Minimal presentation for fundamental group of Complement of hyperplane arrangement.

§. Minimal CW complex

X : a finite CW complex.

Def

X is minimal

$\overset{\Longleftrightarrow}{\text{def}}$ # of k-dim cells $= b_k(X)$

Rem In general : # of k-dim cells $\geq b_k(X)$

Thm (Dimca-Papadima, Randell)

$\mathcal{A} = \{H_1 \cdots H_n\}$ $H_i \subset \mathbb{C}^\ell$ affine hyperplane

$M := M(\mathcal{A}) := \mathbb{C}^\ell \setminus \bigcup H_i$, then

M is homotopy equivalent to a minimal CW complex.

Considering the case $\ell = 2$

$\rightsquigarrow$ minimal presentation for $\pi_1(M)$

$\left\{ \begin{array}{l} \text{\# of generators} = b_1 \\ \text{\# of relations} = b_2 \end{array} \right.$

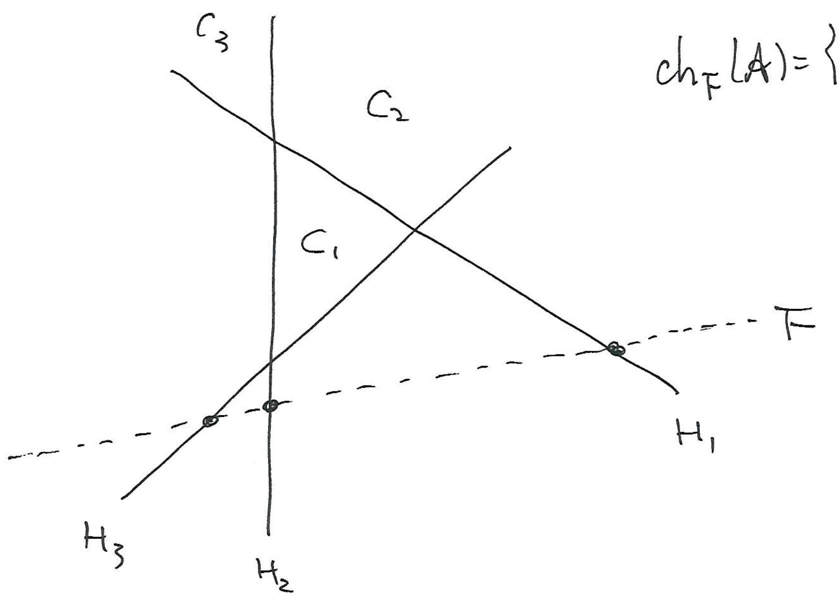
DATE

Rem

$\{(x,y) \in \mathbb{C}^2 \mid x^3 \neq y^2\}$ is NOT minimal.

From now, assume $\ell = 2$. H_i is define over $\mathbb{R}$.

We will give a minimal presentation using combinatorial information.



$\mathrm{ch}_F(\mathcal{A}) = \{C_1, C_2, C_3\}$

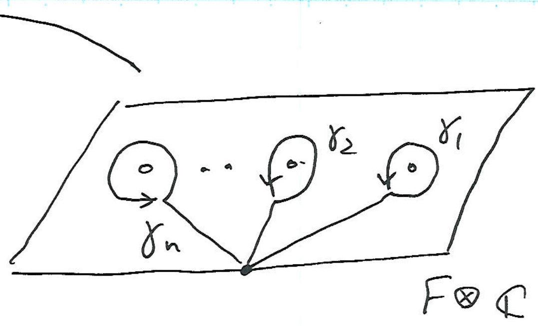
F: a generic line (defined over $\mathbb{R}$)

Def

$$\mathrm{ch}_F(\mathcal{A}) := \{C : \text{chamber} \mid C \cap F = \phi\}$$

$H_i \cap F \longleftrightarrow$ generator

$\mathrm{ch}_F(\mathcal{A}) \longleftrightarrow$ relation



<u>Prop</u> $\mathcal{A} = \{H_1, \cdots, H_n\}$

$$\# \mathrm{ch}_F(\mathcal{A}) = b_2(M)$$

(proof)

$$\begin{array}{ccc} \#\text{ of chambers} & \overset{\text{Zaslawski}}{=\!=} & b_0 + b_1 + b_2 \\ \| & & \begin{array}{cc} \| & \| \\ 1 & n \end{array} \\ n+1+\#\mathrm{ch}_F(\mathcal{A}) & & \end{array}$$

$\square$

<u>Thm</u> For $C \in \mathrm{ch}_F(\mathcal{A})$,

$\exists !$ (unique up to homotopy) continuous map

$$\sigma_C : (D^2, \partial D^2) \longrightarrow (M, M \cap F)$$

s.t.

(i) $\sigma_C(D^2) \pitchfork C$: transversal

(ii) $\sigma_C(D^2) \cap C = \{pt\}$

(iii) $\sigma_C(D^2) \cap C' = \phi$ for $C' \in \mathrm{ch}_F(\mathcal{A}) \setminus \{C\}$

Thm

$$M \underset{\text{homotopy}}{\simeq} F_{\mathbb{C}} \cup_{\sigma_c} \left(\bigsqcup_{c \in ch_F(A)} D^2 \right)$$

By these theorems, $\{ \sigma_c|_{\partial D^2} : S^1 \to F \mid c \in ch_F \}$ gives a relations.

§ An Algorithm obtaining a minimal presentation

(1) draw a generic (oriented) line F near the line at infinity H_∞.

(2) numbering $\mathcal{A}$ s.t. $H_1 \cap F > H_2 \cap F > \cdots > H_n \cap F$

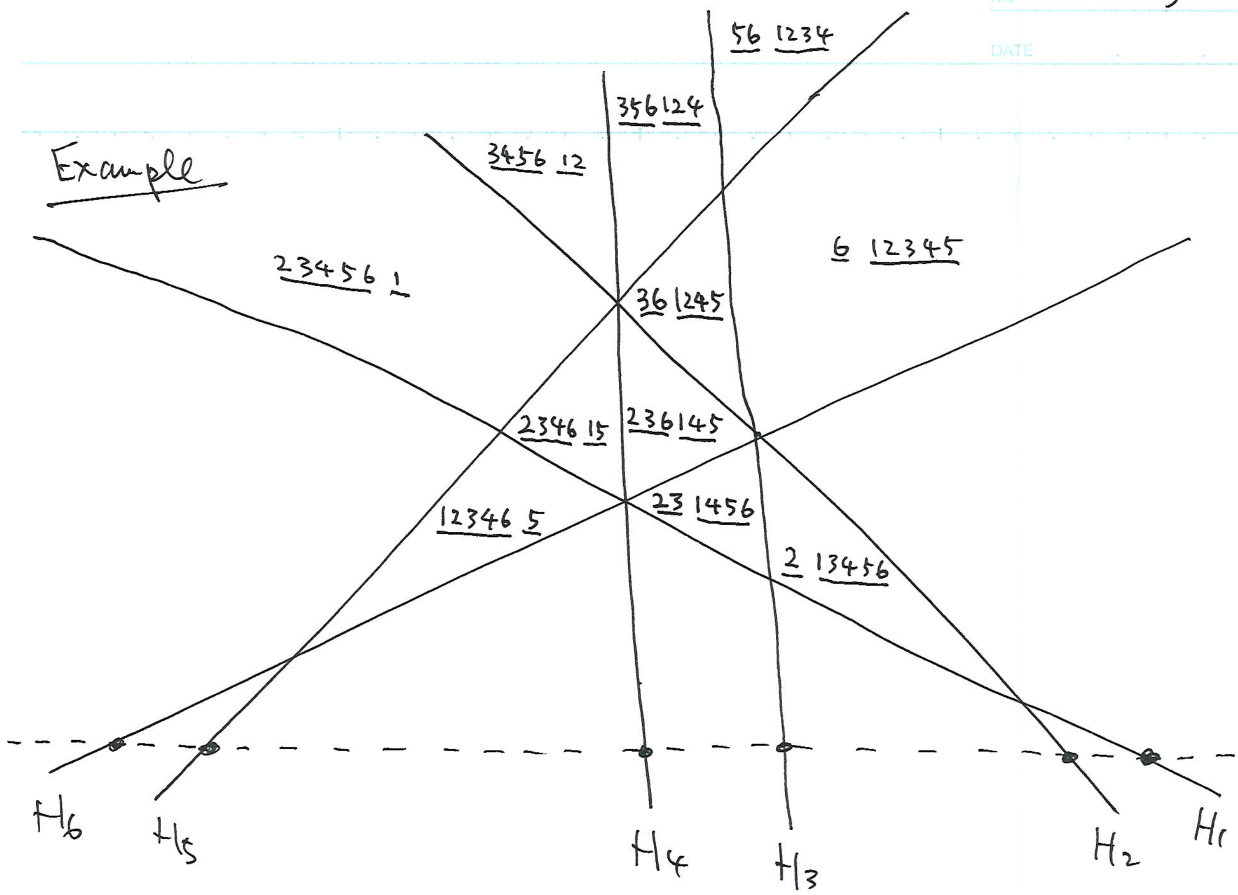
(3) To $c \in ch_F(A)$, attach a permutation $(i_1, i_2 \cdots i_n)$ of $(1, 2, \cdots, n)$ as:

$$\underbrace{i_1 < i_2 < \cdots < i_k}_{\text{right lines}},\quad \underbrace{i_{k+1} < \cdots < i_n}_{\text{left lines}}$$

(4) Associate a relation: $R(C): \gamma_1 \gamma_2 \cdots \gamma_n = \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_n}$

Thm $\pi_1(M) \simeq \langle \gamma_1, \cdots, \gamma_n \mid R(C),\ C \in ch_F(\mathcal{A}) \rangle$

Example



$$\pi_1(M) \cong \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \;\middle|\; \begin{aligned} \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 &= 213456 = 231456 \\ &= 123465 = 234615 \\ &= 236145 = 361245 \\ &= 356124 = 234561 \\ &= 345612 = 561234 = 612345 \end{aligned} \right\rangle$$