Resonant bands, local systems and Milnor fibers of real line arrangements

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Abstract This is a short note on the study of cohomology groups of rank one local systems of real line arrangements via resonant bands. Results on Milnor fibers and several conjectures are also stated.

1 Local systems

Let $\mathscr{A} = \{H_1, H_2, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{C}^2$. We can identify $\mathbb{C}^2$ with $\mathbb{CP}^2 \setminus \overline{H}_0$, where $\overline{H}_0$ is the line at infinity. We define $c\mathscr{A} = \{\overline{H}_0, \overline{H}_1, \dots, \overline{H}_n\}$, where $\overline{H}_i$ is the closure of H_i in $\mathbb{CP}^2$. The complement of lines is denoted by $M(\mathscr{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^n H_i = \mathbb{CP}^2 \setminus \bigcup_{i=0}^n \overline{H}_i$.

We define the character torus by $\mathbb{T}(\mathscr{A}) = \mathrm{Hom}(\pi_1(M(\mathscr{A})), \mathbb{C}^*)$. Since the fundamental group $\pi_1(M(\mathscr{A}))$ is generated by meridians γ_i of H_i ($i = 0, \dots, n$), $\rho \in \mathrm{Hom}(\pi_1(M(\mathscr{A})), \mathbb{C}^*)$ is specified by the images $(\rho(\gamma_0), \rho(\gamma_1), \dots, \rho(\gamma_n)) \in (\mathbb{C}^*)^{n+1}$. By this correspondence, we have the following isomorphism

$$\mathbb{T}(\mathscr{A}) \simeq \{(q_0, q_1, \dots, q_n) \in (\mathbb{C}^*)^{n+1} \mid q_0 q_1 \cdots q_n = 1\}.$$

The character torus $\mathbb{T}(\mathscr{A})$ also can be identified with the moduli space of complex rank one local systems. For given $q = (q_0, q_1, \dots, q_n)$ with $\prod q_i = 1$, we denote by $\mathscr{L}_q$ the associated local system, i.e., the local system which has the monodromy $q_i \in \mathbb{C}^*$ around the line H_i.

The twisted cohomology $H^1(M(\mathscr{A}), \mathscr{L}_q)$ is related to many other problems in topology of $M(\mathscr{A})$. One of the central problem is combinatorial decidability of $H^1(M(\mathscr{A}), \mathscr{L}_q)$.

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2 Chambers and Bands

From now, we assume that each line $H \in \mathscr{A}$ is defined over the real number field $\mathbb{R}$. Our purpose is to describe $H^1(M(\mathscr{A}), \mathscr{L}_q)$ in terms of real structure.

A connected component C of $\mathbb{R}^2 \setminus \bigcup_{i=1}^n H_i$ is called a *chamber*. The set of all chambers is denoted by $\mathrm{ch}(\mathscr{A})$. Let $C, C' \in \mathrm{ch}(\mathscr{A})$. A line $H \in \mathscr{A}$ is said to separate C and C' if C and C' belong opposite half spaces defined by $H \subset \mathbb{R}^2$.

Definition 1. $\mathrm{Sep}(C,C') := \{H \in \mathscr{A} \mid H \text{ separates } C \text{ and } C'\}$.

Definition 2. We call the number of separating lines $d(C,C') := \sharp \mathrm{Sep}(C,C')$ the *distance* of C and C'.

The following object is useful to compute $H^1(M(\mathscr{A}), \mathscr{L}_q)$. See [8, 9] for more details and examples.

Definition 3. A *band* is a region bounded by a pair of consecutive parallel lines.

Let B be a band. Then there are exactly two unbounded chambers in B. We call them $U_1(B)$ and $U_2(B) \in \mathrm{ch}(\mathscr{A})$. The distance $d(U_1(B), U_2(B))$ is called the *length* of the band B, denoted by $|B|$.

Definition 4. Let B be a band bounded by two parallel lines H and H'. The closures $\overline{H}, \overline{H}' \subset \mathbb{RP}^2$ intersects on the line at infinity $\overline{H}_0$. The intersection is denoted by $X(B) := \overline{H} \cap \overline{H}' \in \overline{H}_0$. We also have $X(B) = \overline{B} \cap \overline{H}_0$, where $\overline{B}$ is the closure of B in $\mathbb{RP}^2$.

3 Resonant bands

Let $\mathscr{A} = \{H_1, \dots, H_n\}$ be a line arrangement define over $\mathbb{R}$ as in the previous section. Let $q_1, \dots, q_n \in \mathbb{C}^*$ be nonzero complex numbers. We set $q_0 := (q_1 q_2 \cdots q_n)^{-1}$. For each q_i, we fix $t_i \in \mathbb{C}^*$ such that $t_i^2 = q_i$, $i = 0, 1, \dots, n$.

Definition 5. Let $C, C' \in \mathrm{ch}(\mathscr{A})$. Define

$$\Delta_q(C,C') := \prod_{H_i \in \mathrm{Sep}(C,C')} t_i - \prod_{H_i \in \mathrm{Sep}(C,C')} t_i^{-1}.$$

The following proposition is straightforward.

Proposition 1. $\Delta_q(C,C') = 0$ *if and only if* $\prod_{H_i \in \mathrm{Sep}(C,C')} q_i = 1$.

Definition 6. A band B is said to be $\mathscr{L}_q$*-resonant* if $\Delta_q(U_1(B), U_2(B)) = 0$.

Let B be a band. Note that each line $H \in \mathscr{A}$ is either parallel to B or across B. Hence we have

$$\mathrm{ch}(\mathscr{A}) = (c\mathscr{A})_{X(B)} \sqcup \mathrm{Sep}(U_1(B), U_2(B)), \tag{1}$$

where $(c\mathscr{A})_{X(B)}$ is the set of lines passing through $X(B)$. Using the relation $q_0 q_1 \cdots q_n = 1$, we have the following.

Proposition 2. *A band B is $\mathscr{L}_q$-resonant if and only if* $q_{X(B)} := \prod_{H_i \in (c\mathscr{A})_{X(B)}} q_i = 1$.

Definition 7. Denote by $\mathrm{RB}_{\mathscr{L}_q}(\mathscr{A})$ the set of all $\mathscr{L}_q$-resonant bands.

Next we define a linear map

$$\nabla : \mathbb{C}[\mathrm{RB}_{\mathscr{L}_q}(\mathscr{A})] \longrightarrow \mathbb{C}[\mathrm{ch}(\mathscr{A})] \tag{2}$$

from the vector space spanned by the $\mathscr{L}_q$-resonant bands to the vector space spanned by the chambers.

Definition 8. Let $B \in \mathrm{RB}_{\mathscr{L}_q}(\mathscr{A})$. Define $\nabla(B) \in \mathbb{C}[\mathrm{ch}(\mathscr{A})]$ by the following formula.

$$\nabla(B) := \sum_{C \subset B} \Delta_q(U_1(B), C) \cdot [C].$$

Theorem 1. *Assume that $q_0 \neq 1$. Then*

$$\mathrm{Ker}(\nabla : \mathbb{C}[\mathrm{RB}_{\mathscr{L}_q}(\mathscr{A})] \longrightarrow \mathbb{C}[\mathrm{ch}(\mathscr{A})]) \simeq H^1(M(\mathscr{A}), \mathscr{L}_q).$$

See [9] for proofs and applications. From Theorem 1 we also have the following vanishing result.

Theorem 2. *Assume that $q_0 \neq 1$.*

(i) Suppose that there does not exist point $X \in \overline{H}_0$ such that $|(c\mathscr{A})_X| \geq 3$ and $q_X = 1$. Then $H^1(M(\mathscr{A}), \mathscr{L}_q) = 0$.
(ii)Suppose that there exists unique $X \in \overline{H}_0$ such that $|(c\mathscr{A})_X| \geq 3$ and $q_X = 1$. Then

$$\dim H^1(M(\mathscr{A}), \mathscr{L}_q) = \begin{cases} 0, & \text{if } \exists \overline{H}_i \text{ with } X \notin \overline{H}_i \text{ and } q_i \neq 1, \\ |(c\mathscr{A})_X| - 2, & \text{if } \forall \overline{H}_i \text{ with } X \notin \overline{H}_i \text{ it holds } q_i = 1. \end{cases}$$

Remark 1. By a result by Cohen, Dimca and Orlik [1], Theorem 2 (i) is true for any complex arrangements.

In general, two lines H, H' on the real projective plane $\mathbb{RP}^2$ divides the space into two regions. A pair of lines $\overline{H}_i, \overline{H}_j \in c\mathscr{A} = \{\overline{H}_0, \overline{H}_1, \ldots, \overline{H}_n\}$ is called *sharp pair* if one of two regions does not contain any intersections of $c\mathscr{A}$ in its interior. The existence of sharp pair gives an upper bound of the dimension of the twisted cohomology groups.

Theorem 3. *Suppose that there exists a sharp pair $\overline{H}_i, \overline{H}_j \in c\mathscr{A}$ such that $q_i \neq 1$ and $q_j \neq 1$. Then $\dim H^1(M(\mathscr{A}), \mathscr{L}_q) \leq 1$.*

4 Milnor fibers

The Milnor fiber $F(\mathscr{A})$ of the cone of $c\mathscr{A}$ is a $\mathbb{Z}/(n+1)\mathbb{Z}$ cyclic covering space of $M(\mathscr{A})$. One of the open problems is the combinatorial description of the Betti numbers of $F(\mathscr{A})$, especially $b_1(F(\mathscr{A}))$.

There is a natural automorphism $\rho : F(\mathscr{A}) \longrightarrow F(\mathscr{A})$ so called the monodromy automorphism. Since ρ is order $n+1$, the cohomology group decomposes into the sum of eigen spaces

$$H^k(F(\mathscr{A}),\mathbb{C}) = \bigoplus_{\lambda^{n+1}=1} H^K(F(\mathscr{A}),\mathbb{C})_\lambda, \tag{3}$$

where the sum runs over all complex numbers satisfying $\lambda^{n+1}=1$ and $H^k(F(\mathscr{A}),\mathbb{C})_\lambda$ is the λ-eigenspace.

Let λ be a complex number satisfying $\lambda^{n+1}=1$. Let us denote by $\mathscr{L}_\lambda$ the local system corresponding to $(\lambda,\lambda,\dots,\lambda)\in\mathbb{T}(\mathscr{A})$. It is known [2] that the λ-eigenspace is isomorphic to the twisted cohomology group of $M(\mathscr{A})$, namely, $H^k(F(\mathscr{A}),\mathbb{C}) \simeq H^k(M(\mathscr{A}),\mathscr{L}_\lambda)$. To compute this, we can apply the result in the previous section. Note that $\Delta_\lambda(C,C') = \lambda^{d(C,C')} - \lambda^{-d(C,C')}$.

Now we fix a complex number $\lambda \in \mathbb{C}^*$ of order $k>1$ such that $k|(n+1)$.

Proposition 3. *A band B is $\mathscr{L}_\lambda$-resonant if and only if $k|d(U_1(B),U_2(B))$. Equivalently, $\lambda^{|(c\mathscr{A})_{X(B)}|}=1$.*

Let B be a $\mathscr{L}_\lambda$-resonant band. Then

$$\nabla(B) = \sum_{C\subset B} (\lambda^{d(U_1(B),C)} - \lambda^{-d(U_1(B),C)})\cdot[C]. \tag{4}$$

Theorem 4. $H^1(F(\mathscr{A}),\mathbb{C})_\lambda \simeq \mathrm{Ker}(\nabla : \mathbb{C}[\mathrm{RB}_{\mathscr{L}_\lambda}(\mathscr{A})] \longrightarrow \mathbb{C}[\mathrm{ch}(\mathscr{A})])$.

Using the above theorem, we can prove some vanishing results.

Definition 9. A point $p\in\overline{H}_0$ is said to be a *$\mathscr{L}_\lambda$-resonant edge* if $|(c\mathscr{A})_p|\geq 3$ and $|(c\mathscr{A})_p|$ is divisible by k.

Theorem 5. *If there are no $\mathscr{L}_\lambda$-resonant edge, then $H^1(F(\mathscr{A}))_\lambda = 0$.*

The proof is very easy, actually, we have $\mathrm{RB}_{\mathscr{L}_\lambda}(\mathscr{A})=\emptyset$ by the assumption. Then obviously $\mathrm{Ker}\nabla = 0$. Theorem 5 is due to Libgober [5]. We should note that Libgober's result is more general than Theorem 5, for he proved for any complex arrangements.

We call the affine line arrangement $\mathscr{A}=\{H_1,\dots,H_n\}$ in $\mathbb{R}^2$ is *essential* if there is at least one intersection. This assumption is not a strong restriction. Indeed it avoids only the case "$H_1,\dots,H_n$ are parallel". Under the essential hypothesis, we can strengthen the previous result.

Theorem 6. *Suppose $\mathscr{A}$ is essential. If there exists at most one $\mathscr{L}_\lambda$-resonant edge on $\overline{H}_0$, then $H^1(F(\mathscr{A}))_\lambda = 0$.*

The above theorem says that (if $\mathscr{A}$ is essential) $H^1(F(\mathscr{A}))_\lambda \neq 0$ implies that every line $\overline{H}_i \in c\mathscr{A}$ has at least two points of $\mathscr{L}_\lambda$-resonant edges. It seems natural to ask what happens if there are exactly two $\mathscr{L}_\lambda$-resonant edges on $\overline{H}_0$. The following result answers to it.

Theorem 7. *Suppose that there exist two $\mathscr{L}_\lambda$-resonant edges. If $H^1(F(\mathscr{A}))_\lambda \neq 0$, then $c\mathscr{A}$ is projectively equivalent to the so called A_3-arrangement defined by the equation $xyz(x-z)(y-z)(x-y) = 0$.*

Corollary 1. *Suppose $|c\mathscr{A}| \geq 7$ and $H^1(F(\mathscr{A}))_\lambda \neq 0$. Then each line $\overline{H}_i \in c\mathscr{A}$ has at least three $\mathscr{L}_\lambda$-resonant edges on it.*

Theorem 8. *If $c\mathscr{A}$ has a sparp pair of lines, then* $\dim H^1(F(\mathscr{A}))_\lambda \leq 1$.

5 Conjectures

Conjecture 1. Theorem 6 and Theorem 7 hold for any complex line arrangements.

Conjecture 2. For real arrangement $c\mathscr{A}$, $\dim H^1(F(\mathscr{A}))_\lambda \leq 1$ for any $\lambda \neq 1$. Furthermore, if $\lambda^3 \neq 1$, then $H^1(F(\mathscr{A}))_\lambda = 0$.

For simplicial arrangements ([4]), we have more precise conjecture.

Conjecture 3. Let $c\mathscr{A}$ be a simplicial arrangement on $\mathbb{RP}^2$. Then the following are equivalent.

(1)$H^1(F(\mathscr{A}))_{\neq 1} \neq 0$.
(2)$\dim H^1(F(\mathscr{A}))_{\exp(2\pi\sqrt{-1}/3)} = 1$.
(3)$c\mathscr{A}$ has 3-multinet structure (of multiplicity 1). ([3])
(4)$c\mathscr{A}$ is of type $A(6m,1)$. ([4])

References

1. Cohen, D., Dimca, A., Orlik, P.: Nonresonance conditions for arrangements. *Ann. Inst. Fourier* **53** (2003), 1883–1896.
2. Cohen, D., Suciu, A.: On Milnor fibrations of arrangements. *J. London Math. Soc.* **51** (1995), no. 2, 105-119.
3. Falk, M., Yuzvinsky, S.: Multinets, resonance varieties, and pencils of plane curves. *Compos. Math.* **143** (2007), no. 4, 1069-1088.
4. Grünbaum, B.: A catalogue of simplicial arrangements in the real projective plane. *Ars Math. Contemp.* **2** (2009), no. 1, 1-25.
5. Libgober, A.: Eigenvalues for the monodromy of the Milnor fibers of arrangements. *Trends in singularities*, 141-150, Trends Math., Birkhäuser, Basel, 2002.
6. Suciu, A.: Fundamental groups of line arrangements: enumerative aspects. *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, 43-79, Contemp. Math., 276, Amer. Math. Soc., Providence, RI, 2001.

7. Suciu, A.: Hyperplane arrangements and Milnor fibrations, to appear in Ann. Fac. Sci. Toulouse.
8. Yoshinaga, M.: Milnor fibers of real line arrangements. Journal of Singularities, **7**, 220-237 (2013)
9. Yoshinaga, M.: Resonant bands and local system cohomology groups for real line arrangements. arXiv:1301.1888, Preprint, Jan. 10, 2013.