$\{(x, y) \in P \times P \mid x < y\}$$

is also a semi-algebraic set

(In other words, the order "$\leq$" is semi-algebraically defined.)

3. Euler characteristic reciprocity

<u>Definition</u> $(P, \leq)$ is a semi-algebraic poset

$\underset{\text{def}}{\Longleftrightarrow}$
- $(P, \leq)$ is a poset, and
- P admits a structure of semi-algebraic set such that $\{(x, y) \in P \times P \mid x < y\}$ is also a semi-algebraic set

<u>Examples</u>

(1) A finite poset P is a semi-algebraic poset. $(e(P) = \# P)$

(2) The open interval $(0,1)$ is a semi-alg. poset.
$(e((0,1)) = -1)$

3. Euler characteristic reciprocity

Definition $(P, \leq)$ is a semi-algebraic poset

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Examples (1) A finite poset P is a semi-algebraic poset. $(e(P) = \# P)$

(2) The open interval $(0,1)$ is a semi-alg. poset. $(e((0,1)) = -1)$

Prop. If P and Q are semi-alg. posets, then

$P \times Q$ with lexicographic ordering is also a semi-alg poset.

$$(p_1, q_1) \leq (p_2, q_2) \iff \begin{cases} \circ\ p_1 < p_2 \quad \text{or} \\ \circ\ p_1 = p_2 \ \text{and} \ q_1 \leq q_2 \end{cases}$$

3. Euler characteristic reciprocity

<u>Def.</u> Let Q be a semi-algebraic poset. Then define the negative of Q by

$$-Q := Q \times (0,1).$$

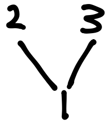
<u>Remarks</u>:

- "$-Q$" is a semi-alg poset with $e(-Q) = e(Q \times (0,1)) = -e(Q)$.
- Even if Q is a finite poset, $-Q$ is no longer finite.
- $(0,1) \times Q \neq \underbrace{Q \times (0,1)}$ (different poset str.)

  only this works for later purposes.
- $-(-Q) \neq Q$.

3. Euler characteristic reciprocity

<u>Prop:</u> Let P be a finite poset, and Q be a semi-alg poset. Then $\mathrm{Hom}^{<}(P,Q)$ is a semi-algebraic set.

<u>Proof for the case</u> $P =$ 

(i.e. $P=\{1,2,3\}$ with ordering $1<2$ and $1<3$. 2 and 3 are not comparable.)

$$\mathrm{Hom}^{<}(P,Q)=\{f:P\to Q \mid f(1)<f(2),\ f(1)<f(3)\}\subset Q^3$$

$$\begin{tikzcd}
 & Q^3 \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{13}"] & \\
\{f(1)<f(2)\}\subset Q^2 & & Q^2 \\
 & & \{f(1)<f(3)\} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

semi-alg. by def.

//

3. Euler characteristic reciprocity

Thm. 1 (Hasebe, Y.)

Let P be a finite poset, and Q be a semi-alg. poset.

Then

$$e(\mathrm{Hom}^{<}(P, \pm Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, \mp Q))$$

Thm. 2 (The role of $\mathcal{O}_P^{\leq}(t) \in \mathbb{Q}[t]$)

Let P be a finite poset, and Q be a totally-ordered semi-algebraic poset. Then

$$e(\mathrm{Hom}^{\leq}(P, Q)) = \mathcal{O}_P^{\leq}(e(Q)).$$

3. Euler characteristic reciprocity

<u>Thm. 1</u> Let P be a finite poset, and Q be a semi-alg. poset.

Then $$e(\mathrm{Hom}^{<}(P, \pm Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, \mp Q))$$

<u>Thm. 2</u> Let P be a finite poset, and Q be a totally-ordered semi-algebraic poset. Then $$e(\mathrm{Hom}^{\leq}(P, Q)) = \mathcal{O}_P^{\leq}(e(Q)).$$

<u>Remark</u>

- Thm 1 + Thm 2 $\xRightarrow{Q=[n]}$ Stanley's reciprocity

- Thm 1 $\xRightarrow[P=[d],\, Q=\{pt\}]{}$ $e(\sigma_d^{<}) = (-1)^d$

3. Euler characteristic reciprocity

<u>Sketch of the proof</u> of

$$e(\mathrm{Hom}^{<}(P, +Q)) = (-1)^{\# P} \cdot e(\mathrm{Hom}^{\leq}(P, -Q)) \qquad Q \times (0,1) = -Q$$

Do the "fiber integral" for

$$\underbrace{\pi : \mathrm{Hom}^{\leq}(P, Q\times(0,1))}_{X} \longrightarrow \mathrm{Hom}^{\leq}(P,Q) = Y$$

$$\mathrm{Hom}^{\leq}(P,Q) \supset \mathrm{Hom}^{<}(P,Q) = Y'$$

$$e(X) = \int_X 1 = \int_Y \pi_* 1 = \underbrace{\int_{Y'} \pi_* 1}_{=(-1)^{\# P} \cdot e(Y)} + \underbrace{\int_{Y \setminus Y'} \pi_* 1}_{=0}$$

"Euler characteristic integral of constructible functions"

0 (need further decomp.)

$\pi|_{\pi^{-1}(Y')}$ is a fibration with fiber $(0,1)^{\# P}$. //

3. Euler characteristic reciprocity

Other results:

- Euler char. reciprocity for chromatic poly. of a simple graph.

  (Hasebe, Y.)

- —— for Ehrhart reciprocity, Not yet. But there is ad hoc Euler char. explanation.

- —— for flow polynomials of a graph.

  (Ongoing. Hasebe, Miyatani. Y.)

Reference: T. Hasebe, M. Yoshinaga, Euler characteristic reciprocity for chromatic and order polynomials. (arXiv: 1601.00254)