Milnor fibers of real line arrangements.

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Complex Analysis and Topology of
Discrete Groups and Hyperbolic Spaces

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Abstract : We study the topology of the Milnor fiber of a hyperplane arrangement by using "real structure".

Hyperplane arrangement

$$\{Q(x,y,z)=\prod_i \alpha_i(x,y,z)=0\}$$

(Non-isolated Singularity)

$\rightsquigarrow$

Milnor fiber

$$F=\{Q=1\}\subseteq \mathbb{C}^3$$

$\rightsquigarrow$

Real Structures.

$\Longrightarrow$???

Contents

1. Milnor fibers of line arrangements

2. "Resonant band" algorithm and applications.

3. Milnor fiber and 3-multinets.

4. Idea of proof.

1. Milnor Fibers of Line arrangements

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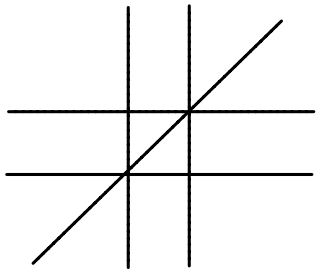
Given datum



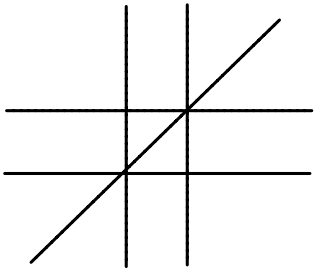
$$x(x-1)y(y-1)(x-y) = 0$$

a line arrangement

$\mathcal{A} = \{H_1, \cdots, H_n\}$

1. Milnor Fibers of Line arrangements

Given datum $\rightsquigarrow$ Its Cone



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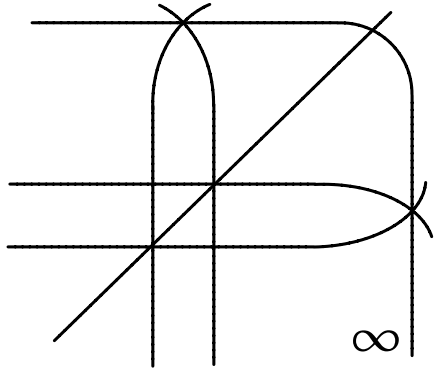

$Q := x(x-z)y(y-z)(x-y)\cdot z$
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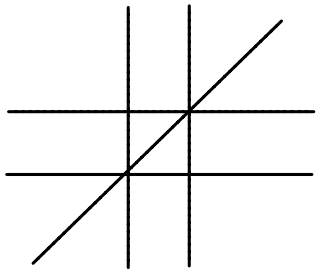

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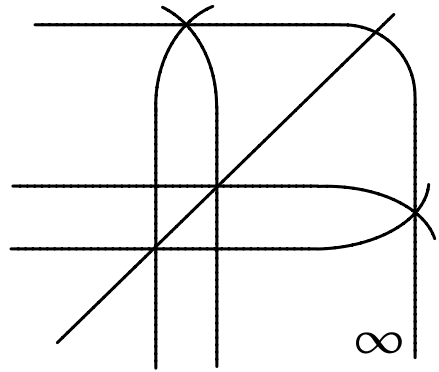

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1. Milnor Fibers of Line arrangements

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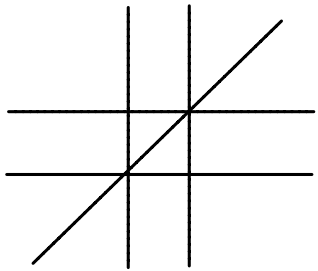

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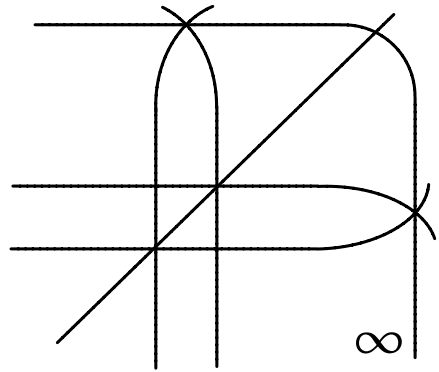

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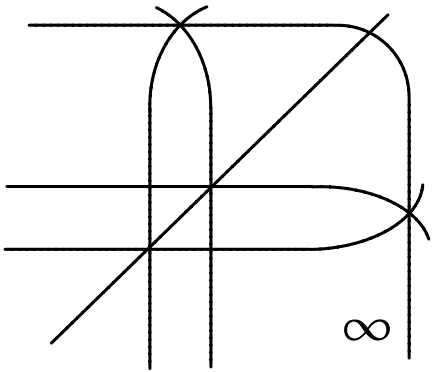
"Milnor Fiber"

$$F = F_{\mathcal{A}} := Q^{-1}(1) = \{(x,y,z) \in \mathbb{C}^3 \mid Q(x,y,z) = 1\} \subset \mathbb{C}^3$$

1. Milnor Fibers of Line arrangements

Remark: $Q(x,y,z)$ is homogeneous of deg $= n+1$.



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$$F = \{(x,y,z) \in \mathbb{C}^3 \mid Q(x,y,z) = 1\}$$
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Monodromy action

$$\begin{array}{ccc} \rho : F & \longrightarrow & F \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x,y,z) & \longmapsto & (\zeta x, \zeta y, \zeta z), \end{array}$$

where $\zeta = e^{2\pi i/n+1}$

1. Milnor Fibers of Line arrangements

$$\rho : F \longrightarrow F : (x, y, z) \longmapsto (\zeta x, \zeta y, \zeta z),$$

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induces a linear automorphism

$$\rho^* : H^1(F) \longrightarrow H^1(F).$$

We have eigendecomposition

$$H^1(F) = \bigoplus_{\lambda^{n+1}=1} H^1(F)_\lambda$$

λ-eigen space

Since $\rho^{n+1} = id.$

1. Milnor Fibers of Line arrangements

Easy part: $\lambda = 1$.

Nontrivial part: $\lambda \neq 1$

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ρ-invariant part

$$H^1(F)_1 = H^1(F)^{\rho} \cong H^1(F/\langle\rho\rangle) \cong H^1(M(\mathcal{A})) \cong \mathbb{C}^n.$$

(Recall: $F/\langle\rho\rangle = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i$)

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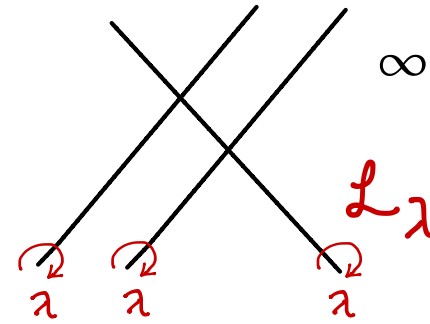
$$H^1(F)_\lambda \cong H^1(M(\mathcal{A}), \mathcal{L}_\lambda)$$

rank 1 local system on $M(\mathcal{A})$, s.t. each monodromy around H_i is λ.



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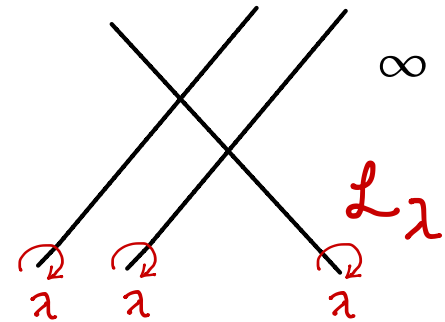
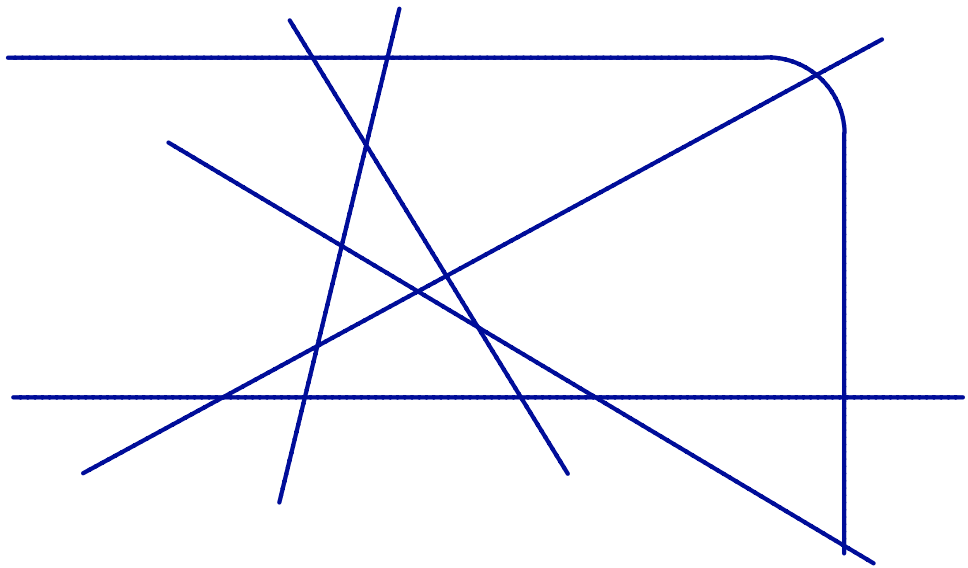


$$H^1(F)_{\neq 1} := \bigoplus_{\lambda \neq 1} H^1(F)_\lambda$$: non-trivial eigenspace.

1. Milnor Fibers of Line arrangements

Known Fact and examples

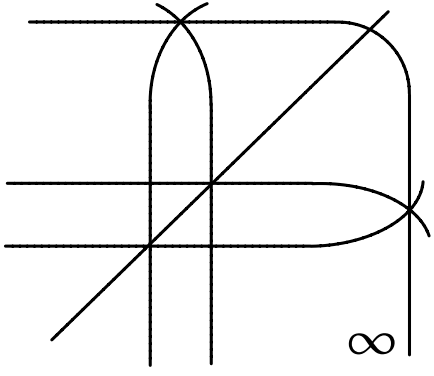
Thm. (Orlik-Randell, Hattori) If $\mathcal{A}=\{H_1,\cdots,H_n,H_\infty\}$ is generic (= only double points), then $H^1(F)_{\neq 1}=0$. (i.e. $b_1(F)=n$)



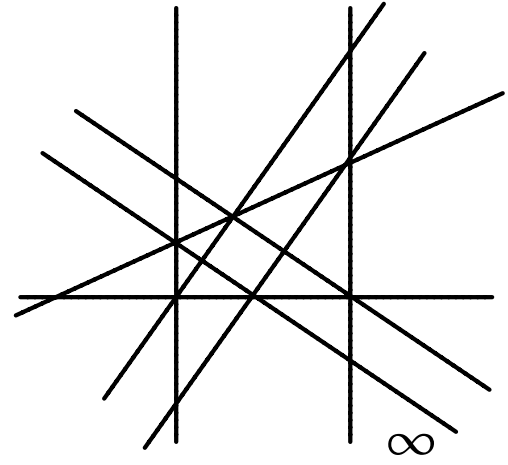
1. Milnor Fibers of Line arrangements

Non-generic cases:

$$\boxed{\zeta = e^{2\pi i/3}}$$



A_3-arr.



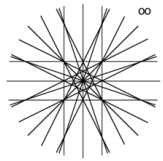
Pappus arr.

$$H^1(F) = \underset{\mathbb{C}^5}{\underset{\cong}{H^1(F)_1}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_\zeta}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_{\zeta^2}}}$$

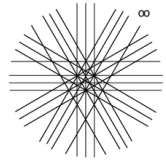
$$H^1(F) = \underset{\mathbb{C}^8}{\underset{\cong}{H^1(F)_1}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_\zeta}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_{\zeta^2}}}$$

1. Milnor Fibers of Line arrangements

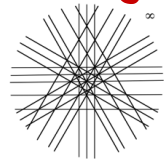
NOT All "non-generic arr" have non-trivial eigenspaces.



$\mathcal{A}(21,4)$



$\mathcal{A}(22,4)$



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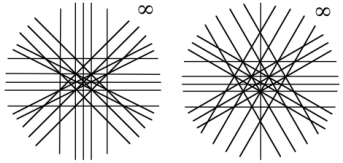
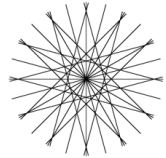
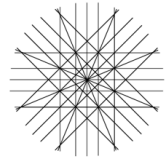


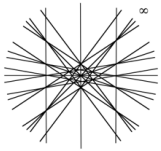
$\mathcal{A}(21,5)$



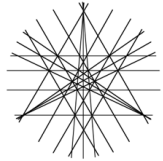
$\mathcal{A}(24,1)$



$\mathcal{A}(24,2)$



$\mathcal{A}(21,6)$



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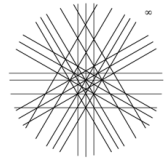

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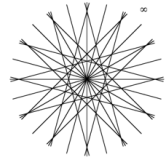
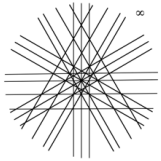


$\mathcal{A}(25,1)$



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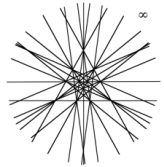

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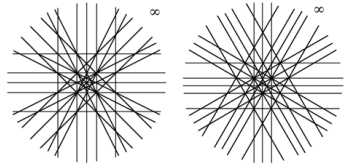

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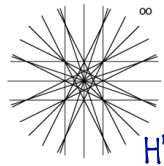
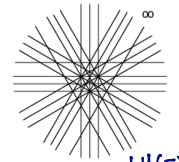


$\mathcal{A}(21,4)$ $H^1(F)_{\neq 1}=0$



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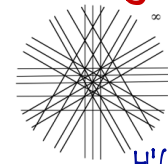

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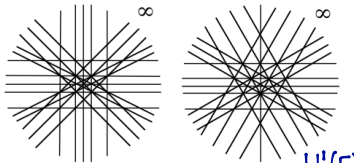

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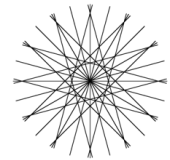
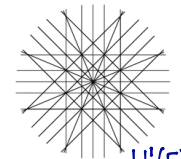
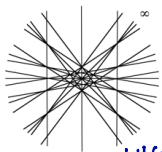


$\mathcal{A}(24,1)$



$\mathcal{A}(24,2)$ $H^1(F)_{\neq 1}=0$



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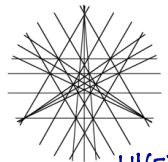

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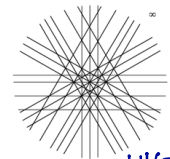

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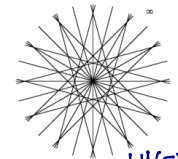

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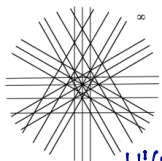

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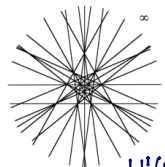

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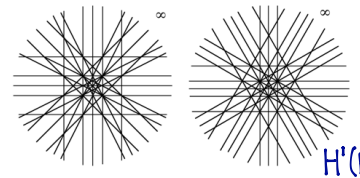

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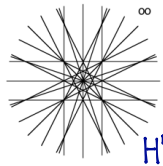
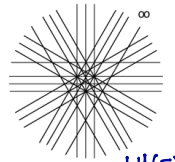
1. Milnor Fibers of Line arrangements

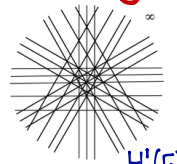
NOT All "non-generic arr" have non-trivial eigenspaces.



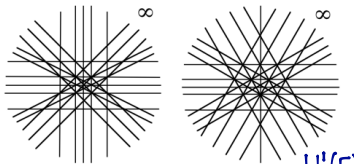
$\mathcal{A}(21,4)$ $H^1(F)_{\neq 1}=0$



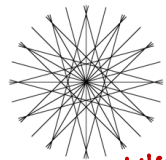
$\mathcal{A}(22,4)$ $H^1(F)_{\neq 1}=0$



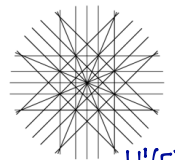
$\mathcal{A}(23,1)$ $H^1(F)_{\neq 1}=0$



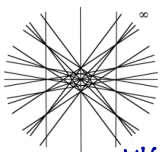
$\mathcal{A}(21,5)$ $H^1(F)_{\neq 1}=0$



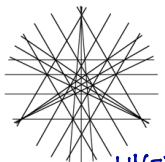
$\mathcal{A}(24,1)$ $H^1(F)_{\neq 1}\neq 0$



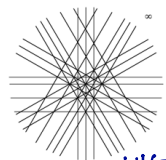
$\mathcal{A}(24,2)$ $H^1(F)_{\neq 1}=0$



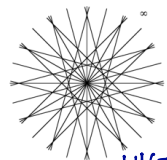
$\mathcal{A}(21,6)$ $H^1(F)_{\neq 1}=0$



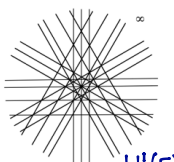
$\mathcal{A}(21,7)$ $H^1(F)_{\neq 1}=0$



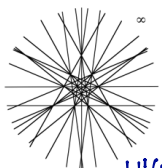
$\mathcal{A}(24,3)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(25,1)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(22,2)$ $H^1(F)_{\neq 1}=0$



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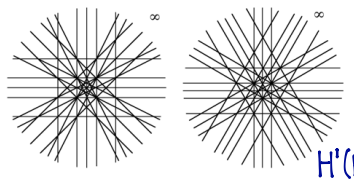

$\mathcal{A}(25,2)$ $H^1(F)_{\neq 1}=0$

1. Milnor Fibers of Line arrangements

$\boxed{\text{Problem}}$ Compute $\dim H^1(F)_\lambda = \dim H^1(M, \mathcal{L}_\lambda)$.

(Combinatorial formula? Unknown.)

<u>Rem.</u> $\dim H^1(F)_\lambda$ is related to many other things.

- Betti numbers of certain covering spaces of $M(\mathcal{A})$.
- Alexander polynomial of $\pi_1(M)$.
- Counting certain plane curves.
- Hodge str. of $H^1(F, \mathbb{C})$.

2. Resonant band algorithm

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Now we assume $\mathcal{A}=\{H_1,\cdots,H_n\}$ is defined $/\mathbb{R}$.

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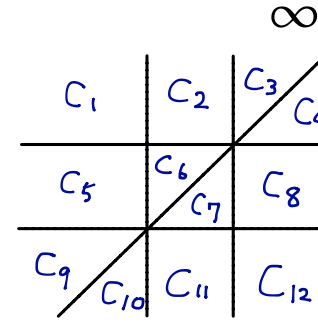
<u>Notation</u>:

$ch(\mathcal{A})$: the set of chambers.



$ch(\mathcal{A})=\{C_1,\cdots,C_{12}\}$

Adjacency distance:

For $C, C' \in ch(\mathcal{A})$,

$d(C,C') :=$ # of lines which separates C & C'.

e.g. $d(C_5, C_{12})=4$

2. Resonant band algorithm

Fix a positive integer $k \mid (n+1)$. Set $\lambda = e^{2\pi i/k}$.

($k > 1$, mainly $k = 3$.)

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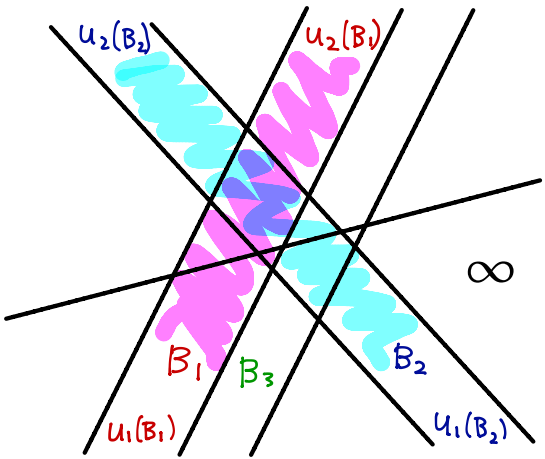
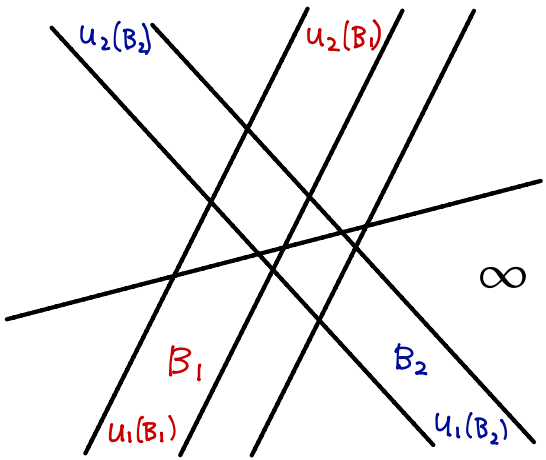


A **band** is a region bounded by a pair of consecutive parallel lines.

In a band B, there are exactly two unbounded chambers, set $u_1(B)$ and $u_2(B)$.

Next we define k-resonance.

2. Resonant band algorithm



A **band** is a region bounded by a pair of consecutive parallel lines.

In a band B, there are exactly two unbounded chambers, set $u_1(B)$ and $u_2(B)$.

A band B is **k-resonant** $\overset{\text{def}}{\Longleftrightarrow}$ $k \mid d(u_1(B), u_2(B))$.

E.g. $d(u_1(B_1), u_2(B_1)) = 3 \rightsquigarrow B_1$ is 3-resonant.

$d(u_1(B_2), u_2(B_2)) = 4 \rightsquigarrow B_2$ is not 3-resonant.

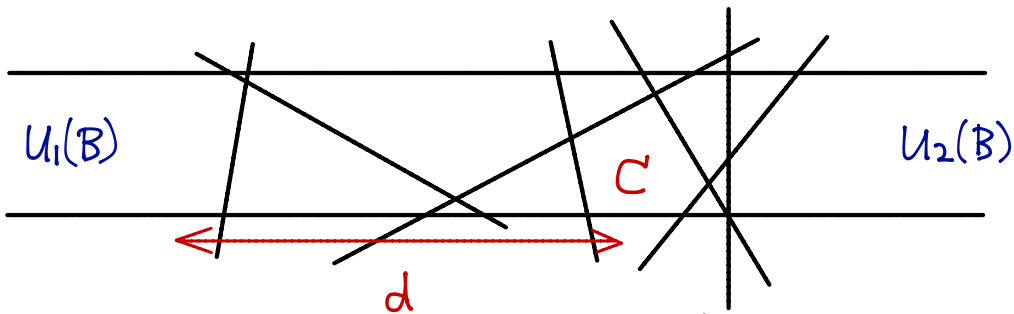
2. Resonant band algorithm

Discrete Wave on the band B:

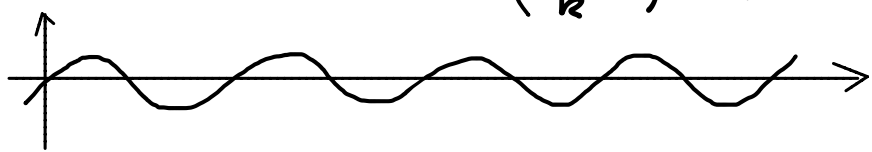
Suppose that B is k-resonant. $\left(\Longleftrightarrow \sin\left(\frac{\pi \cdot d(u_1, u_2)}{k}\right) = 0\right)$

vector space spanned by chambers.

$$\nabla(B) := \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) \cdot [C] \in \mathbb{C}[ch(A)]$$



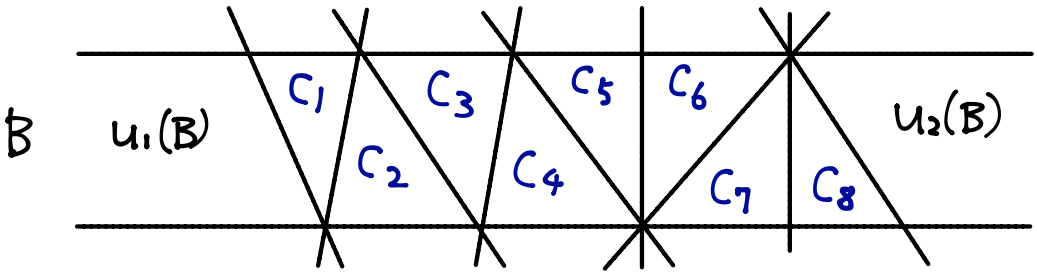
$$\nabla(B) = \cdots + \sin\left(\frac{\pi}{k} \cdot d\right) \cdot [C] + \cdots$$



2. Resonant band algorithm

$$\nabla(B) := \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) \cdot [C] \in \mathbb{C}[\mathrm{ch}(A)]$$

<u>Ex.</u> $d(u_1(B), u_2(B)) = 9, \quad k = 3.$



$$\nabla(B) = \sum_{p=1}^{8} \sin\left(\frac{\pi}{3} \cdot p\right) \cdot [C_p]$$

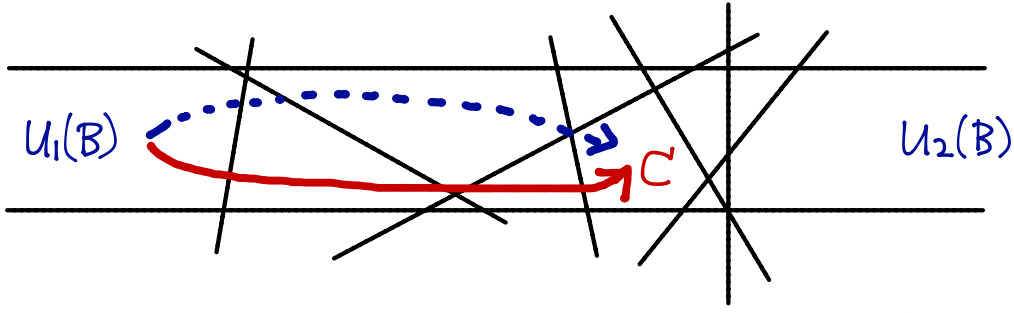
$d(u_1(B), C_p) = p.$

$$= \frac{\sqrt{3}}{2}[C_1] + \frac{\sqrt{3}}{2}[C_2] + 0 \cdot [C_3] - \frac{\sqrt{3}}{2}[C_4] - \frac{\sqrt{3}}{2}[C_5] + 0[C_6] + \frac{\sqrt{3}}{2}[C_7] + \frac{\sqrt{3}}{2}[C_8].$$

2. Resonant band algorithm

Another interpretation:

$$\nabla(B) := \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) \cdot [C] \in \mathbb{C}[\mathrm{ch}(A)]$$



$$\sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) = \frac{1}{2i}\left(e^{\frac{\pi i d}{k}} - e^{-\frac{\pi i d}{k}}\right)$$

is the difference of two parallel translations.

(positive and negative)

2. Resonant band algorithm

$$\nabla(B) = \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) \cdot [C]$$

Notation: $RB_k(A)$ denotes the set of k-resonant bands.

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Notation: $RB_k(A)$ denotes the set of k-resonant bands.

<u>Theorem</u>. (Y.) Let $\mathcal{A} = \{H_1, \cdots, H_n\}$ be a line arr. in $\mathbb{R}^2$, $k \mid (n+1)$, $k > 1$, $\lambda = e^{2\pi i/k}$ (or its Galois conj). Then

$$\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(A)] \to \mathbb{C}[ch(A)]\right) \cong H^1(F)_\lambda.$$

In particular, $\dim H^1(F)_\lambda$ is equal to the number of linear relations among discrete waves $\nabla(B)$, $B \in RB_k(A)$.

2. Resonant band algorithm

<u>Theorem</u>. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(A)] \to \mathbb{C}[ch(A)]\right) \cong H^1(F)_\lambda$.

(Hence $\dim H^1(F)_\lambda$ is equal to # of linear rel's among $\nabla(B)$'s.)

<u>Example</u> (Pappus arr.)

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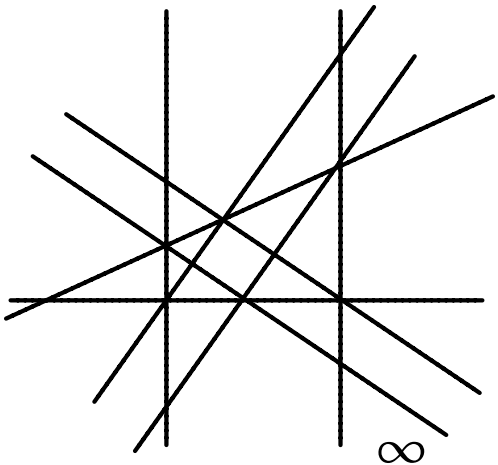

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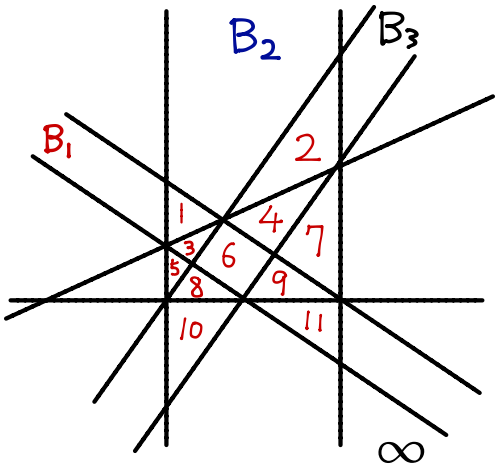

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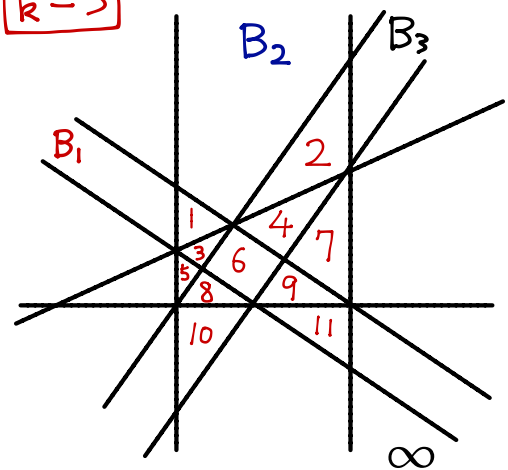
$\boxed{k=3}$



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Example (Pappus arr.) $\frac{2}{\sqrt{3}}\nabla(B_1) = [C_1] + [C_3] - [C_9] - [C_{11}]$.

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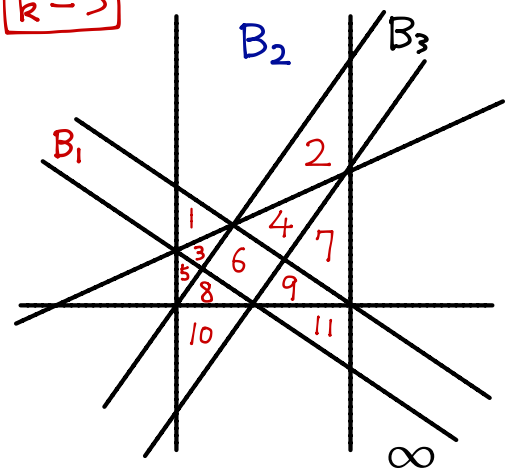

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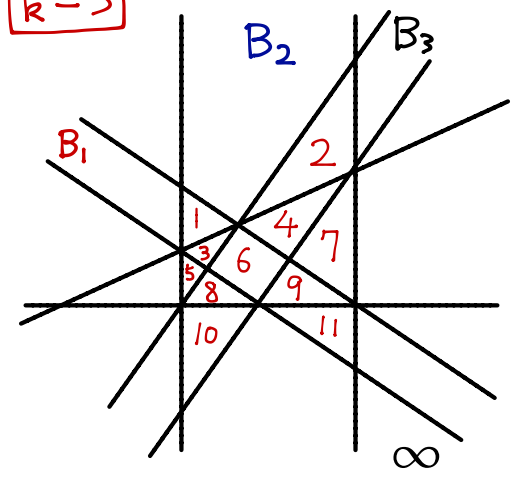

$$\frac{2}{\sqrt{3}}\nabla(B_1) = [C_1]+[C_3]-[C_9]-[C_{11}].$$

$$\frac{2}{\sqrt{3}}\nabla(B_3) = [C_2]+[C_4]-[C_8]-[C_{10}]$$

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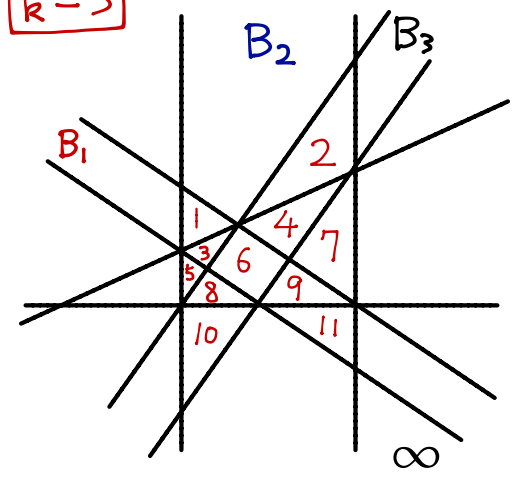

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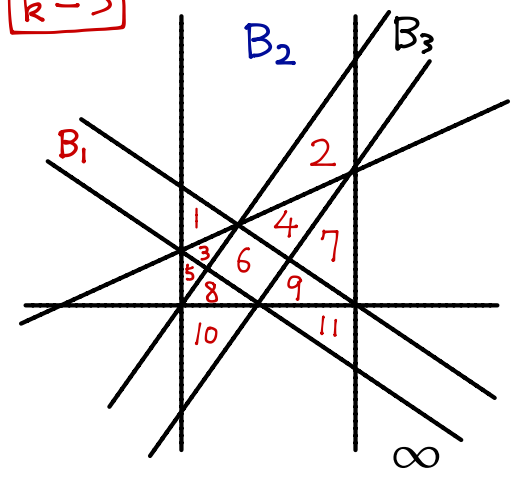

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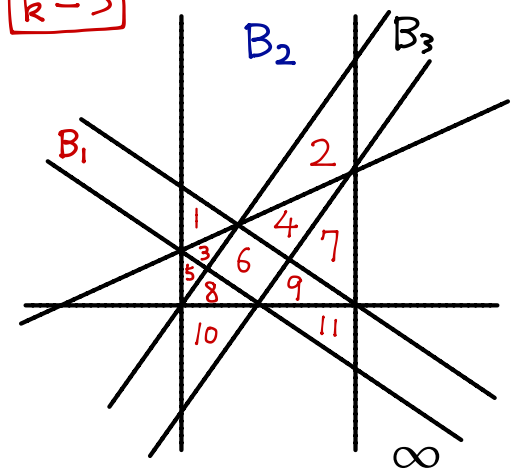

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$$\begin{aligned}\frac{2}{\sqrt{3}}\nabla(B_2) &= [C_1]+[C_2]+[C_3]+[C_4] \\ &\quad -[C_8]-[C_9]-[C_{10}]-[C_{11}] \\ &= \frac{2}{\sqrt{3}}\nabla(B_1)+\frac{2}{\sqrt{3}}\nabla(B_3).\end{aligned}$$

$$\therefore \dim H^1(F)_{e^{2\pi i/3}} = 1.$$

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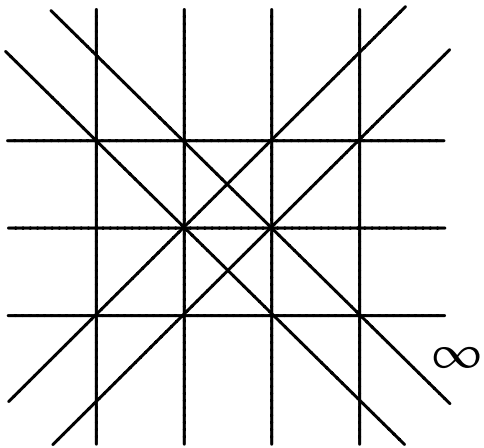
Example ($\mathcal{A}(12,2)$, Grünbaum's Catalogue)



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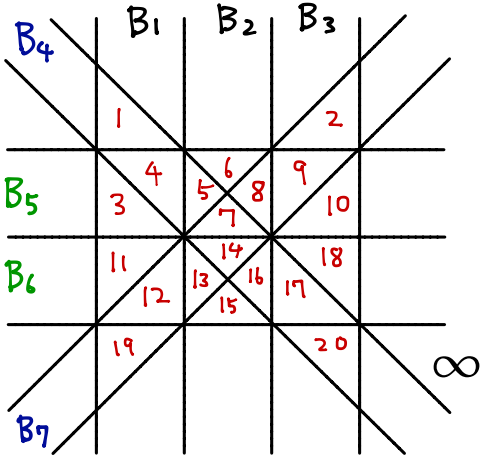

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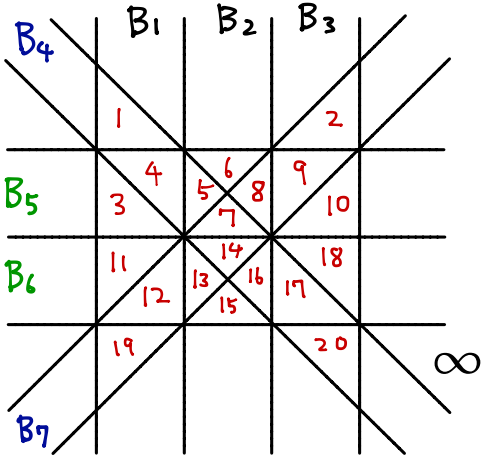


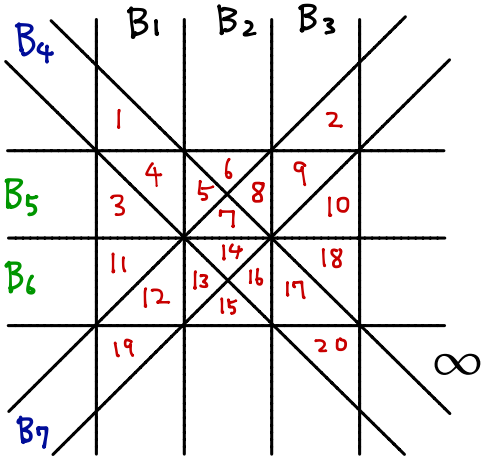
$n+1 = 12$. $k = 2, 3, 4, 6, 12$.

2. Resonant band algorithm

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(A)] \to \mathbb{C}[ch(A)]\right) \cong H^1(F)_\lambda$.

(Hence $\dim H^1(F)_\lambda$ is equal to # of linear rel's among $\nabla(B)$'s.)

Example ($\mathcal{A}(12,2)$, Grünbaum's Catalogue)



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Bands	Length
B_1, B_2, B_3	7
B_4, B_7	9
B_5, B_6	8

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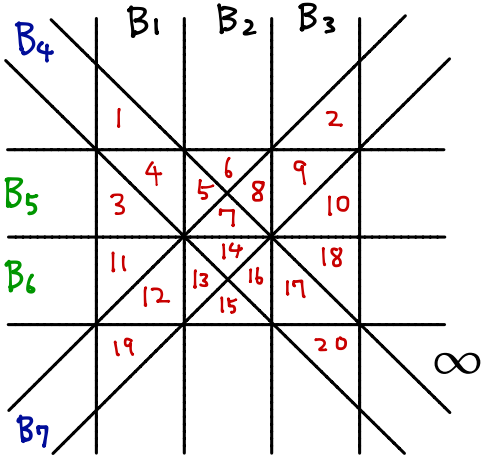

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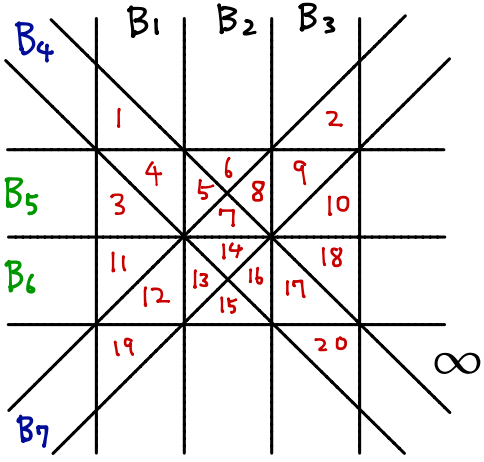

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Bands	Length	
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2. Resonant band algorithm

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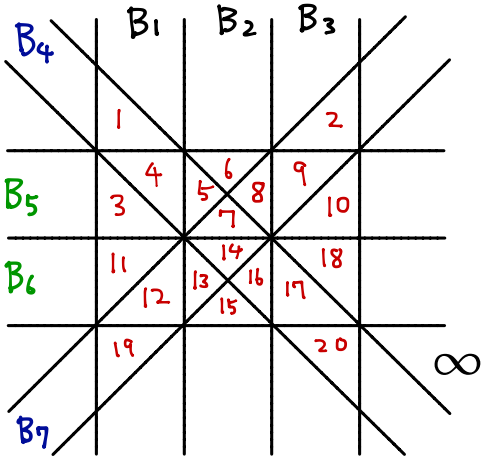

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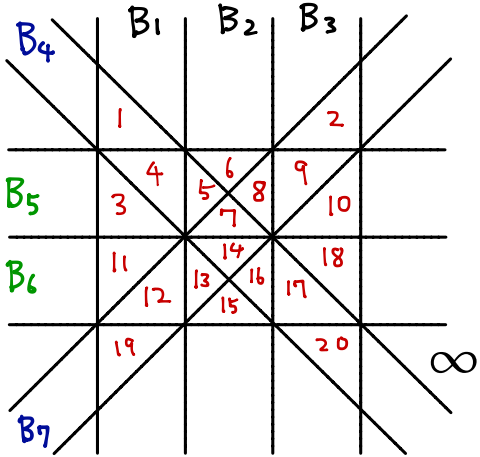

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2. Resonant band algorithm

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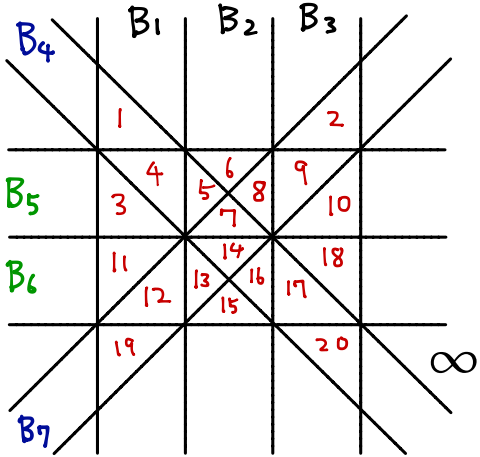

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$$\nabla(B_4)=\frac{\sqrt{+3}}{2}[C_1]+\cdots,$$

$$\nabla(B_7)=\frac{\sqrt{+3}}{2}[C_{19}]+\cdots.$$

$[C_1]$ does not appear.

Hence $\nabla(B_4)$ and $\nabla(B_7)$ are linearly indep. Hence $H^1(F)_{e^{2\pi i/3}}=0$.

2. Resonant band algorithm

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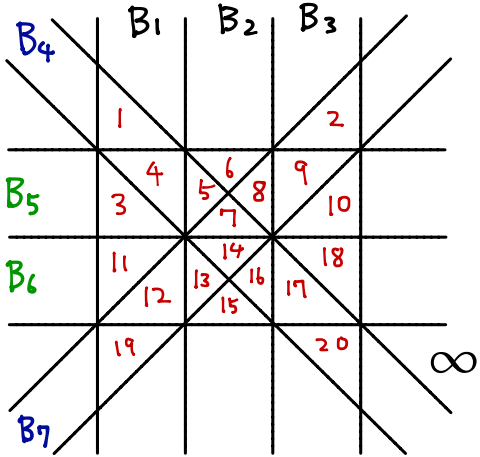

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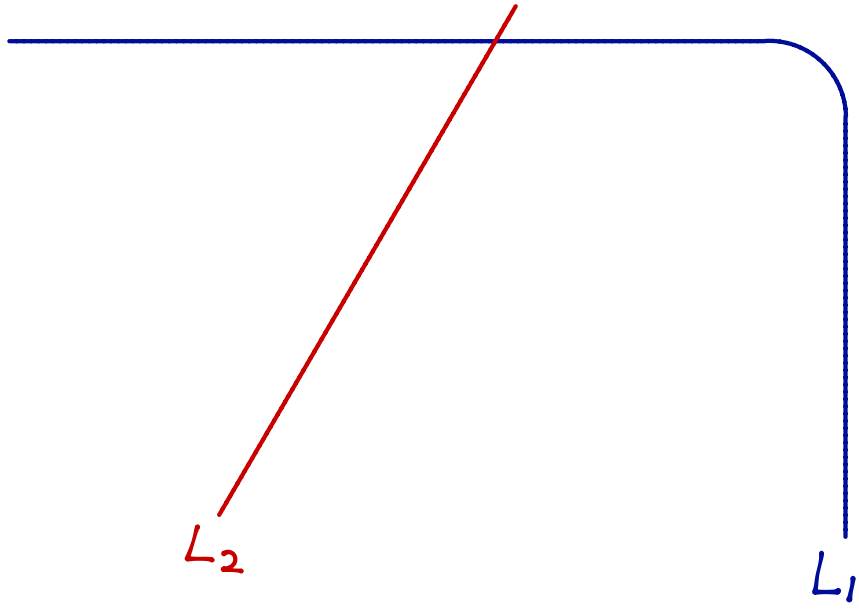
Bands	Length	
B_4, B_7	9	← 3-resonant
B_5, B_6	8	← 2 & 4-resonant.

parallel

Similarly, $RB_2 = RB_4 = \{B_5, B_6\}$, $\nabla(B_5)$ and $\nabla(B_6)$ are linearly indep. (No overlaps!), hence $H^1(F)_{\neq 1} = 0$.

2. Resonant band algorithm

Recall that two lines L_1 and L_2 divides $\mathbb{RP}^2$ into two regions.



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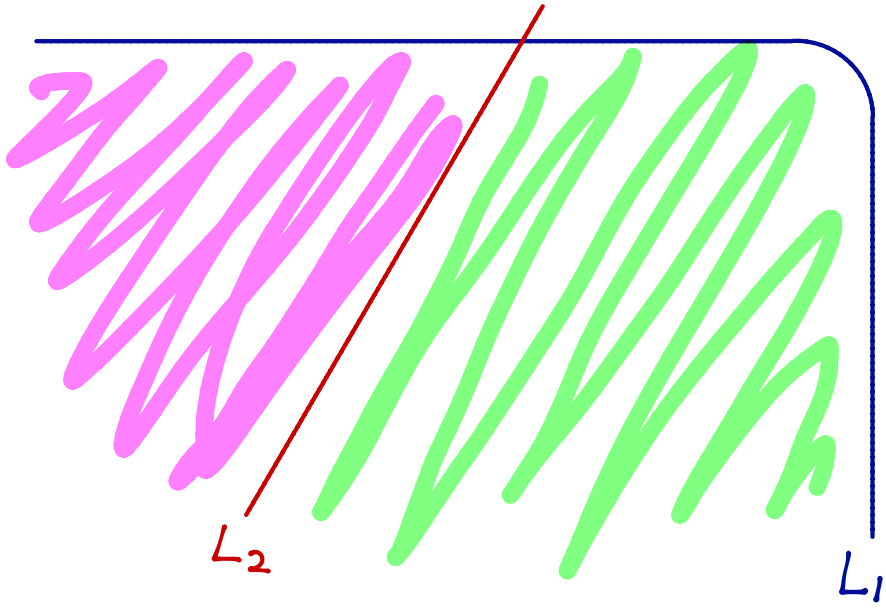
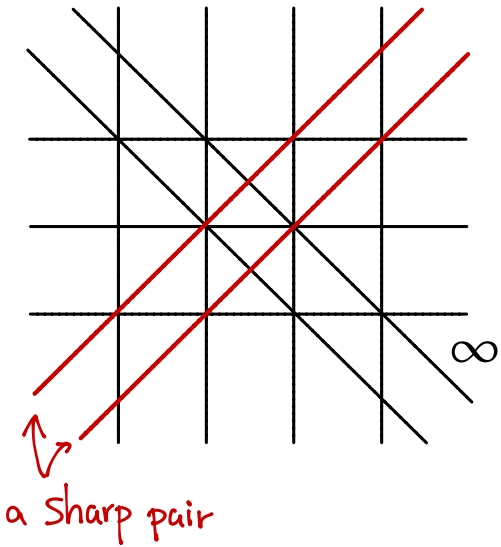
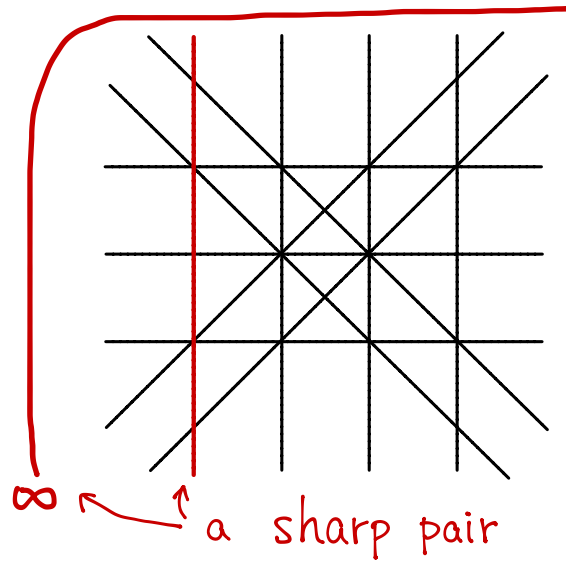


2. Resonant band algorithm

Recall that two lines L_1 and L_2 divides $\mathbb{RP}^2$ into two regions.

<u>Def</u>. The pair $H_1, H_2 \in \mathcal{CA}$ is called "sharp pair" if one of two regions does not contain intersections (in its interior).





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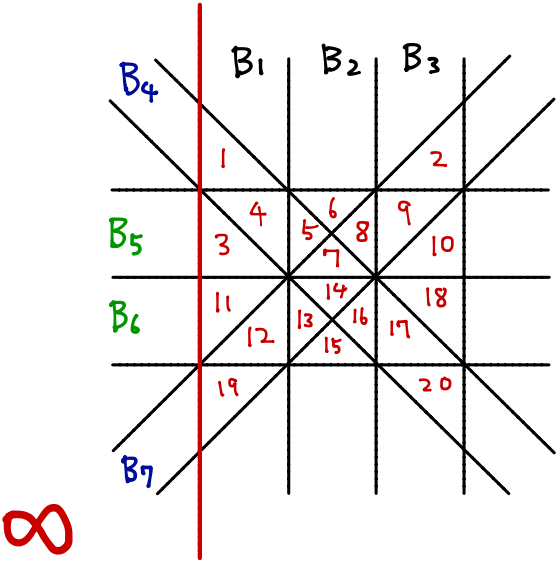

<u>Thm.</u> (Y.)

If $\mathcal{A}$ has a sharp pair, then

$$\dim H^1(F)_\lambda \leq 1 \quad \text{for } {}^\forall \lambda \neq 1.$$

Idea of proof: Suppose $\sum_i a_i \cdot \nabla(B_i) = 0$.

(In the left figure) a_1 determines a_4, a_5, a_6, a_7.

Then a_2, a_3 are also uniquely determined. (Q.E.D.)

2. Resonant band algorithm

<u>Thm.</u> (Libgober)

If $\exists H_i \in \mathcal{A}$ s.t. all intersections on H_i has multiplicity 2 (i.e. No higher multiple points exist on H_i)

$$\Longrightarrow H^1(F)_\lambda = 0 \quad \text{for } \lambda \neq 1.$$

(proof)

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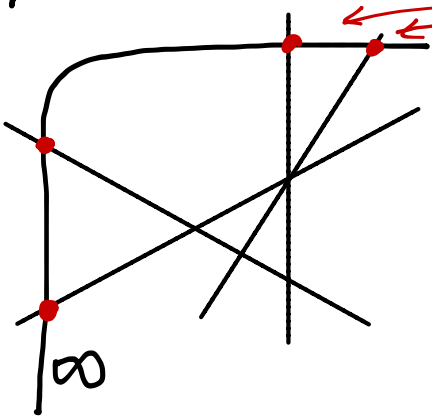

All intersections on H_∞ have multiplicity $=2$

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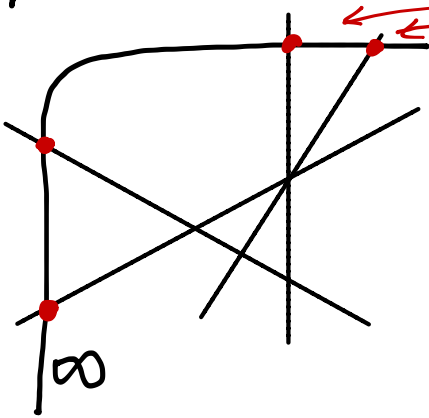

All intersections on H_∞ have multiplicity $=2$

$\Longleftrightarrow$ There are no bands.

$RB_\ell(\mathcal{A}) = \phi$. Hence $\ker(\mathbb{C}[RB_\ell] \to \mathbb{C}[ch])$

$= 0$ (Q.E.D.)

2. Resonant band algorithm

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$$\Longrightarrow H^1(F)_\lambda = 0 \quad \text{for } \lambda \neq 1.$$

<u>Rem.</u> Libgober proved the above result for any complex line arrangements. Our new proof works only for those defined $/\mathbb{R}$. However ⋯

2. Resonant band algorithm

We can strengthen for real arrangements

<u>Prop.</u> Suppose $\mathcal{A}$ is not a <u>pencil</u>. If $\exists H_i \in \mathcal{A}$ s.t.

There is at most one higher mult. pt. on H_i.

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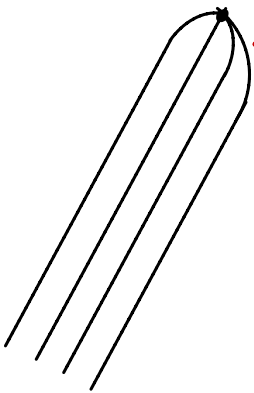

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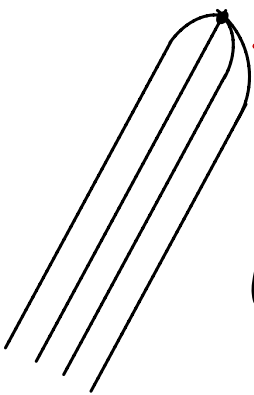

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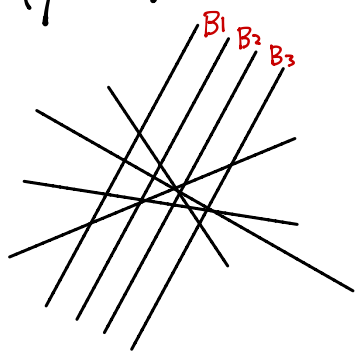
$$\Longrightarrow H^1(F)_\lambda = 0 \quad \text{for } \lambda \neq 1$$

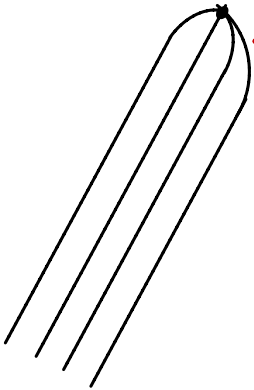
(proof) In this case, all bands are parallel.



Since parallel bands have no overlaps, $\nabla(B_1), \cdots$ are linearly indep. So $H^1(F)_{\neq 1} = 0$ (Q.E.D.)

2. Resonant band algorithm

<u>Thm.</u> (Y.) Suppose $\mathcal{A}$ is not a pencil. If $\exists H_i \in \mathcal{A}$ s.t.



There are exactly two higher mult. pt. on H_i,

2. Resonant band algorithm

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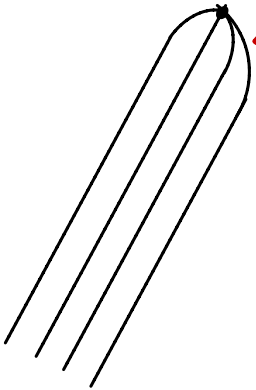

There are exactly two higher mult. pt. on H_i,

and $H^1(F)_{\neq 1} \neq 0$, then

$\Longrightarrow$

2. Resonant band algorithm

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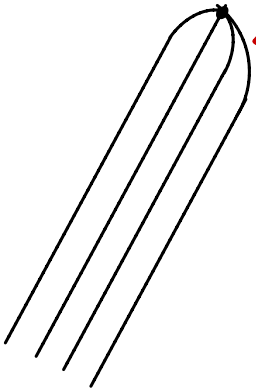
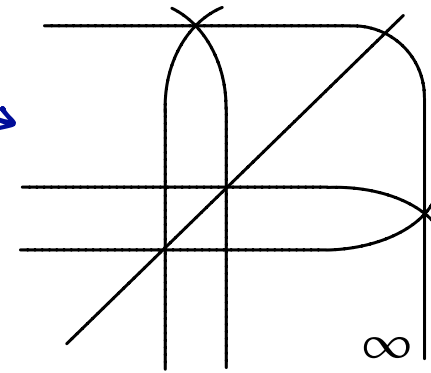


There are exactly two higher mult. pt. on H_i, and $H^1(F)_{\neq 1} \neq 0$, then

$$\Longrightarrow$$

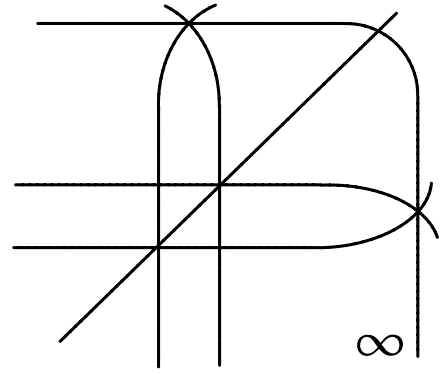
$\mathcal{A}$ is A_3-arr

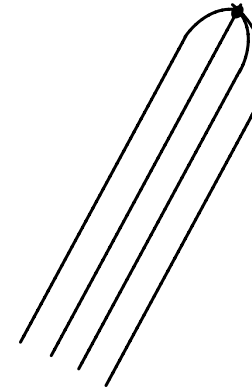


proof: Laborious combinatorial arguments.

2. Resonant band algorithm

<u>Cor.</u> Except for A_3-arr. and pencils,





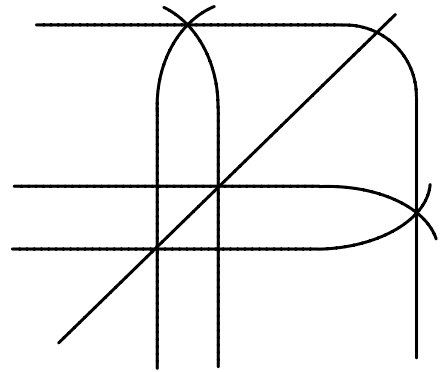
If $\exists$ line $H_i \in \mathcal{A}$

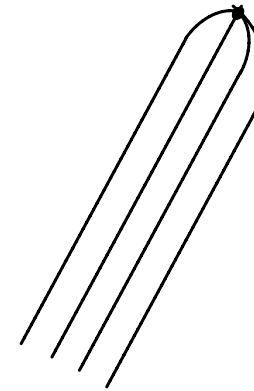
s.t. There are at most two points of multiplicity ≥ 3 on H_i,

then $H^1(F)_{\neq 1} = 0$.

2. Resonant band algorithm

<u>Cor.</u> Except for A_3-arr. and pencils,





$H^1(F)_{\neq 1} \neq 0$ implies

that each line $H \in c\mathcal{A}$ has at least 3 multiple intersections on it.

(Roughly, $H^1(F)_{\neq 1} \neq 0$ implies $c\mathcal{A}$ is far from generic.)

2. Resonant band algorithm

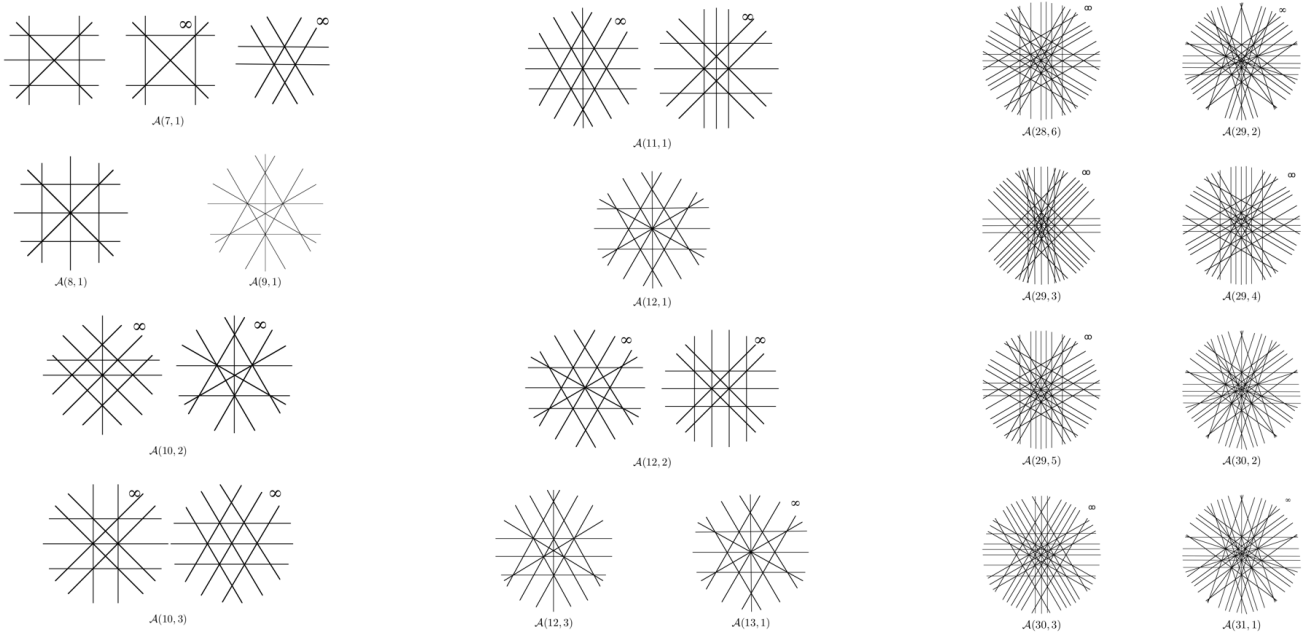
We have computed non trivial eigen space $H^1(F)_{\neq 1}$ for arrangements which are appeared in

- B. Grünbaum, A catalogue of simplicial arrangements in the real projective plane.
  (and some additional information by Michael Cuntz)

- B. Grünbaum, Configurations of points and lines. (Book, AMS)

2. Resonant band algorithm

Def. $\mathcal{A}$ is called simplicial if it gives a triangulation of $\mathbb{RP}^2$.



From Grunbaum's paper "A catalogue of simplicial arrangements in the projective plane"

2. Resonant band algorithm

<u>Def.</u> $\mathcal{A}$ is called simplicial if it gives a triangulation of $\mathbb{RP}^2$.

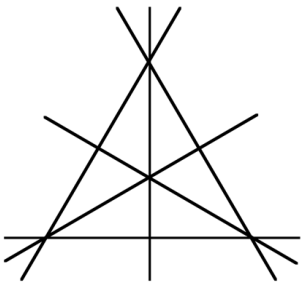
Notes:

- There are some infinite series.
- It is generally believed that there exists only finitely many sporadic simplicial arr's. (Not yet classified.)
- $M(\mathcal{A})$, $F_{\mathcal{A}}$ are $K(\pi,1)$-spaces. (Deligne)
- Theory of free arrangements started with these examples. (Terao.)

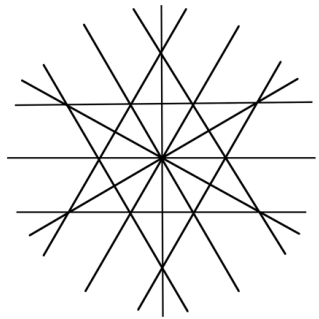
2. Resonant band algorithm

<u>Def.</u> $\mathcal{A}$ is called simplicial if it gives a triangulation of $\mathbb{RP}^2$.

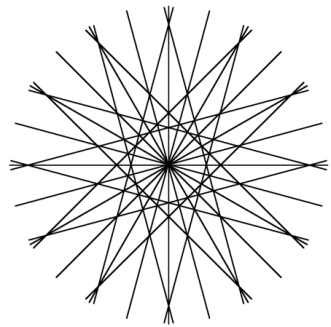
<u>Def.</u> For $m \geq 1$, $A(6m,1) =$ $3m$ edges of regular $3m$-gon + $3m$ lines of symmetry.



A(6,1)



$\mathcal{A}(12,1)$



$\mathcal{A}(24,1)$

2. Resonant band algorithm

<u>Conjecture 1</u>. For real line arrangements (except for pencils)

(i) $\dim H^1(F)_\lambda \leqq 1$ if $\lambda \neq 1$.

(ii) $H^1(F)_\lambda = 0$ if $\lambda^3 \neq 1$.

<u>Conjecture 2</u>. Suppose $\mathcal{A}$ is simplicial. Then

$H^1(F)_{\neq 1} \neq 0$ iff $\mathcal{A}$ is of type $A(6m,1)$.

(Conj 2 is verified for all known examples. In particular, true for $|\mathcal{A}| \leq 28$.)

3. Milnor fiber and 3-multinets

3. Milnor fiber and 3-multinets

<u>Def.</u> A projective line arrangement $\mathcal{A}$ is said to have 3-multinets structure (of multiplicity 1) if $\exists$ decomposition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3$ s.t.

(i) $|\mathcal{A}_1| = |\mathcal{A}_2| = |\mathcal{A}_3|$, and

(ii) For any intersection p, $(\mathcal{A})_p = \{H \in \mathcal{A} \mid p \in H\}$ is either

- consisting one color, or
- $|(\mathcal{A}_1)_p| = |(\mathcal{A}_2)_p| = |(\mathcal{A}_3)_p|$.

<u>Rem.</u> Having a 3-multinet str. is a combinatorial property.

3. Milnor fiber and 3-multinets

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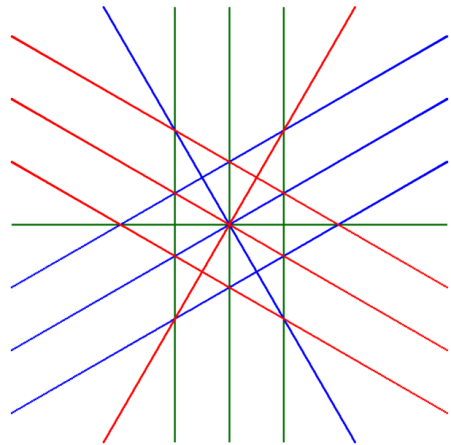
<u>Ex</u>. $A(12,1)$.



3. Milnor fiber and 3-multinets

<u>Thm.</u> (Libgober, again for any complex arr. $\mathcal{A}$)

Suppose $\mathcal{A}$ has only double and triple points.

Then $H^1(F)_{\neq 1} \neq 0$ iff $\mathcal{A}$ has 3-multinet str.

(and then nontrivial eigen values are $e^{\pm 2\pi i/3}$)

<u>Rem.</u> (0) "if" part has been known (Falk-Yuzvinsky).

(1) Hence for double and triple points arrangements, $H^1(F)_{\neq 1} = 0$ or $\neq 0$ is combinatorial.

(2) An example of $\mathcal{A}$ with no 3-multinet str. and $H^1(F)_{\sqrt{-1}} \neq 0$ is known. (defined $/\mathbb{C}$)

3. Milnor fiber and 3-multinets

If we restrict to the real cases,

all examples (from Grünbaum's articles) supports the following.

<u>Conjecture 3</u>. For any real arrangement $\mathcal{A}$,

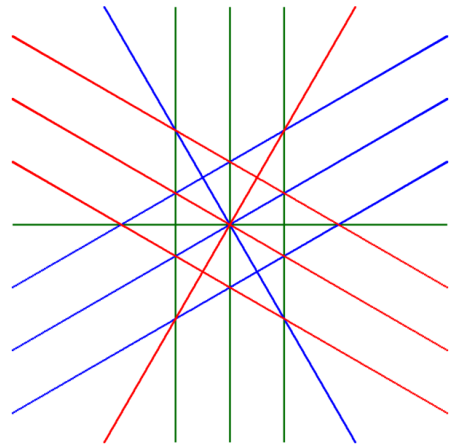
$H^1(F)_{\neq 1} \neq 0$ iff $\mathcal{A}$ has 3-multinet str.

(i.e. Libgober's resut holds for all real arr's without the assumption "at most triple points".)

3. Milnor fiber and 3-multinets

<u>Conjecture 4</u>. Suppose $\mathcal{A}$ is a simplicial arrangement. Then $\mathcal{A}$ has 3-multinet iff $\mathcal{A} = A(6m,1)$ for some $m \geq 1$.

(This is purely combinatorial problem)



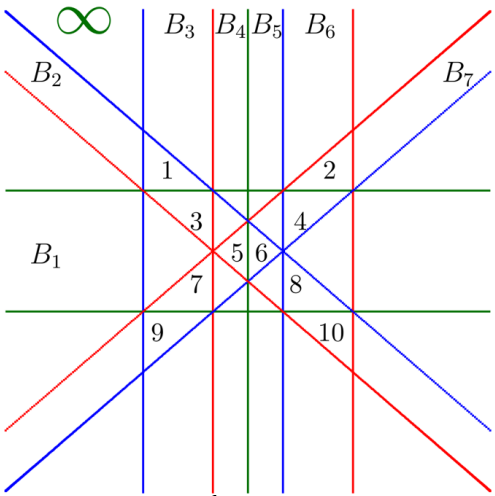
$A(12,1)$

3. Milnor fiber and 3-multinets

(Work in progress with Michele Torielli)

Towards "Conj.3: $H^1(F)_{\neq 1} \neq 0 \iff$ 3-nets".

Observation:



A(12.1)

In A(12.1) (left), unique lin. rel. ($k=3$)

$$\nabla(B_2) - \nabla(B_3) + \nabla(B_6) - \nabla(B_7) = 0$$

Only the bands with blue and red walls appear.

(Bands with green walls do not appear.)

∞.

$RB_3 = \{B_1, B_2, \cdots, B_7\}$

4. Idea of the proof.

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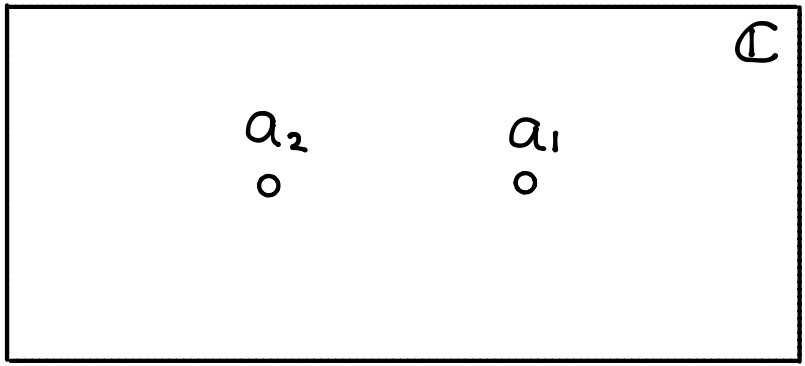
— Construct a good stratification for $M(A)$.
("minimal stratification")

— Computing $H^1(M(A), \mathcal{L}_\lambda)$ by using the stratification.

4. Idea of the proof.

1-dimensional example: $A = \{a_1, a_2\} \subset \mathbb{R}$.



Semi-algebraic subset defined by "ratio of consective linear forms $\in \mathbb{R}_{<0}$"

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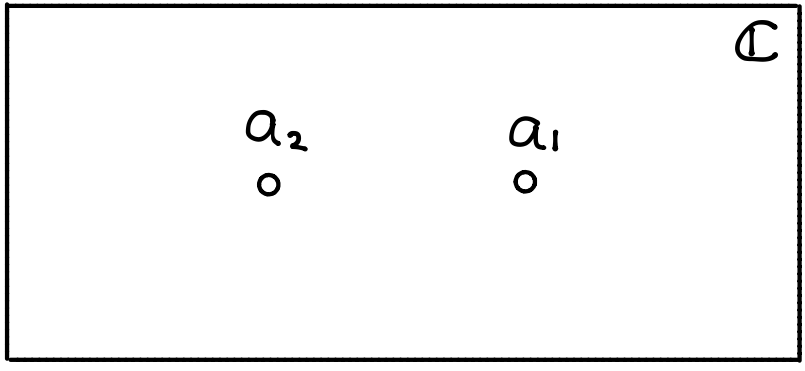

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$$S_1 = \left\{ z \in \mathbb{C} \;\middle|\; \frac{z - a_1}{z - a_2} \in \mathbb{R}_{<0} \right\}$$

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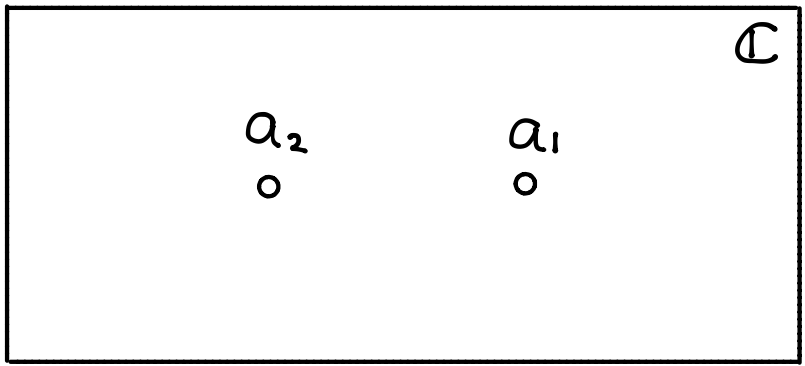

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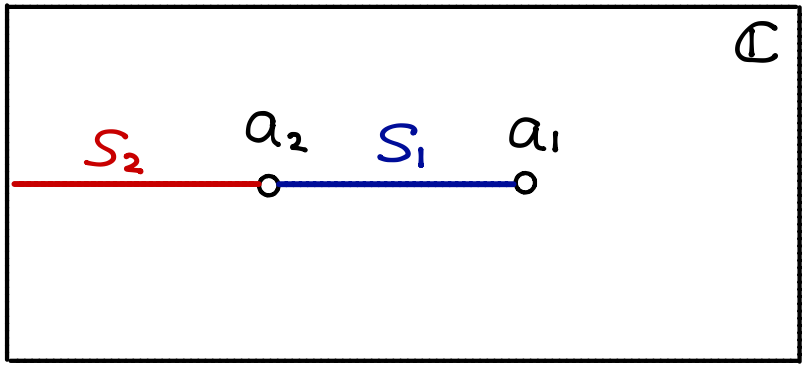

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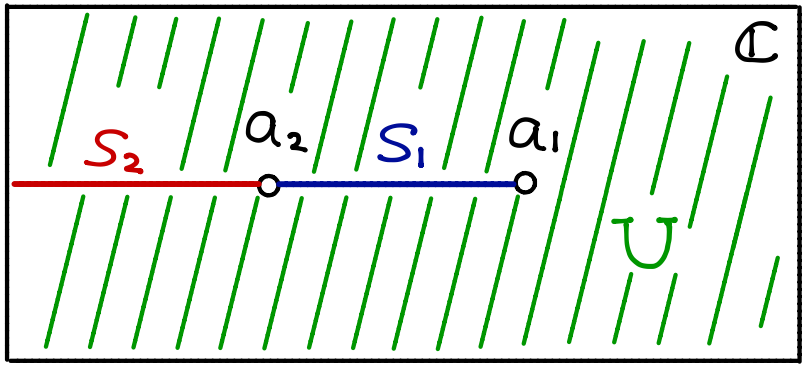

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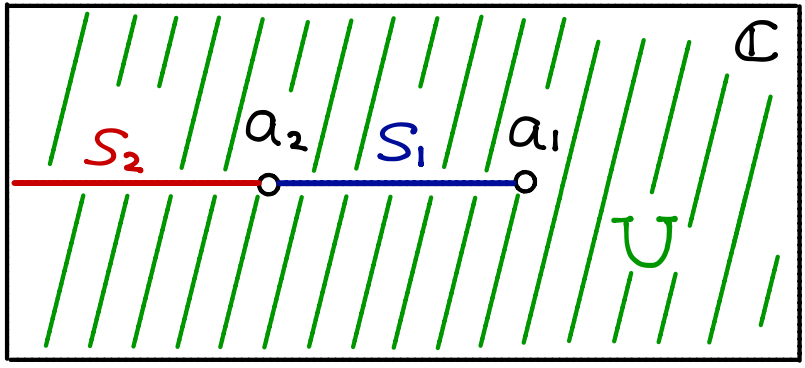

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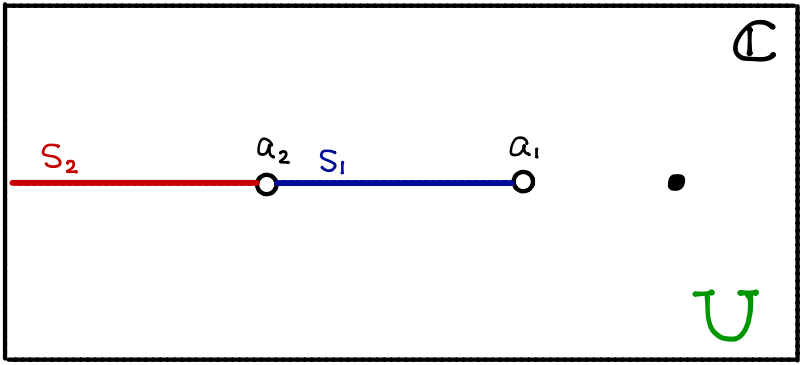
$$U := M(A) \setminus S_1 \sqcup S_2.$$

$$M(A) = U \sqcup S_1 \sqcup S_2$$

Partition by contractible pieces!!

4. Idea of the proof.

Consider the local system $\mathcal{L}_\lambda$ (monodromy around a_i is $\lambda \in \mathbb{C}^*$)



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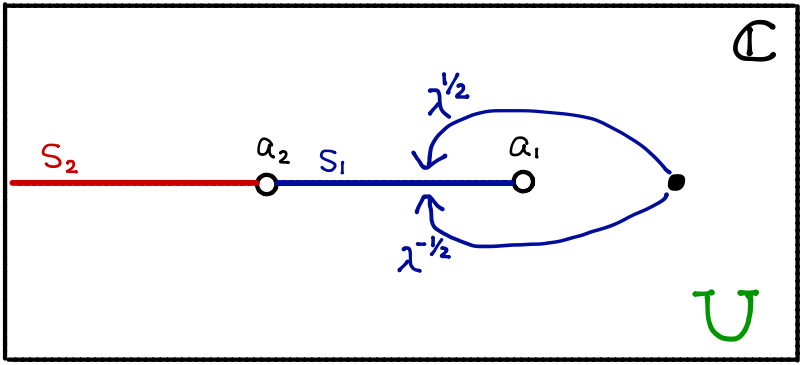

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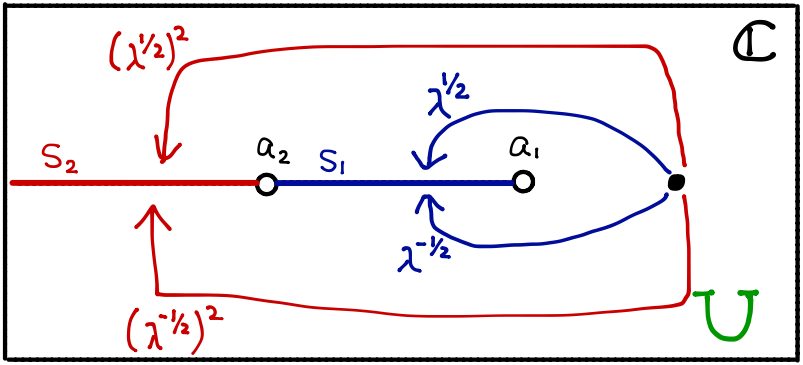

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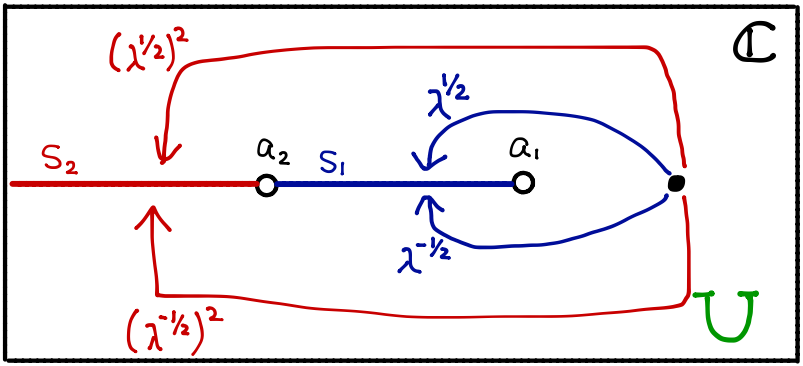

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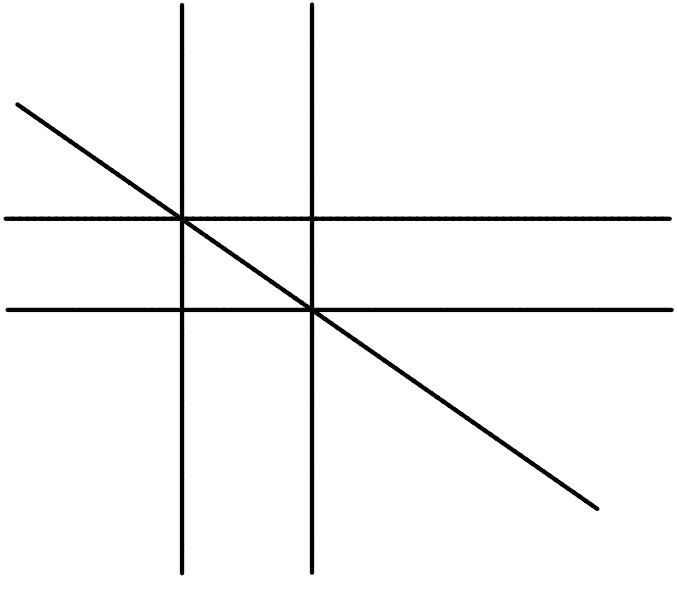
$$= 2\cdot\sqrt{-1}\cdot\left\{ \sin\left(\frac{\pi}{k}\cdot 1\right)\cdot[S_1] + \sin\left(\frac{\pi}{k}\cdot 2\right)\cdot[S_2] \right\}$$

$\boxed{\text{Similar construction works for 2-dim case.}}$

4. Idea of the proof.

Key Construction



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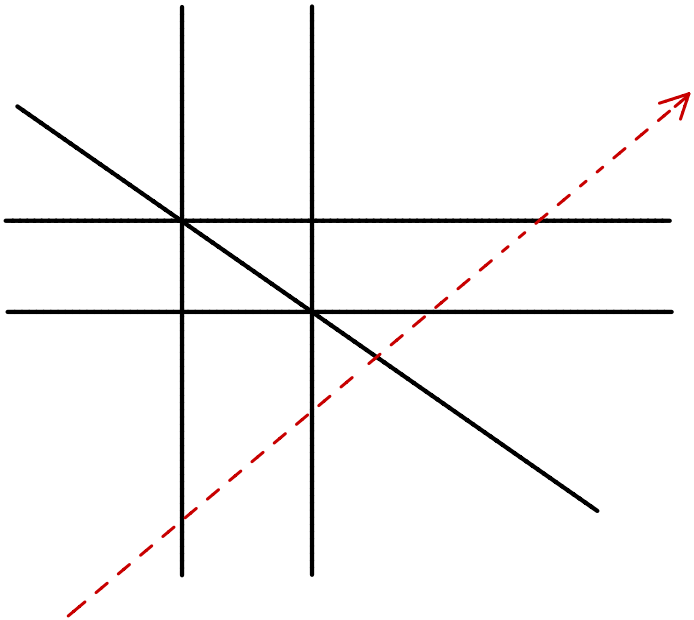
$\boxed{\text{Key Construction}}$

(i) Fix a generic line (w. ori.)



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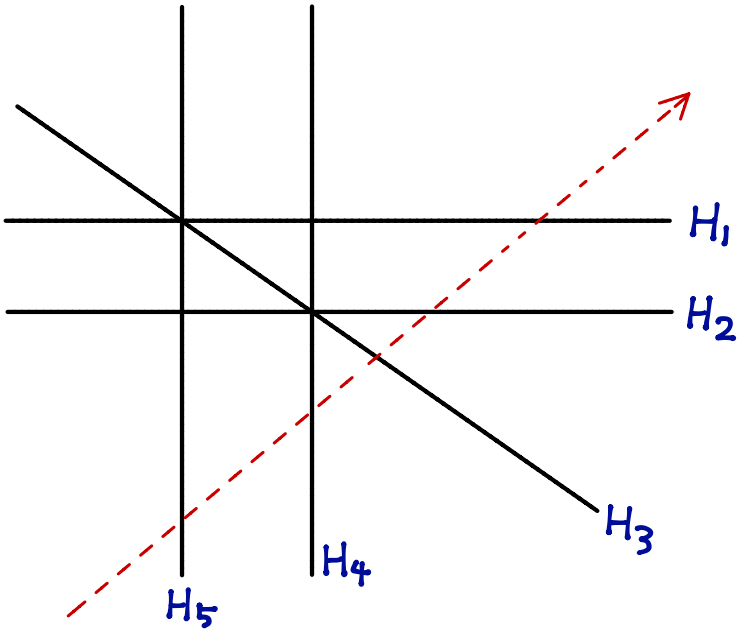

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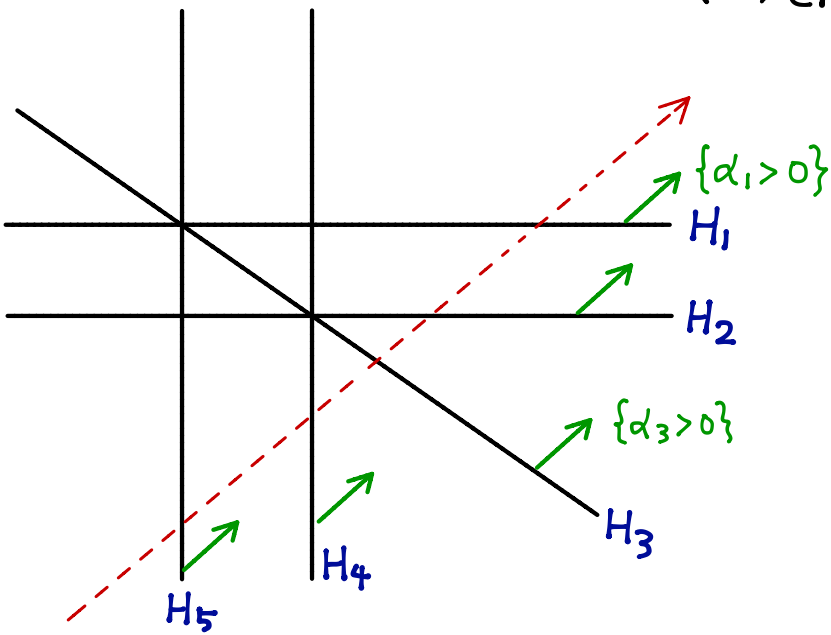
(iii) Choose the sign of the defining linear eq. α_i as in the figure.



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Def.

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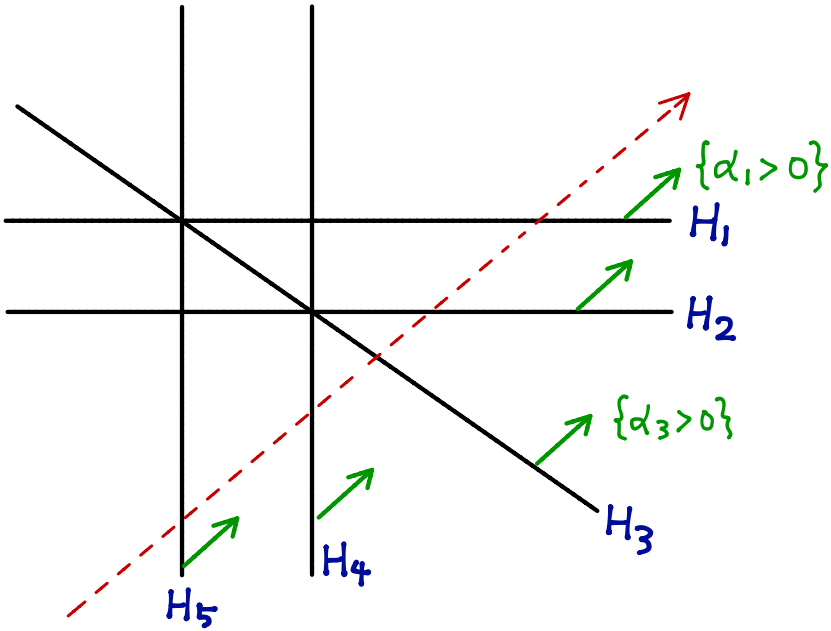

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Def.

$$S_i := \left\{ z \in M(A) \;\middle|\; \frac{\alpha_i(z)}{\alpha_{i+1}(z)} \in \mathbb{R}_{<0} \right\}$$

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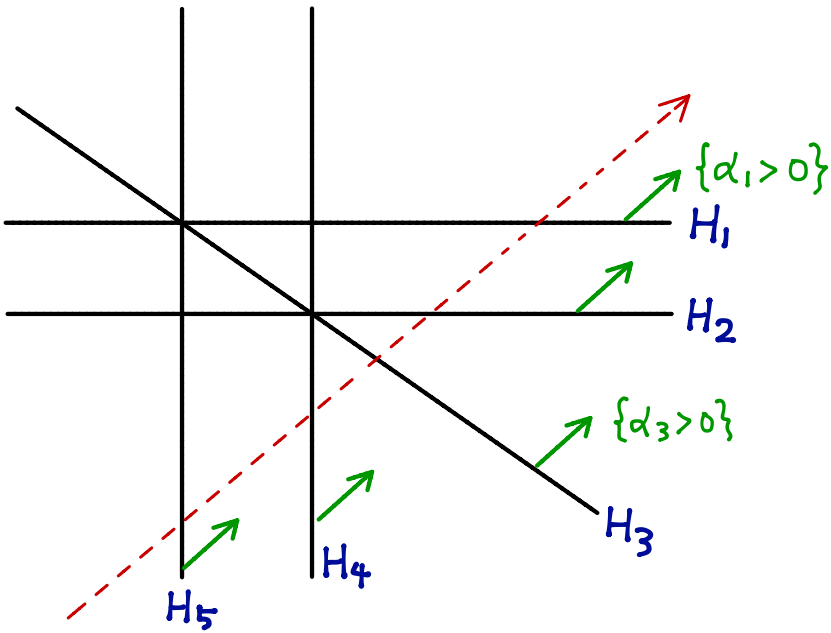

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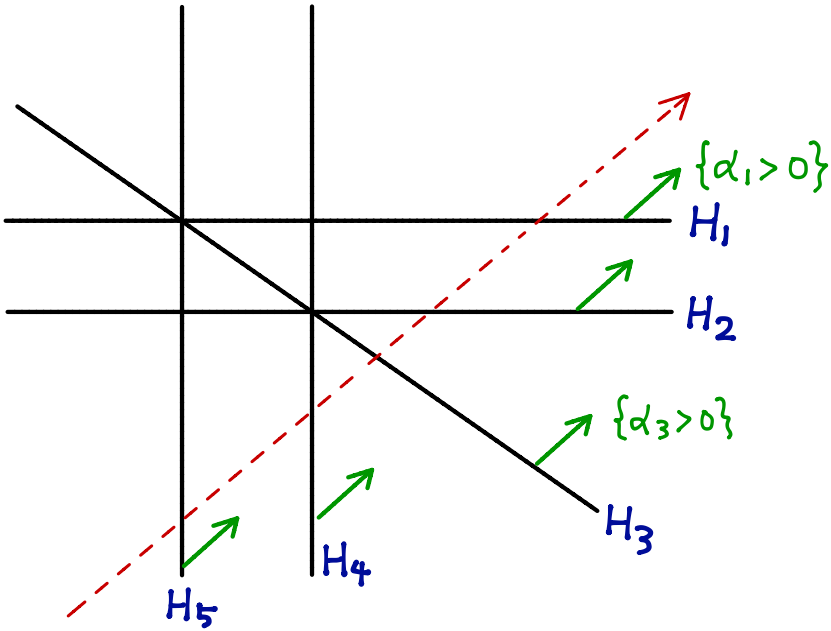

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$S_i \subset M(A)$ is a 3-dim submfd.

Intersecting nicely, among others.

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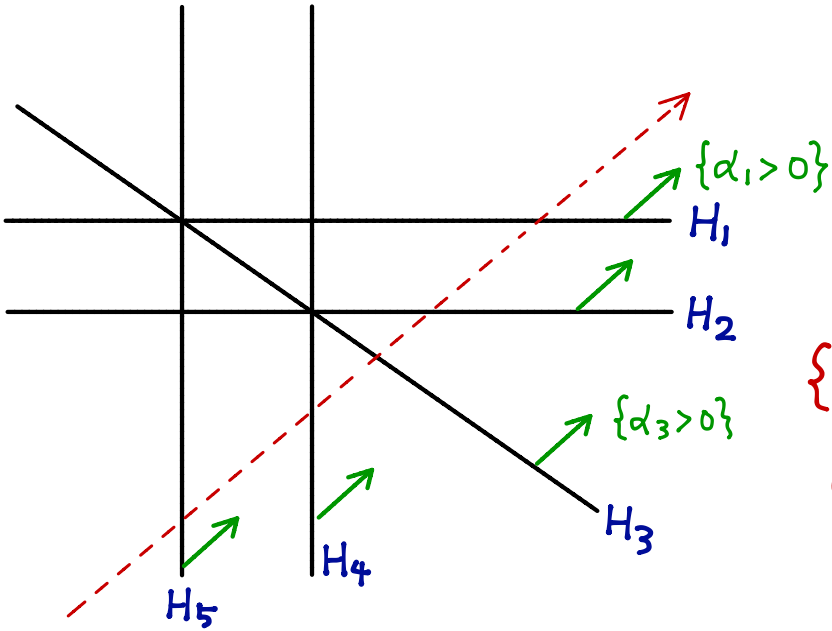

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$\{S_i\}$ induces a partition of $M(A)$ into contractible pieces.

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Thm. (Y.)

$\{S_i\}$ induces a partition of $M(A)$ into contractible pieces.

- Compute $H^1(M(A), \mathcal{L}_\lambda)$ using this partition.
- Simplify the cochain complex (purely algebraically).

Cor. $H^1(F)_\lambda = \mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(A)] \longrightarrow \mathbb{C}[ch(A)]\right)$.

Rem. The above partition is a "dual" description of Dimca-Papadima's minimal structure.

Summary

- Some topological problems on $\mathcal{A}$ are interpreted by using the notion of "resonant bands".
- The study of minimal structure is crucial.

Reference:

Y. Milnor fibers of real line arrangements.
J. of Singularities. vol. 7 (2013) 220–237.

Y. Minimal stratification for line arrangements and ····