Minimal stratifications of line arrangements and Milnor fibers

Masahiko Yoshinaga (Hokkaido University)

24 October 2014, Kinosaki

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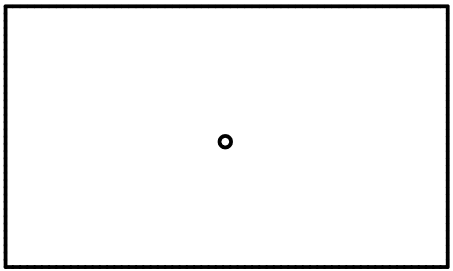
Examples of cell decompostions:

- $\mathbb{CP}^n$

- $\mathbb{C}^* =$ 

1. Cell decompositions of affine variety

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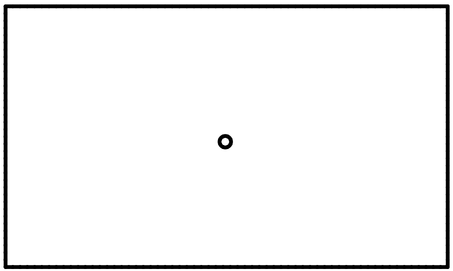
- $\mathbb{C}^* =$ 

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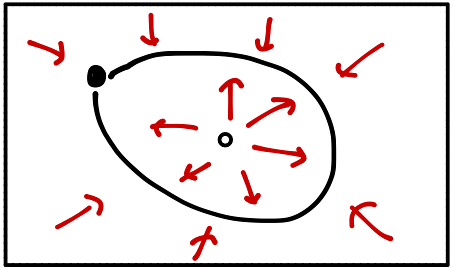
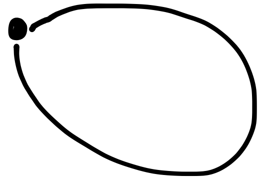
- $\mathbb{C}^* =$  $\approx$ 

$(\dim_{\mathbb{R}} \mathbb{C}^* = 2)$

Homotopic to a 1-$\dim_{\mathbb{R}}$ CW-cpx

More generally, every complex affine variety is homotopic to a finite CW-cpx of half $\dim_{\mathbb{R}}$.

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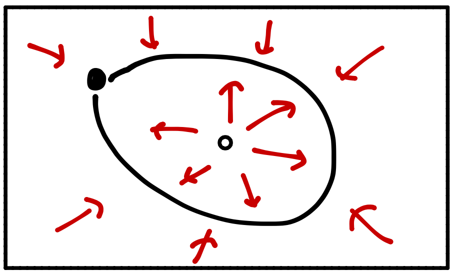
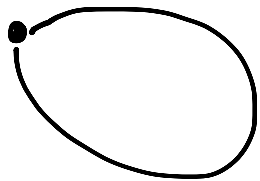
- $\mathbb{C}^* =$  $\approx$ 

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1. Cell decompositions of affine variety

Theorem (Lefschetz?)

Let M be a smooth affine variety $(/\mathbb{C})$ of $\dim_{\mathbb{C}} M = n$.

Then M is homotopic to a finite CW-cpx of $\dim_{\mathbb{R}} \leq n$.

(Proof) Suppose $M \subseteq \mathbb{C}^N$: closed.

Choose $p \in \mathbb{C}^N \setminus M$, generically.

Consider the distance function

$$f : M \longrightarrow \mathbb{R}_{\geq 0}, \quad x \longmapsto \mathrm{dist}(p, x)$$

Then f is Morse with index $\leq n$.

(Q.E.D.)

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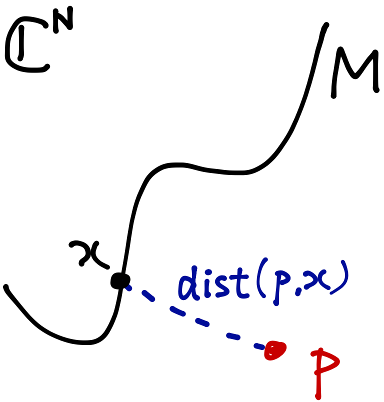

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An explicit example

Let $M := \mathbb{C}\setminus\{0,1\}$.

Consider

$$\begin{aligned} f : M &\longrightarrow \mathbb{R}_{\geq 0} \\ \rotatebox{90}{$\in$}\; & \qquad\quad \rotatebox{90}{$\in$} \\ z &\longmapsto \left| \frac{(z+1)^2}{\sqrt{z(z-1)}} \right| \end{aligned}$$

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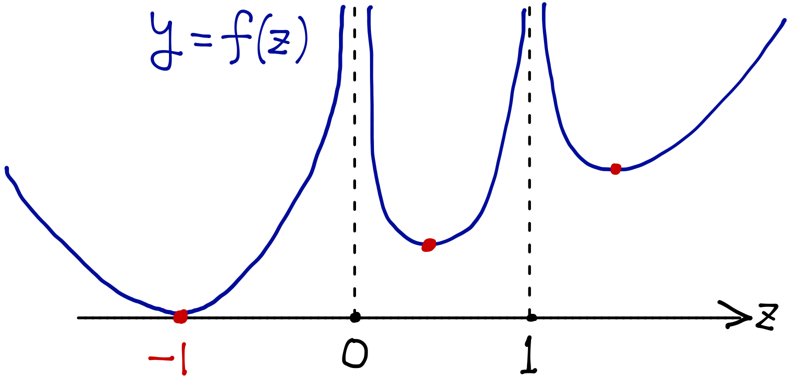

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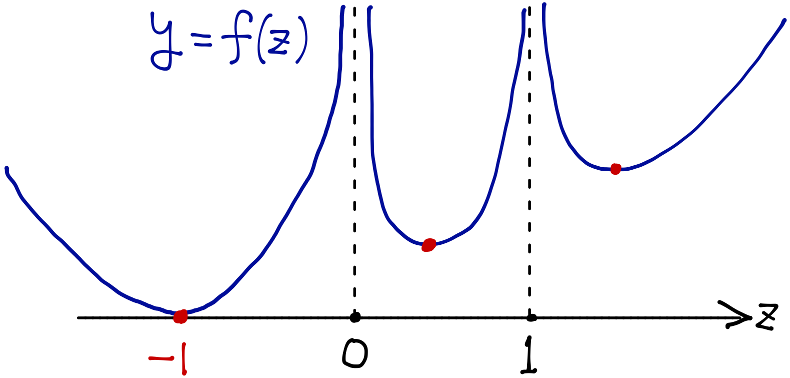

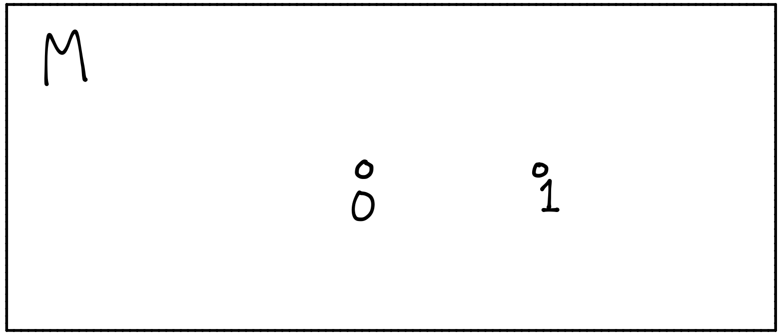

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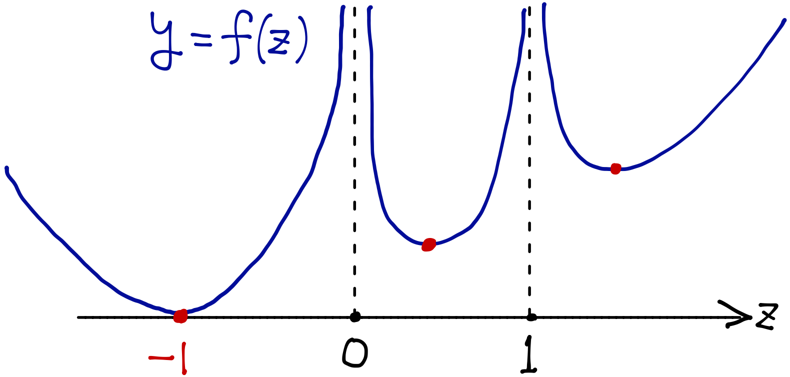

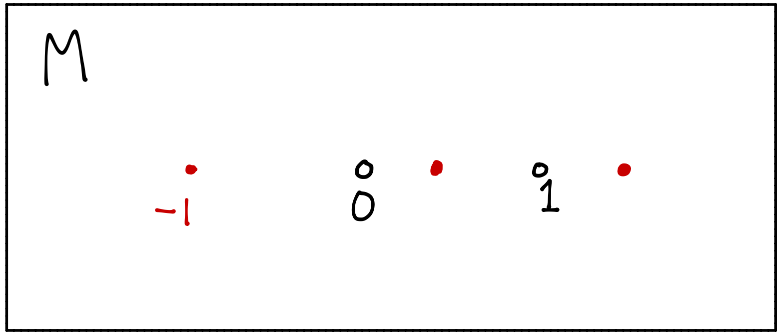

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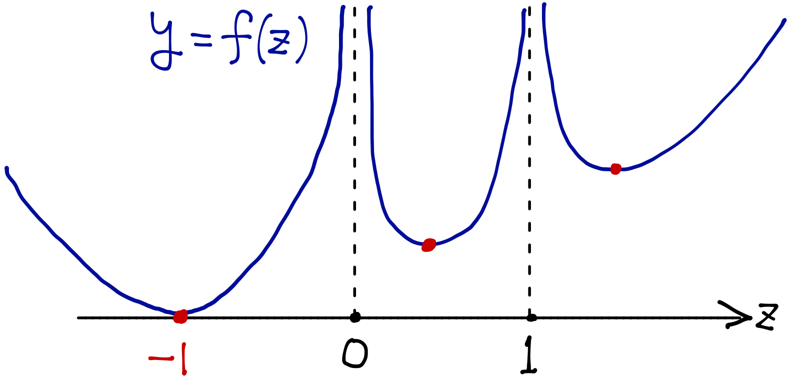
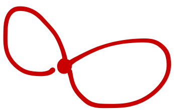
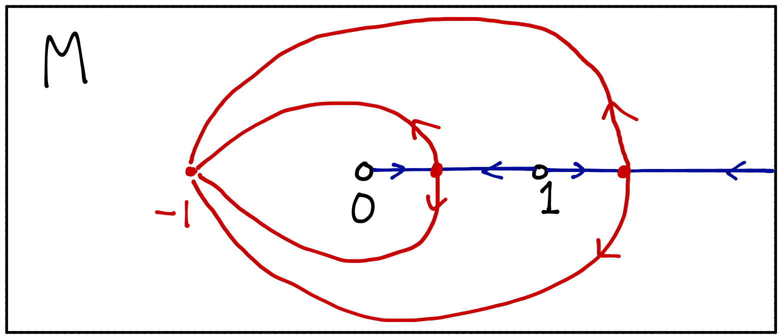


The grad flow $-\mathrm{grad}(f)$ tells the homotopy equiv.

$M \approx$ 



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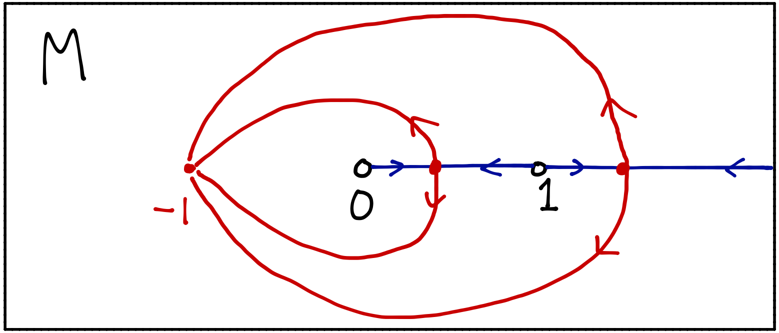

Observations

- The unstable cells are highly transcendental.

- But, the stable cells are "Semi-algebraic".

  $(0,1)$ & $(1,\infty)$

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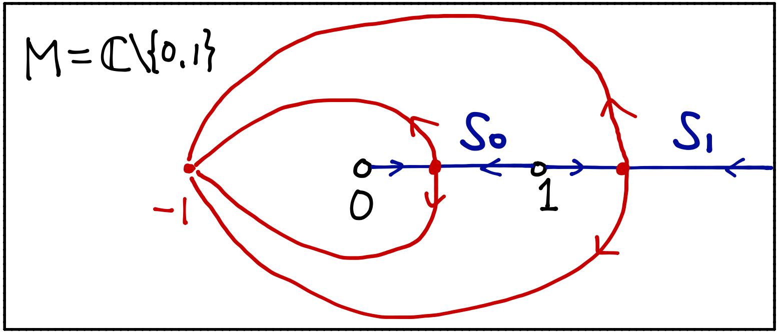


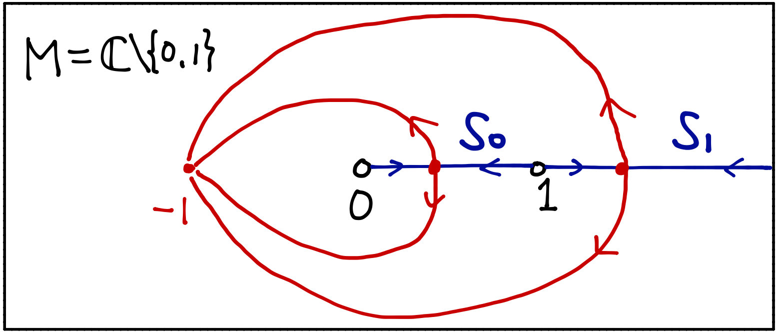
Another description of Stable cells:

$$S_0 = (0,1) = \left\{ z \in M \,\middle|\, \frac{z-1}{z} \in \mathbb{R}_{<0} \right\}$$

"ratio of linear forms" $\in \mathbb{R}_{<0}$

$$S_1 = (1,\infty) = \left\{ z \in M \,\middle|\, \frac{-1}{z-1} \in \mathbb{R}_{<0} \right\}$$

1. Cell decompositions of affine variety



Def.

$$U := M \setminus (S_0 \cup S_1)$$

$$M = U \sqcup S_0 \sqcup S_1$$

CW-cpx

One 0-cell

Two 1-cells

Cell decomp

(Transcendental)

← dual description →

Stratification (or Partition)

One 0-codim stratum (U)

Two 1-codim strata (S_0, S_1)

Strata are contractible.

(Semi-algebraic)

1. Cell decompositions of affine variety

<u>General setting</u> $f_1, \cdots, f_k \in \mathbb{C}[z_1, \cdots, z_n]$

$M := \mathbb{C}^n \setminus \{f_1 f_2 \cdots f_k = 0\}$ hypersurface complement.

<u>Question</u> Can one detect homotopy type of M by using (real) semi-algebraic stratification defined by $f_1, \cdots, f_k$?

<u>A very partial answer</u>

If (1) $n=2$,
(2) $\deg f_i = 1$, and
(3) $f_i \in \mathbb{R}[z_1, z_2]$

$\Longrightarrow$ Then the above idea works. (We can construct a good semi-algebraic stratification)

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2. Minimal Stratification

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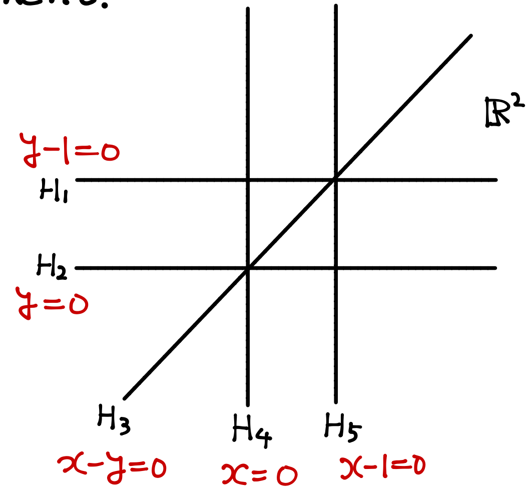
<u>Notations</u> $\mathcal{A}$: a real line arrangement.

i.e.

$\mathcal{A} = \{H_1, \cdots, H_n\}$, $H_i \subset \mathbb{R}^2$: line

$H_i = \{\alpha_i = 0\}$. α_i : defining eq.

$(\alpha_i \in \mathbb{R}[z_1, z_2],\ \deg \alpha_i = 1)$



$M = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i \otimes \mathbb{C}$

: the complexified complement.

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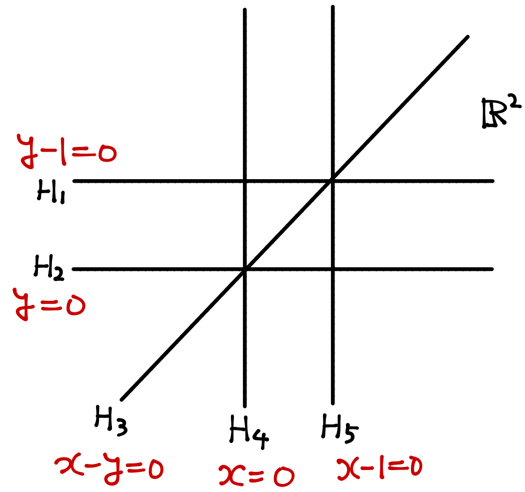


Notation

$\mathcal{A}=\{H_1, H_2, \cdots, H_n\}$, $H_i=\{\alpha_i=0\}$

$M=M(\mathcal{A})=\mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i \otimes \mathbb{C}$

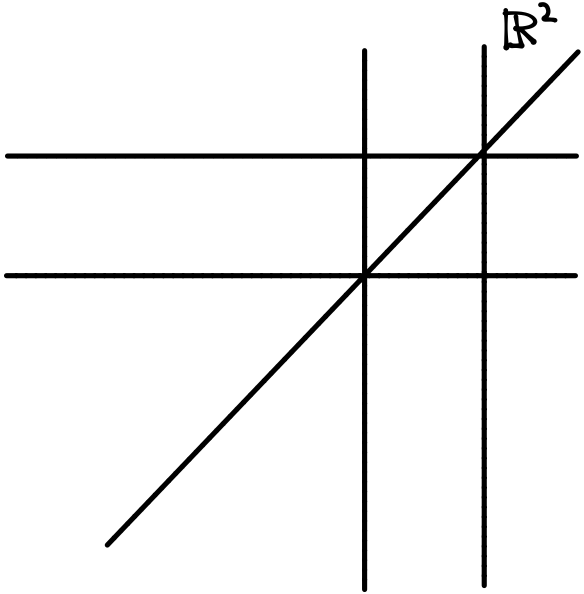
Consider the subset defined by

$$\left\{ (z_1, z_2) \in M(\mathcal{A}) \,\middle|\, \frac{\alpha_i(z)}{\alpha_j(z)} \in \mathbb{R}_{<0} \right\} \subset M(\mathcal{A})$$

In order to get a good stratification, we need more setting.

2. Minimal Stratification

More settings:



① Fix a generic line F (oriented)

② Re-numbering as
$H_1 \cap F < H_2 \cap F < \cdots < H_n \cap F$

③ Fix the sign of α_i so that the half space $H_i^+ := \{\alpha_i > 0\}$ covers positive side of F.

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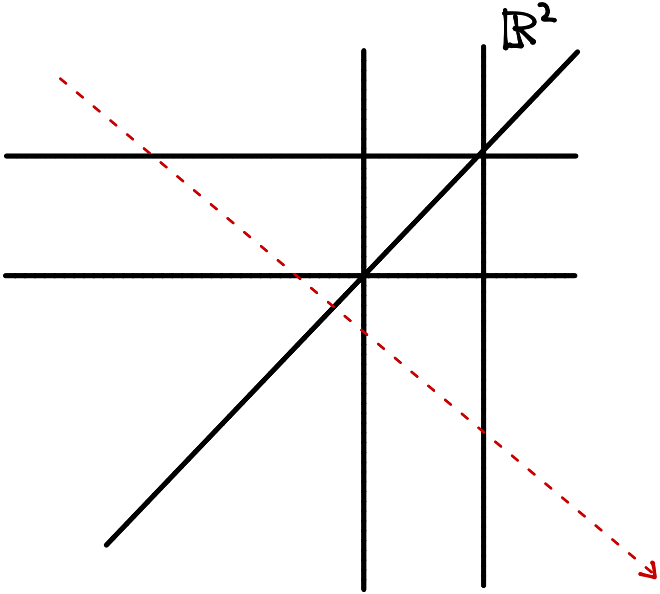

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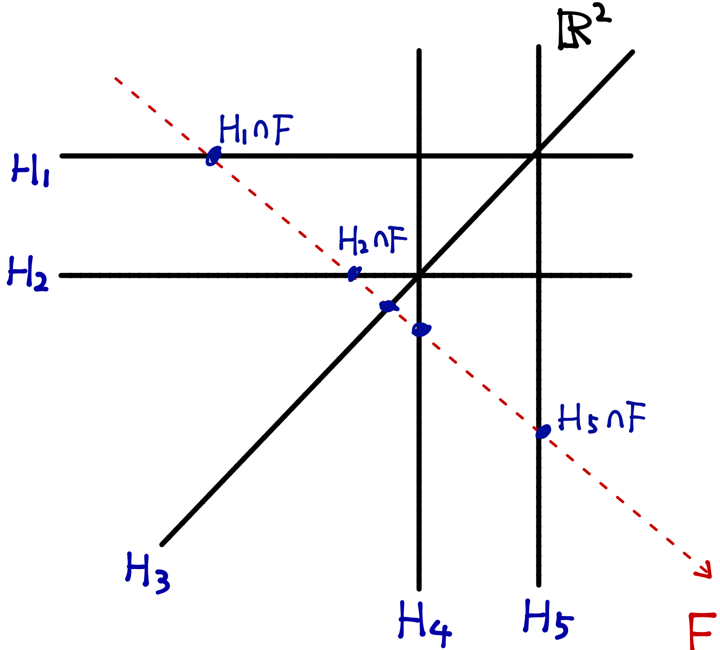

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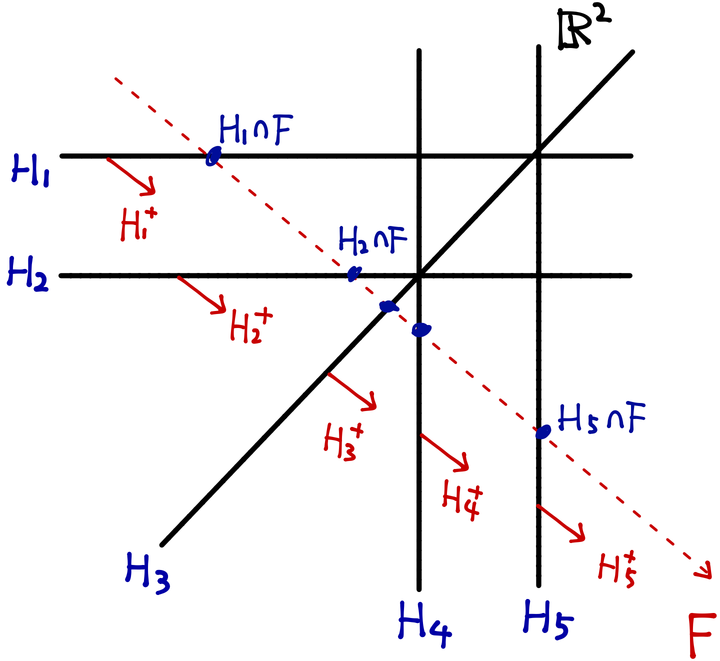

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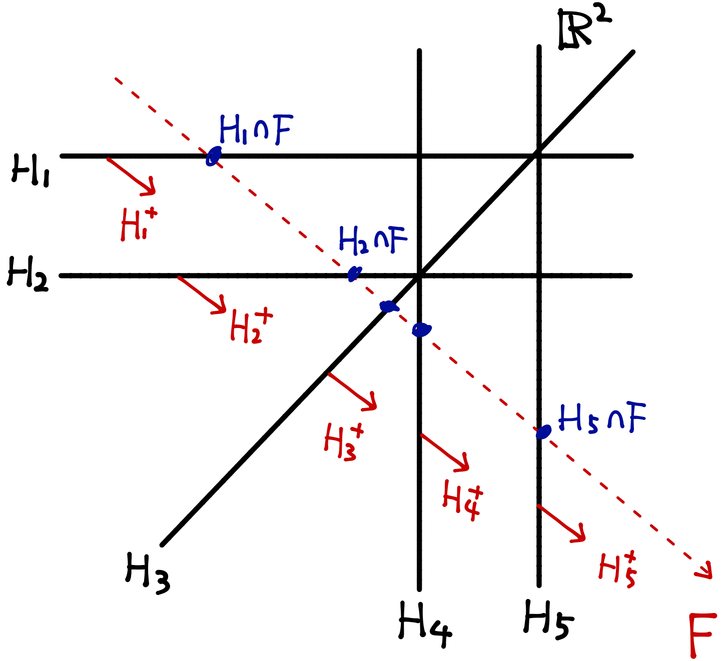

Def. For $i = 1, 2, \cdots, n$.

$$S_i := \left\{ z \in M(\mathcal{A}) \,\middle|\, \frac{\alpha_{i+1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\}$$

where $\alpha_{n+1} = -1$

Then:

Prop.

- $\dim_{\mathbb{R}} S_i = 3$
- $S_i \pitchfork S_j$

(Therefore $\dim_{\mathbb{R}} S_i \cap S_j = 2$)

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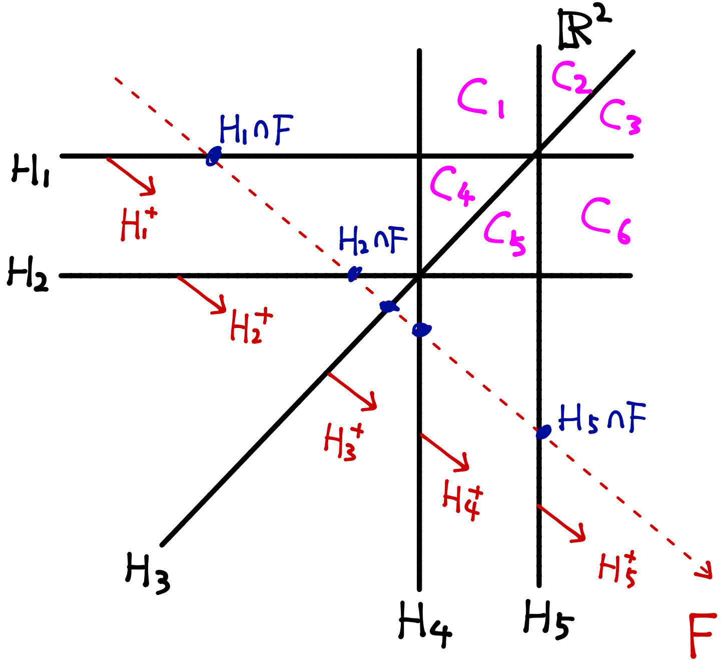


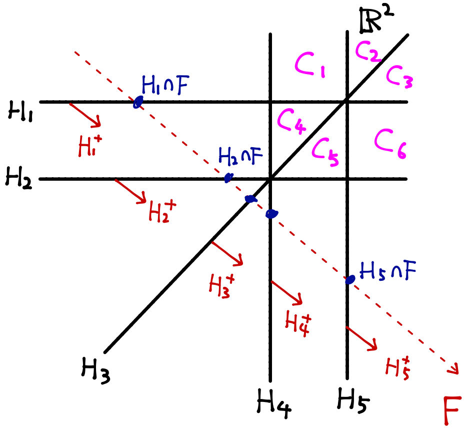
Def.

$$ch_F(\mathcal{A}) := \{ C : \text{chamber} \mid C \cap F = \phi \}$$

Ex. $ch_F(\mathcal{A}) = \{ C_1, C_2, \cdots, C_6 \}$

Prop. $\# ch_F(\mathcal{A}) = b_2(M)$.

2. Minimal Stratification



<u>Def. & Notation</u> $\mathcal{A} = \{H_1, \cdots, H_n\}$, $H_i = \{\alpha_i = 0\}$

- $M = \mathbb{C}^2 \setminus \bigcup H_i \otimes \mathbb{C}$
- $S_i := \left\{ z \in M \,\middle|\, \dfrac{\alpha_{i+1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\}$
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<u>Theorem</u> (Homological level) (Y. 2012)

Borel-Moore homology.

(1) $[S_1], \cdots, [S_n]$ form a basis of $H_3^{BM}(M, \mathbb{Z})$.

(2) $[C]$, $C \in ch_F(\mathcal{A})$ form a basis of $H_2^{BM}(M, \mathbb{Z})$

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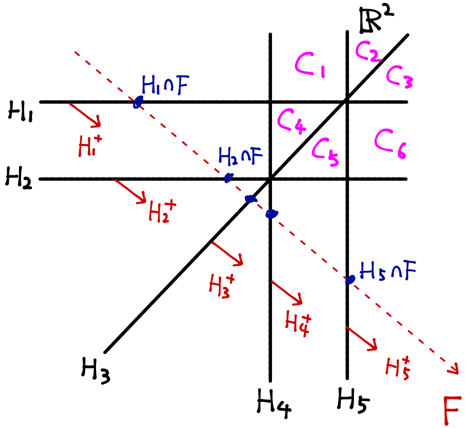

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Theorem (Homotopical level) (Y. 2012)

(i) $S_i \pitchfork S_j$, and $S_i \cap S_j$ is a disjoint union of some chambers in $ch_F(\mathcal{A})$.

(ii) Denote $S_i^\circ := S_i \setminus \bigcup_{C \in ch_F(\mathcal{A})} C$, $\mho = M \setminus \bigcup_i S_i$. Then S_i°, $\mho$ are contractible and $M = \mho \sqcup \bigsqcup_i S_i \sqcup \bigsqcup_{ch_F(\mathcal{A})} C$.

(iii) It is minimal ($b_1 = n$, $b_2 = \# ch_F(\mathcal{A})$).

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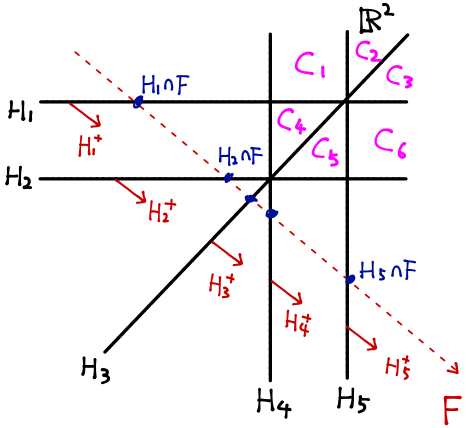

Theorem

(1) $[S_1], \cdots, [S_n]$ form a basis of $H_3^{BM}(M, \mathbb{Z})$.

(2) $[C]$, $C \in ch_F(A)$ form a basis of $H_2^{BM}(M, \mathbb{Z})$

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Proof. ① Homotopical version implies homological ver.

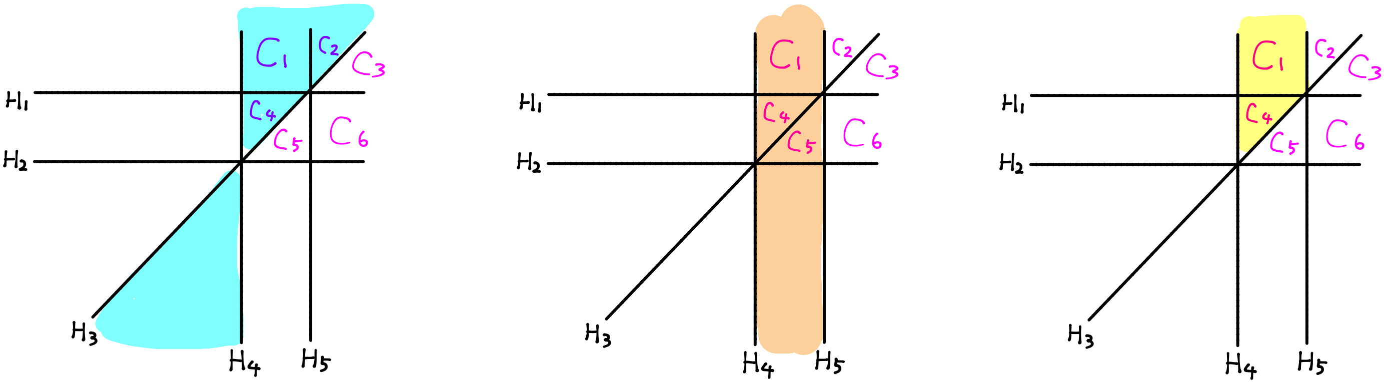
② Proof of homotopical version:

Explicit construction of contractions (complicated!)

Conjecture: Similar construction works in any dim.

2. Minimal Stratification

<u>Example</u> Intersection product in BM homology :



$$S_3 = \left\{ \frac{\alpha_4}{\alpha_3} \in \mathbb{R}_{<0} \right\} \qquad S_4 = \left\{ \frac{\alpha_5}{\alpha_4} \in \mathbb{R}_{<0} \right\} \qquad S_3 \cap S_4 = C_1 \sqcup C_4$$

$$\begin{array}{ccccc} \cap & & \cap & & \cap \\ H_3^{BM}(M,\mathbb{Z}) & \times & H_3^{BM}(M,\mathbb{Z}) & \longrightarrow & H_2^{BM}(M,\mathbb{Z}) \end{array}$$

3. Application to Milnor fibers

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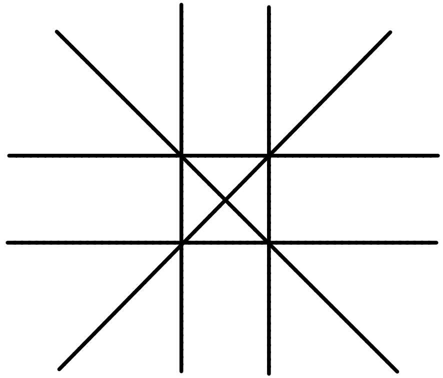
General Problems

"combinatorial decidability"



$\mathcal{A} = \{H_1, \cdots, H_n\}$ hyperplanes

Geometric invariant.

Combinatorial Str. $\Longrightarrow$ $H^*(M,\mathbb{Z})$: combinatorial

(Mainly intersection lattice) $\not\Longrightarrow$

$\updownarrow$?? between these two.

$\pi_1(M)$: Not combinatorial.

(Works by Orlik-Solomon, Rybnikov)

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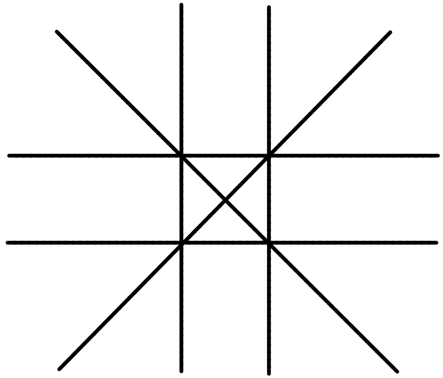

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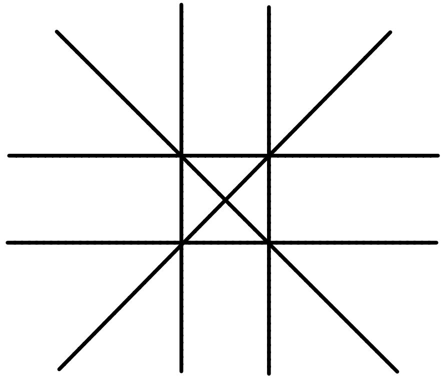

$\mathcal{A} = \{H_1, \cdots, H_n\}$ hyperplanes

Combinatorial Str.

(Mainly intersection lattice)

(Works by Orlik-Solomon, Rybnikov)

Geometric invariant.

$\Longrightarrow H^*(M, \mathbb{Z})$: combinatorial

$\updownarrow$?? between these two.

$\pi_1(M)$: Not combinatorial.

3. Application to Milnor fibers

General Problems

"combinatorial

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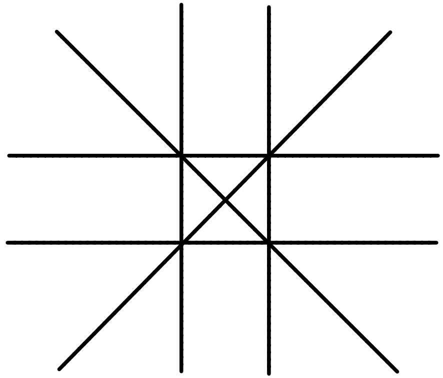

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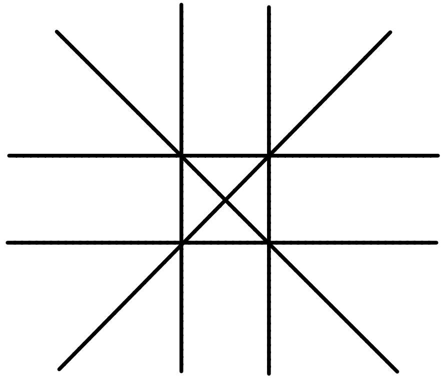

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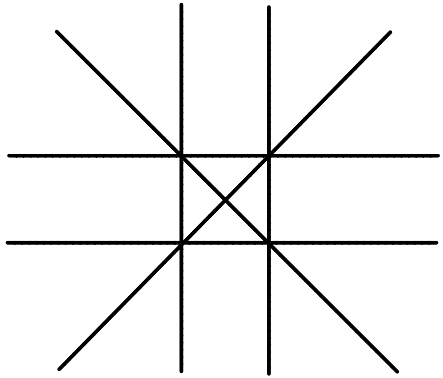

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3. Application to Milnor fibers

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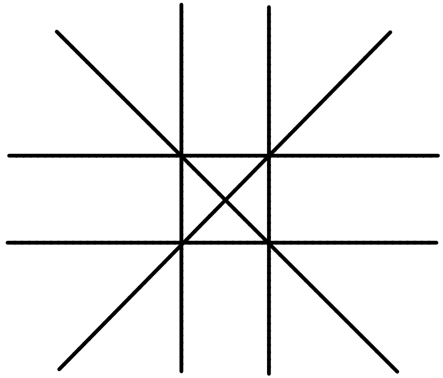

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3. Application to Milnor fibers

<u>General Problems</u> "combinatorial decidability"

$H^*(M,\mathbb{Z})$: combinatorial

- $H^1(F,\mathbb{C}) \circlearrowleft \rho$

  F : Milnor fiber of (the cone of) $\mathcal{A}$

  $\circlearrowleft \rho$: monodromy action.

Between these two

recent target

- $H^*(M,\mathcal{L})$: local system cohomology.

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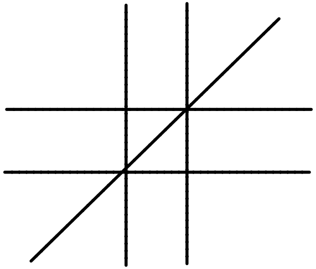
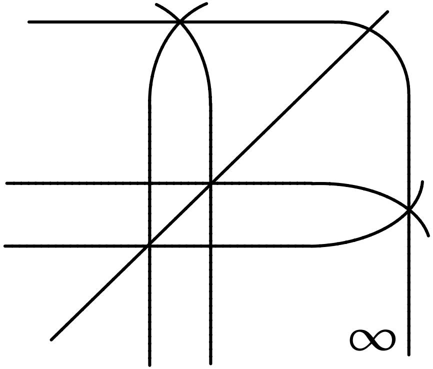
3. Application to Milnor fibers

Affine lines $\rightsquigarrow$ Projectivize, cone.



$x(x-1)y(y-1)(x-y) = 0$



$Q := x(x-z)y(y-z)(x-y)\cdot z = 0.$

a line arrangement

$\mathcal{A} = \{H_1, \cdots, H_n\}$

$c\mathcal{A} := \{H_1, \cdots, H_n, H_\infty\}$

$$M(\mathcal{A}) := \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i = \mathbb{P}^2_{\mathbb{C}} \setminus Q^{-1}(0)$$

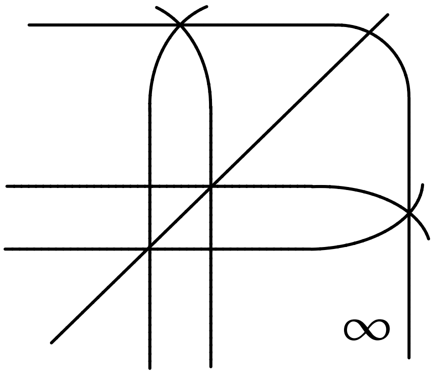
"Milnor Fiber"

$$F = F_{\mathcal{A}} := Q^{-1}(1) = \{(x,y,z) \in \mathbb{C}^3 \mid Q(x,y,z) = 1\} \subset \mathbb{C}^3$$

$\boxed{\text{Problem: } b_1(F) = ?}$

3. Application to Milnor fibers

Remark : $Q(x,y,z)$ is homogeneous of $\deg = n+1$.



$Q := x(x-z)y(y-z)(x-y)\cdot z$

$= 0.$

$F = \{ (x,y,z) \in \mathbb{C}^3 \mid Q(x,y,z) = 1 \}$

$= Q^{-1}(1)$: Milnor fiber (affine surf.)

Monodromy action

$$\begin{array}{ccc} \rho : F & \longrightarrow & F \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x,y,z) & \longmapsto & (\zeta x, \zeta y, \zeta z), \end{array}$$

where $\zeta = e^{2\pi i / n+1}$

3. Application to Milnor fibers

$$\rho : F \longrightarrow F : (x, y, z) \longmapsto (\zeta x, \zeta y, \zeta z),$$

where $\zeta = e^{2\pi i/n+1}$

induces a linear automorphism

$$\rho^* : H^1(F) \longrightarrow H^1(F).$$

We have eigendecomposition

λ-eigen space

$$H^1(F) = \bigoplus_{\lambda^{n+1}=1} H^1(F)_\lambda$$

Since $\rho^{n+1} = id.$

3. Application to Milnor fibers

Easy part: $\lambda = 1$,

ρ-invariant part

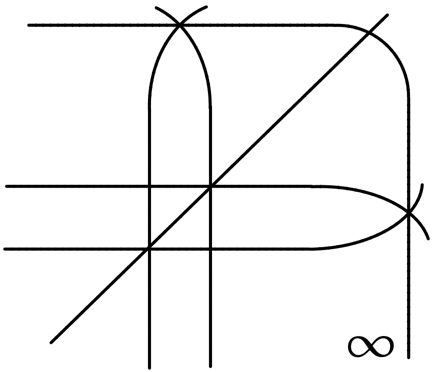
$$H^1(F)_1 = H^1(F)^{\rho} \cong H^1(F/\langle\rho\rangle) \cong H^1(M(\mathcal{A})) \cong \mathbb{C}^n.$$

In general,

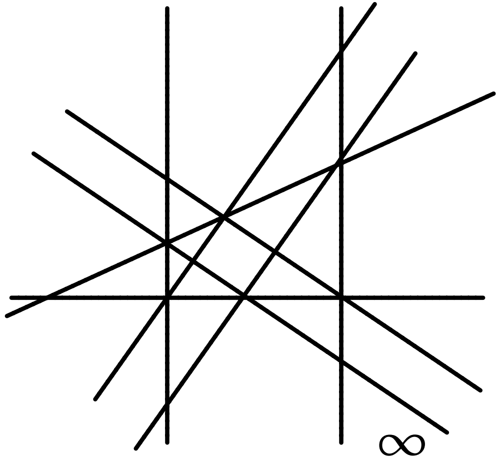
$$H^1(F) = \underset{\text{1-eigen sp.}}{\mathbb{C}^n} \oplus \underbrace{\bigoplus_{\lambda \neq 1} H^1(F)_\lambda}_{\text{non-trivial eigen space.}}$$

3. Application to Milnor fibers

<u>Thm.</u> (Y. 2013) We can formulate an algorithm computing $H^1(F)_\lambda$ by using minimal stratification.



A_3-arr.



Pappus arr.

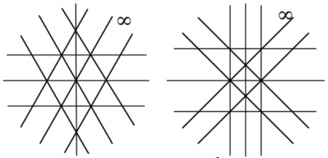
$$\underset{}{H^1(F)} = \underset{\mathbb{C}^5}{\underset{\cong}{H^1(F)_1}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_\zeta}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_{\zeta^2}}}$$

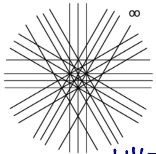
$(\zeta = e^{2\pi i/3})$

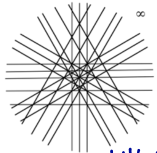
$$H^1(F) = \underset{\mathbb{C}^8}{\underset{\cong}{H^1(F)_1}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_\zeta}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_{\zeta^2}}}$$

3. Application to Milnor fibers

<u>Examples</u> (Grünbaum's Catalogue of simplicial arrangements)


$\mathcal{A}(11,1)$ $H^1(F)_{\neq 1}=0$


$\mathcal{A}(22,4)$ $H^1(F)_{\neq 1}=0$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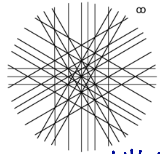
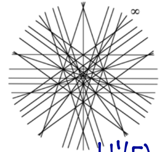
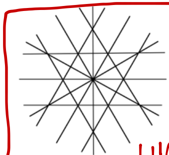
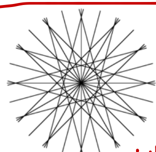
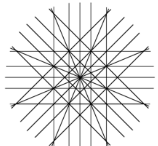

$\mathcal{A}(28,6)$ $H^1(F)_{\neq 1}=0$

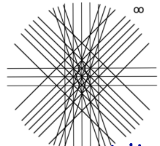

$\mathcal{A}(29,1)$ $H^1(F)_{\neq 1}=0$

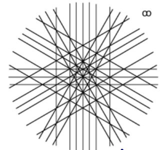

$\mathcal{A}(12,1)$ $H^1(F)_\zeta \cong \mathbb{C}$

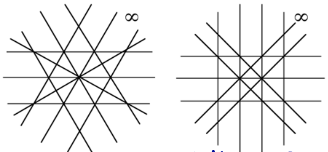
$\zeta = e^{2\pi i/3}$

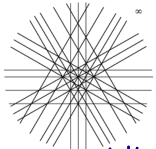

$\mathcal{A}(24,1)$ $H^1(F)_\zeta \cong \mathbb{C}$

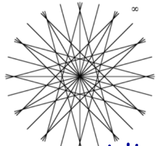

$\mathcal{A}(24,2)$ $H^1(F)_{\neq 1}=0$

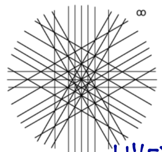

$\mathcal{A}(29,3)$ $H^1(F)_{\neq 1}=0$

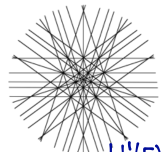

$\mathcal{A}(29,4)$ $H^1(F)_{\neq 1}=0$

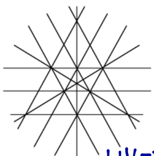

$\mathcal{A}(12,2)$ $H^1(F)_{\neq 1}=0$


$\mathcal{A}(24,3)$ $H^1(F)_{\neq 1}=0$


$\mathcal{A}(25,1)$ $H^1(F)_{\neq 1}=0$


$\mathcal{A}(29,5)$ $H^1(F)_{\neq 1}=0$


$\mathcal{A}(30,2)$ $H^1(F)_{\neq 1}=0$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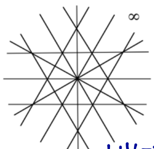
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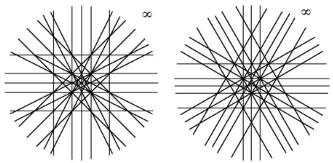
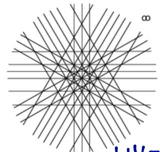

$\mathcal{A}(25,2)$ $H^1(F)_{\neq 1}=0$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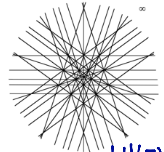

$\mathcal{A}(31,1)$ $H^1(F)_{\neq 1}=0$

3. Application to Milnor fibers

Conjectually, the combinatorial structure called multinet controls nontrivial eigen spaces

$$H^1(F)_{\neq 1} := \bigoplus_{\lambda \neq 1} H^1(F)_\lambda$$

Def. Let $\overline{\mathcal{A}} = \{H_1, \cdots, H_n, H_\infty\}$ be lines on $\mathbb{CP}^2$.

A k-multinet str. ($k \geq 3$) is a partition (coloring) $\overline{\mathcal{A}} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_k$ s.t.

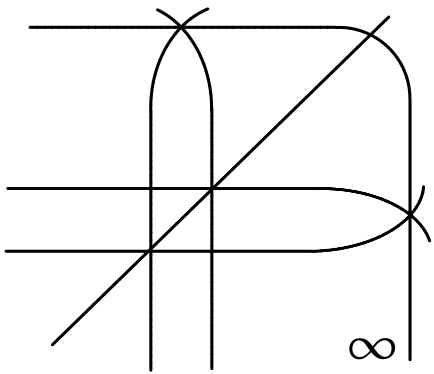
- $|\mathcal{A}_1| = |\mathcal{A}_2| = \cdots = |\mathcal{A}_k|$, and
- at each intersection $p \in \mathbb{P}^2$, either $\overline{\mathcal{A}}_p$ is mono-color or $|(\mathcal{A}_1)_p| = |(\mathcal{A}_2)_p| = \cdots = |(\mathcal{A}_k)_p|$.

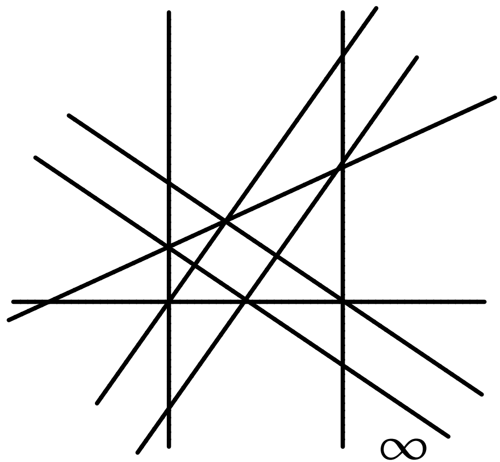
The set of lines passing p.

3. Application to Milnor fibers

Def. A k-multinet str. on $\bar{\mathcal{A}} = \{H_1, \cdots, H_n, H_\infty\}$ is a partition (or coloring)
$\bar{\mathcal{A}} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_k$ s.t. $|\mathcal{A}_1| = |\mathcal{A}_2| = \cdots = |\mathcal{A}_k|$ and at each intersection $p \in \mathbb{P}^2$, $\bar{\mathcal{A}}_p$ is either mono-color, or $|(\mathcal{A}_1)_p| = |(\mathcal{A}_2)_p| = \cdots = |(\mathcal{A}_k)_p|$.

Examples. The following two have 3-multinet str.





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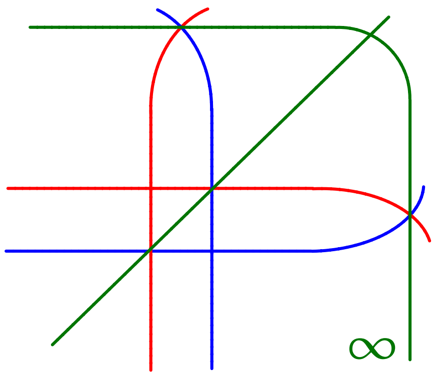

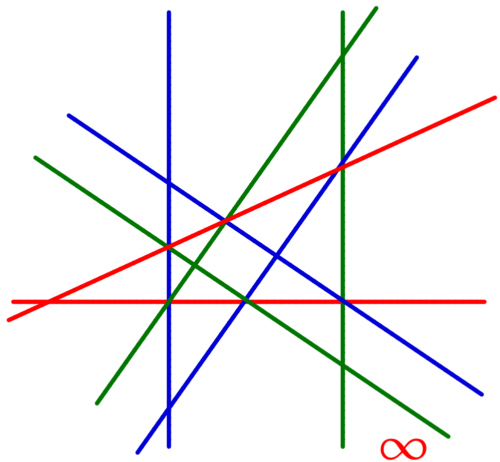

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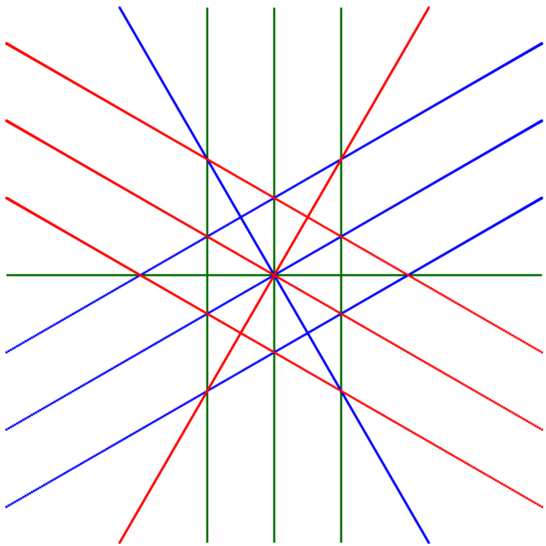
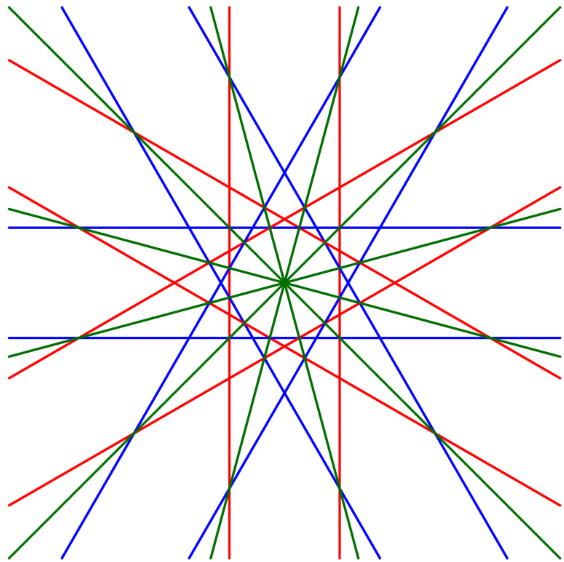




3. Application to Milnor fibers

<u>Example</u> Only one 4-multinet is known.

Let $f_\lambda(z_1,z_2,z_3)=3z_1z_2z_3-\lambda(z_1^3+z_2^3+z_3^3)$, $(\lambda\in\mathbb{C})$

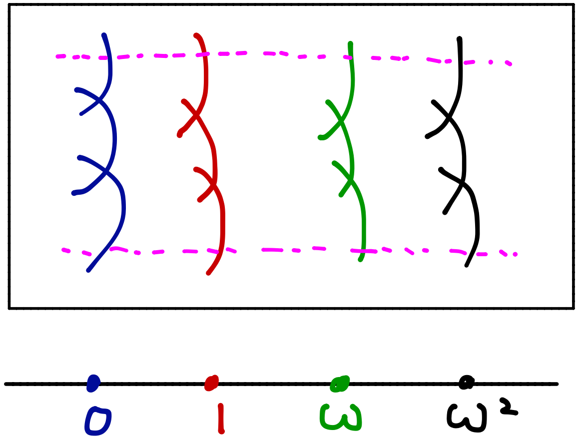
The <u>Hessian arrangement</u>:

$$f_0(z)\cdot f_1(z)\cdot f_\omega(z)\cdot f_{\omega^2}(z)=0 \qquad (\omega=e^{2\pi i/3})$$

has a 4-multinet.

blow up 9 base pts.

$$\mathbb{P}^2 \xrightarrow{[3xyz:x^3+y^3+z^3]} \mathbb{P}^1$$



3. Application to Milnor fibers

Conjecture Let $\lambda \in \mathbb{C}^*$, with order k. Then

$$H^1(F)_\lambda \neq 0 \iff \bar{\mathcal{A}} \text{ has } k\text{-multinet str.}$$

Rem. RHS is purely combinatorial.

Supporting results:

- "$\Leftarrow$" holds (Dimca - Papadima)
- If $\bar{\mathcal{A}}$ has at most triple points, then the conj. is true. (Libgober).

Our recent results intend to contribute to this conj.

3. Application to Milnor fibers

<u>Conjecture</u> Let $\lambda \in \mathbb{C}^*$, with order k. Then

$$H^1(F)_\lambda \neq 0 \iff \bar{\mathcal{A}} \text{ has } k\text{-multinet str.}$$

<u>Thm.</u> (M. Torielli – Y. 2014)

$\bar{\mathcal{A}} / \mathbb{R}$. F: the Milnor fiber. Then

① $\bar{\mathcal{A}}$ can not carry 4-multinet str.

② $H_1(F, \mathbb{Z})$ does not have torsion.

<u>Rem.</u> ① was first proven by Cordovil–Forge (2003)

Proof is heavily relying on minimal stratification.

Reference

- Y. Minimal stratification for line arrangements and positive presentation for π_1. (2012)
- Y. Milnor fibers of real line arrangements. (2013)
- M. Torielli, Y. Resonant bands, Aomoto complex and real 4-nets. (2014)
- M. Torielli, Y. in preparation.