Milnor fibers of real line arrangements.

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1st Franco-Japanese-Vietnamese Symposium on Singularities.

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Abstract: We introduce a new discrete-geometric device "resonant bands" for a real line arrangement.
This arises from the study of minimal CW str.
Then applying it to topology of Milnor fibers.

Contents

1. Milnor fibers of line arrangements

2. "Resonant band" algorithm and applications.

3. Milnor fiber and 3-multinets.

4. Idea of proof.

1. Milnor Fibers of Line arrangements

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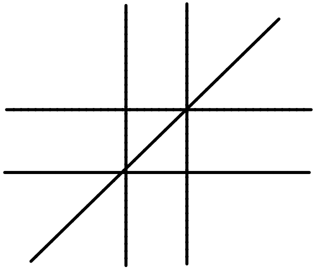
Given datum



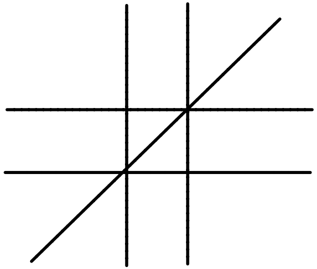
$x(x-1)y(y-1)(x-y) = 0$

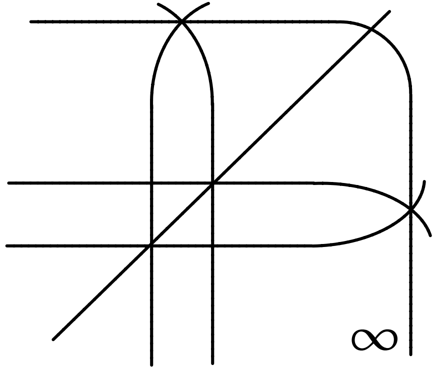
a line arrangement

$\mathcal{A} = \{H_1, \cdots, H_n\}$

1. Milnor Fibers of Line arrangements

Given datum $\rightsquigarrow$ Its Cone





$x(x-1)y(y-1)(x-y)$
$= 0$

$Q := x(x-z)y(y-z)(x-y)\cdot z$
$= 0.$

a line arrangement

$\mathcal{A} = \{H_1, \cdots, H_n\}$

$c\mathcal{A} := \{H_1, \cdots, H_n, H_\infty\}$

1. Milnor Fibers of Line arrangements

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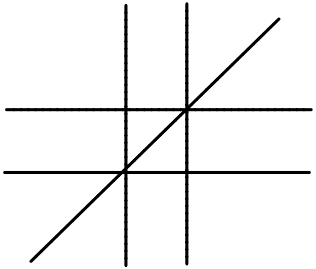

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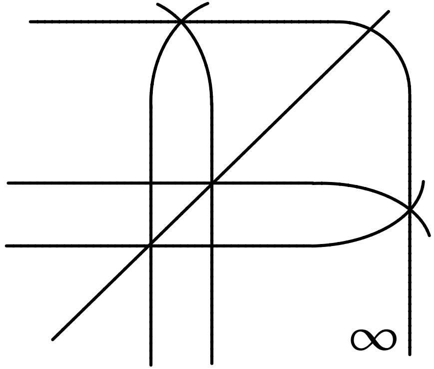


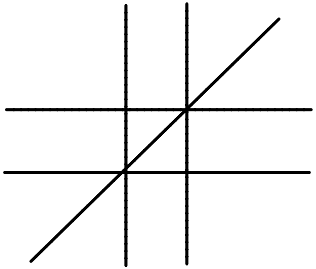
$Q := x(x-z)y(y-z)(x-y)\cdot z = 0.$

$c\mathcal{A} := \{H_1, \cdots, H_n, H_\infty\}$

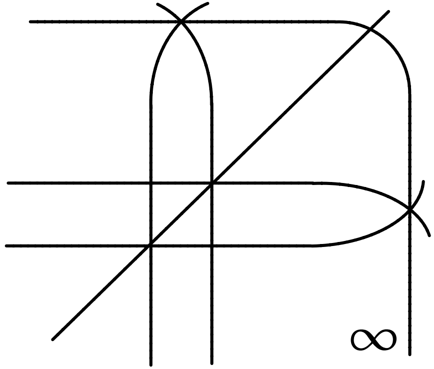
$$M(\mathcal{A}) := \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i = \mathbb{P}^2_{\mathbb{C}} \setminus Q^{-1}(0)$$

1. Milnor Fibers of Line arrangements

Given datum $\rightsquigarrow$ Its Cone — "Milnor Fiber"



$x(x-1)y(y-1)(x-y) = 0$



$Q := x(x-z)y(y-z)(x-y)\cdot z = 0.$

$$F = F_{\mathcal{A}} := Q^{-1}(1) = \{(x,y,z) \in \mathbb{C}^3 \mid Q(x,y,z) = 1\} \subset \mathbb{C}^3$$

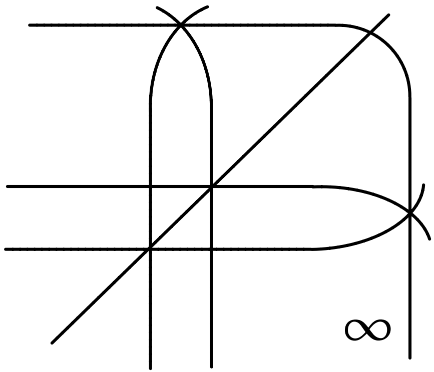
a line arrangement $\mathcal{A} = \{H_1, \cdots, H_n\}$

$c\mathcal{A} := \{H_1, \cdots, H_n, H_\infty\}$

$$M(\mathcal{A}) := \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i = \mathbb{P}^2_{\mathbb{C}} \setminus Q^{-1}(0)$$

1. Milnor Fibers of Line arrangements

Remark: $Q(x,y,z)$ is homogeneous of $\deg = n+1$.



$Q := x(x-z)y(y-z)(x-y)\cdot z$

$= 0.$

$F = \{(x,y,z) \in \mathbb{C}^3 \mid Q(x,y,z) = 1\}$

$= Q^{-1}(1)$

Monodromy action

$$\begin{array}{cccc} \rho : & F & \longrightarrow & F \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (x,y,z) & \longmapsto & (\zeta x, \zeta y, \zeta z), \end{array}$$

where $\zeta = e^{2\pi i / n+1}$

1. Milnor Fibers of Line arrangements

$$\rho : F \longrightarrow F : (x,y,z) \longmapsto (\zeta x, \zeta y, \zeta z),$$

where $\zeta = e^{2\pi i/n+1}$

induces a linear automorphism

$$\rho^* : H^1(F) \longrightarrow H^1(F).$$

We have eigendecomposition

λ- eigen space

$$H^1(F) = \bigoplus_{\lambda^{n+1}=1} H^1(F)_\lambda$$

Since $\rho^{n+1} = id.$

1. Milnor Fibers of Line arrangements

Easy part: $\lambda = 1$,

Nontrivial part: $\lambda \neq 1$

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ρ-invariant part

$$H^1(F)_1 = H^1(F)^{\rho} \cong H^1(F/\langle\rho\rangle) \cong H^1(M(\mathcal{A})) \cong \mathbb{C}^n.$$

$\left(\text{Recall}: F/\langle\rho\rangle = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i\right)$

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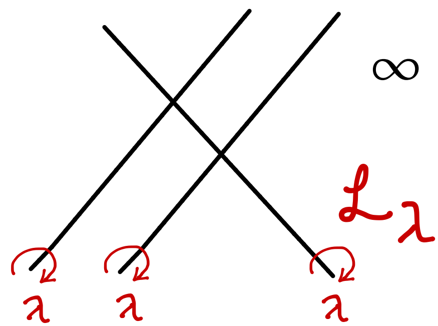
Nontrivial part: $\lambda \neq 1$

$$H^1(F)_\lambda \cong H^1(M(\mathcal{A}), \mathcal{L}_\lambda)$$

rank 1 local system on $M(\mathcal{A})$, s.t. each monodromy around H_i is λ.



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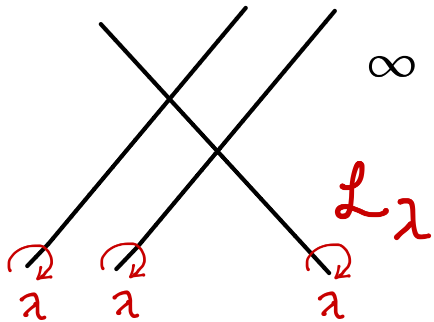
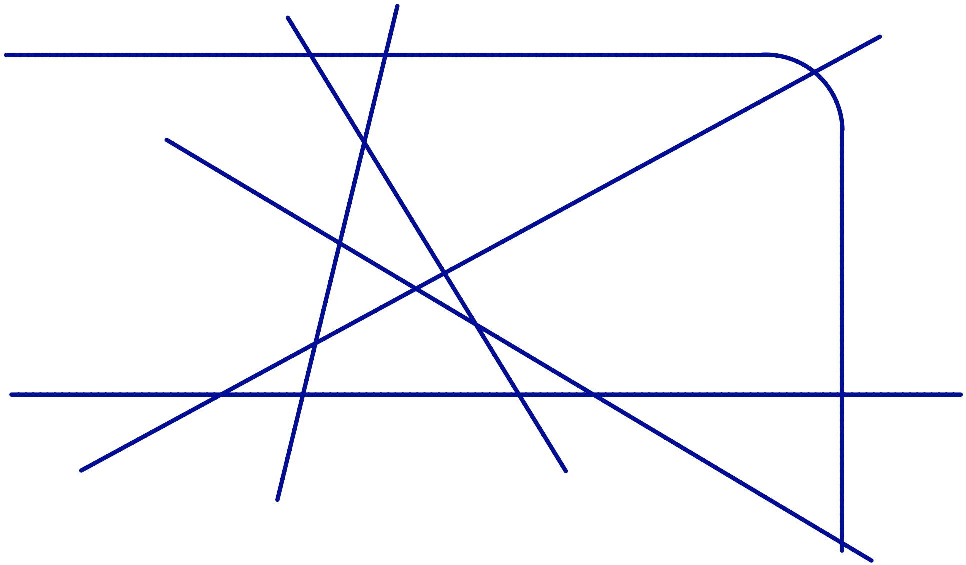


$$H^1(F)_{\neq 1} := \bigoplus_{\lambda \neq 1} H^1(F)_\lambda \; : \text{non-trivial eigenspace.}$$

1. Milnor Fibers of Line arrangements

Known Fact and examples

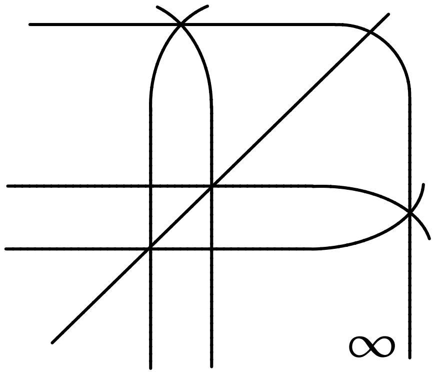
Thm. (Orlik-Randell, Hattori) If $\mathcal{A} = \{H_1, \cdots, H_n, H_\infty\}$ is generic (= only double points), then $H^1(F)_{\neq 1} = 0$. (i.e. $b_1(F) = n$)



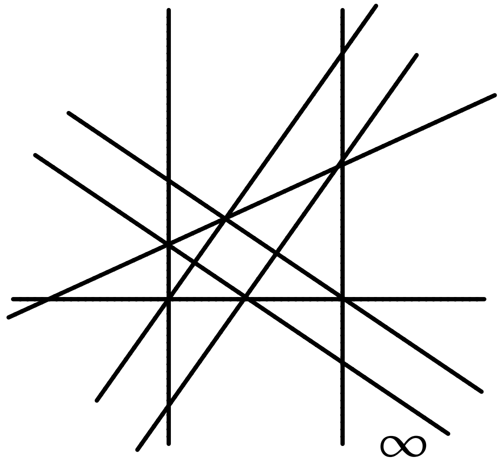
1. Milnor Fibers of Line arrangements

Non-generic cases:

$$\boxed{\zeta = e^{2\pi i/3}}$$



A_3-arr.



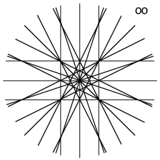
Pappus arr.

$$H^1(F) = \underset{\mathbb{C}^5}{\underset{\cong}{H^1(F)_1}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_\zeta}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_{\zeta^2}}}$$

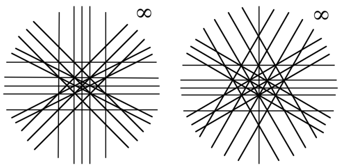
$$H^1(F) = \underset{\mathbb{C}^8}{\underset{\cong}{H^1(F)_1}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_\zeta}} \oplus \underset{\mathbb{C}}{\underset{\cong}{H^1(F)_{\zeta^2}}}$$

1. Milnor Fibers of Line arrangements

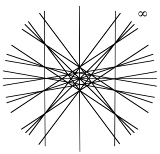
NOT All "non-generic arr" have non-trivial eigenspaces.



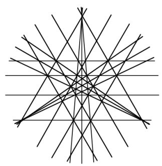
$\mathcal{A}(21,4)$



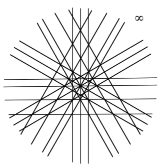
$\mathcal{A}(21,5)$



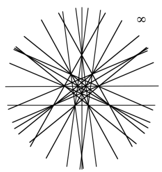
$\mathcal{A}(21,6)$



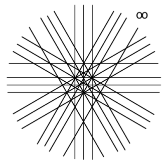
$\mathcal{A}(21,7)$



$\mathcal{A}(22,2)$



$\mathcal{A}(22,3)$



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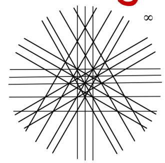
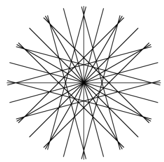
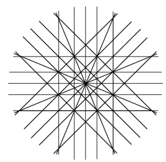


$\mathcal{A}(23,1)$



$\mathcal{A}(24,1)$



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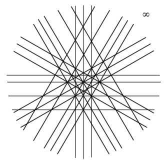

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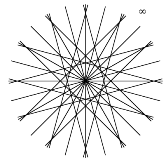
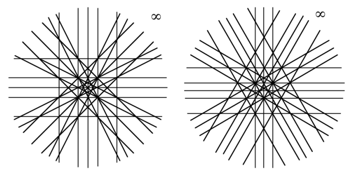


$\mathcal{A}(25,1)$



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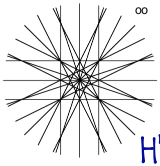
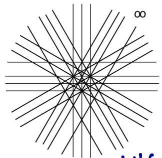
1. Milnor Fibers of Line arrangements

NOT All "non-generic arr" have non-trivial eigenspaces.



$\mathcal{A}(21,4)$ $H^1(F)_{\neq 1}=0$



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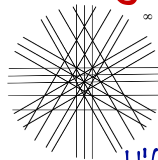
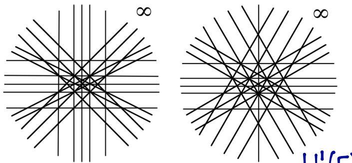


$\mathcal{A}(23,1)$ $H^1(F)_{\neq 1}=0$



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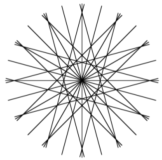

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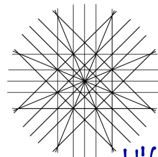
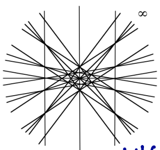
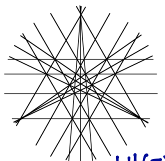


$\mathcal{A}(24,2)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(21,6)$ $H^1(F)_{\neq 1}=0$



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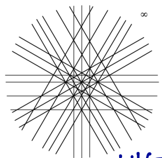
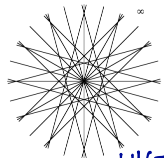
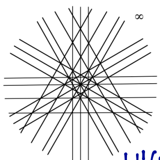


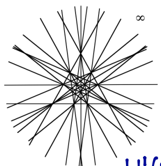
$\mathcal{A}(24,3)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(25,1)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(22,2)$ $H^1(F)_{\neq 1}=0$



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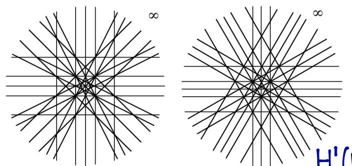

$H^1(F)_{\neq 1}=0$

$\mathcal{A}(25,2)$

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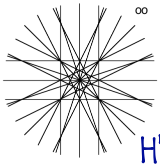
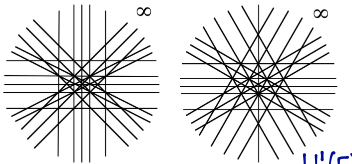


$\mathcal{A}(21,4)$ $H^1(F)_{\neq 1}=0$



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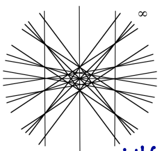

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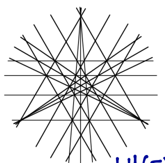

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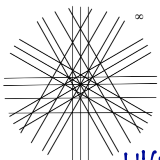
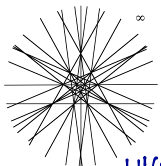


$\mathcal{A}(22,2)$ $H^1(F)_{\neq 1}=0$



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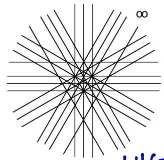

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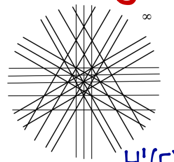
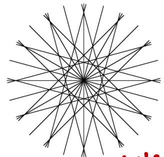
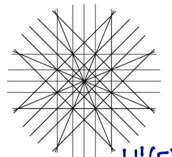


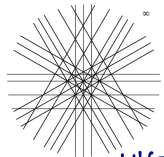
$\mathcal{A}(23,1)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(24,1)$ $H^1(F)_{\neq 1}\neq 0$



$\mathcal{A}(24,2)$ $H^1(F)_{\neq 1}=0$



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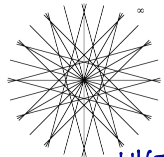
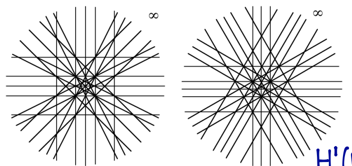


$\mathcal{A}(25,1)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(25,2)$ $H^1(F)_{\neq 1}=0$

1. Milnor Fibers of Line arrangements

$\boxed{\text{Problem}}$ Compute $\dim H^1(F)_\lambda = \dim H^1(M, \mathcal{L}_\lambda)$.

(Combinatorial formula? Unknown.)

<u>Rem.</u> $\dim H^1(F)_\lambda$ is related to many other things.

- Betti numbers of certain covering spaces of $M(\mathcal{A})$.
- Alexander polynomial of $\pi_1(M)$.
- Counting certain plane curves.
- Hodge str. of $H^1(F, \mathbb{C})$.

2. Resonant band algorithm

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Now we assume $\mathcal{A}=\{H_1,\cdots,H_n\}$ is defined $/\mathbb{R}$.

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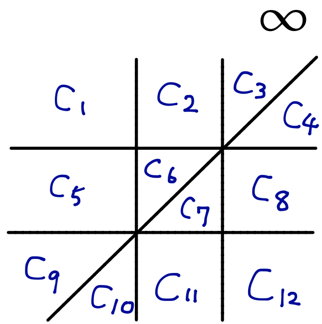

Notation:

$ch(\mathcal{A})$: the set of chambers.

$ch(\mathcal{A})=\{C_1,\cdots,C_{12}\}$

Adjacency distance:

For $C, C' \in ch(\mathcal{A})$,

$d(C,C') :=$ # of lines which separates C & C'.

e.g. $d(C_5, C_{12})=4$

2. Resonant band algorithm

Fix a positive integer $k \mid (n+1)$. Set $\lambda = e^{2\pi i/k}$.

($k > 1$, mainly $k=3$.)

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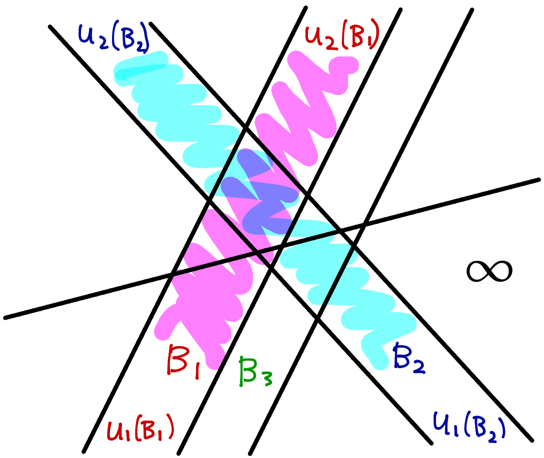
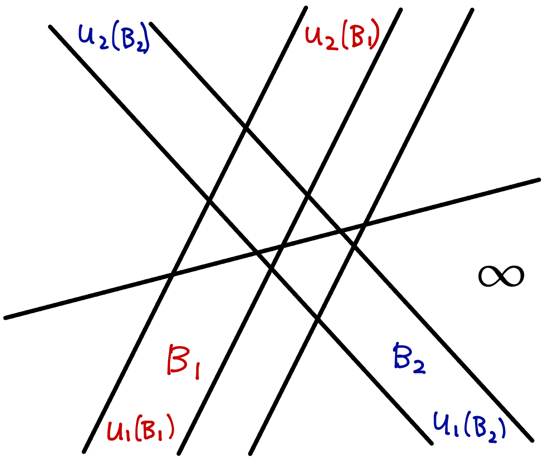


A **band** is a region bounded by a pair of consecutive parallel lines.

In a band B, there are exactly two unbounded chambers, set $u_1(B)$ and $u_2(B)$.

Next we define k-resonance.

2. Resonant band algorithm



A band is a region bounded by a pair of consecutive parallel lines.

In a band B, there are exactly two unbounded chambers, set $u_1(B)$ and $u_2(B)$.

A band B is k-resonant $\overset{\text{def}}{\iff}$ $k \mid d(u_1(B), u_2(B))$.

E.g. $d(u_1(B_1), u_2(B_1)) = 3 \rightsquigarrow B_1$ is 3-resonant.

$d(u_1(B_2), u_2(B_2)) = 4 \rightsquigarrow B_2$ is not 3-resonant.

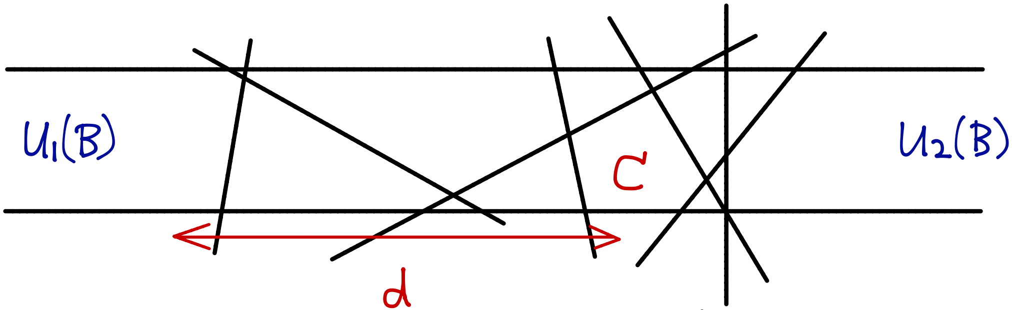
2. Resonant band algorithm

Discrete Wave on the band B:

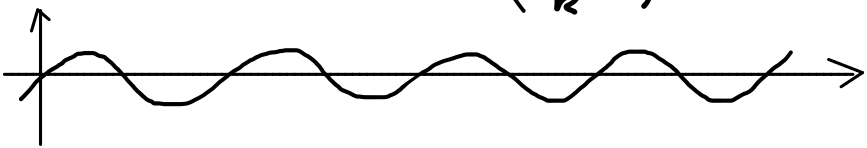
Suppose that B is k-resonant. ($\Longleftrightarrow \sin\left(\frac{\pi\cdot d(U_1,U_2)}{k}\right)=0$)

vector space spanned by chambers.

$$\nabla(B) := \sum_{C\subseteq B} \sin\left(\frac{\pi\cdot d(U_1(B),C)}{k}\right)\cdot[C] \in \mathbb{C}[ch(\mathcal{A})]$$



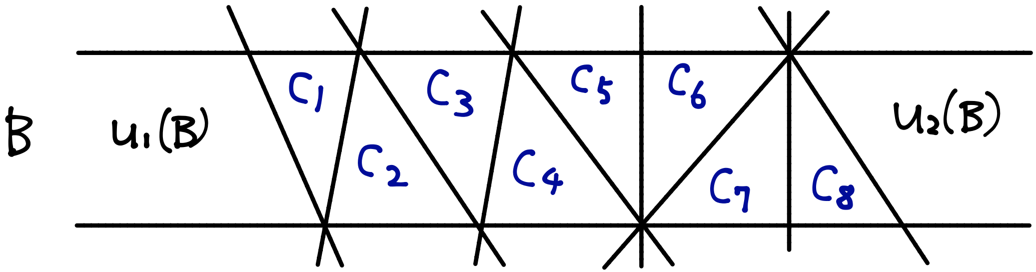
$$\nabla(B)=\cdots+\sin\left(\frac{\pi}{k}\cdot d\right)\cdot[C]+\cdots$$



2. Resonant band algorithm

$$\nabla(B) := \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) \cdot [C] \in \mathbb{C}[ch(A)]$$

<u>Ex.</u> $d(u_1(B), u_2(B)) = 9, \quad k = 3.$



$$\nabla(B) = \sum_{p=1}^{8} \sin\left(\frac{\pi}{3} \cdot p\right) \cdot [C_p]$$

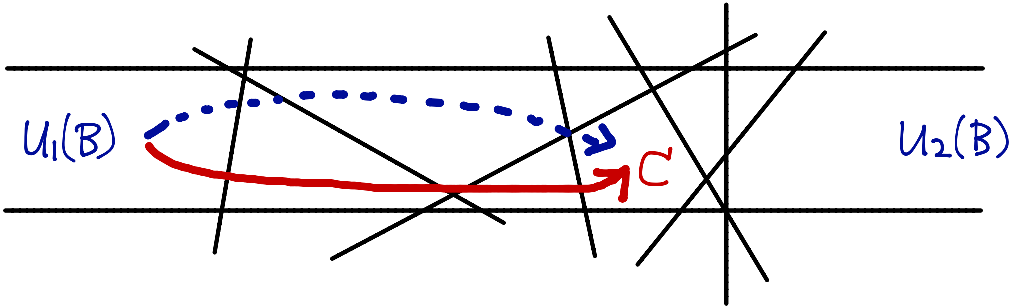
$d(u_1(B), C_p) = p.$

$$= \frac{\sqrt{3}}{2}[C_1] + \frac{\sqrt{3}}{2}[C_2] + 0 \cdot [C_3] - \frac{\sqrt{3}}{2}[C_4] - \frac{\sqrt{3}}{2}[C_5] + 0[C_6] + \frac{\sqrt{3}}{2}[C_7] + \frac{\sqrt{3}}{2}[C_8].$$

2. Resonant band algorithm

Another interpretation:

$$\nabla(B) := \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) \cdot [C] \in \mathbb{C}[\mathrm{ch}(A)]$$



$$\sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) = \frac{1}{2i}\left(e^{\frac{\pi i d}{k}} - e^{-\frac{\pi i d}{k}}\right)$$

is the difference of two parallel translations.

(positive and negative)

2. Resonant band algorithm

Notation: $RB_k(A)$ denotes the set of k-resonant bands.

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<u>Theorem</u> (Y.) Let $\mathcal{A} = \{H_1, \cdots, H_n\}$ be a line arr. in $\mathbb{R}^2$, $k \mid (n+1)$, $k > 1$, $\lambda = e^{2\pi i/k}$ (or its Galois conj). Then

$$\operatorname{Ker}\left(\nabla : \mathbb{C}[RB_k(A)] \longrightarrow \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda.$$

In particular, $\dim H^1(F)_\lambda$ is equal to the number of linear relations among discrete waves $\nabla(B)$, $B \in RB_k(A)$.

2. Resonant band algorithm

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_{\mathbb{R}}(\mathcal{A})] \longrightarrow \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda$.

(Hence $\dim H^1(F)_\lambda$ is equal to # of linear rel's among $\nabla(B)$'s.)

Example (Pappus arr.)

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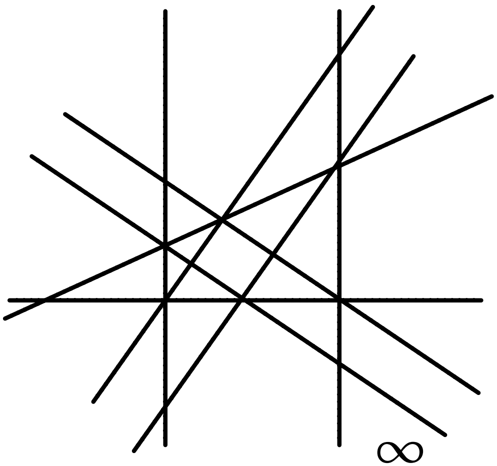

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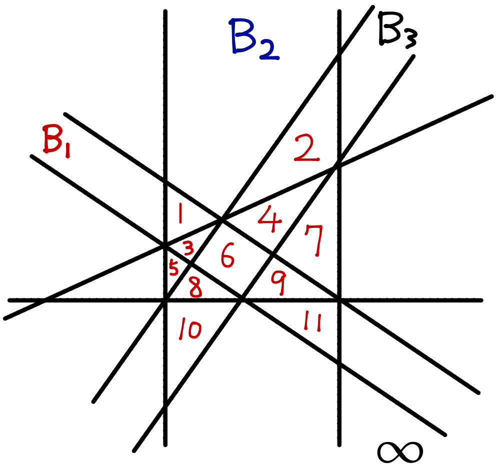

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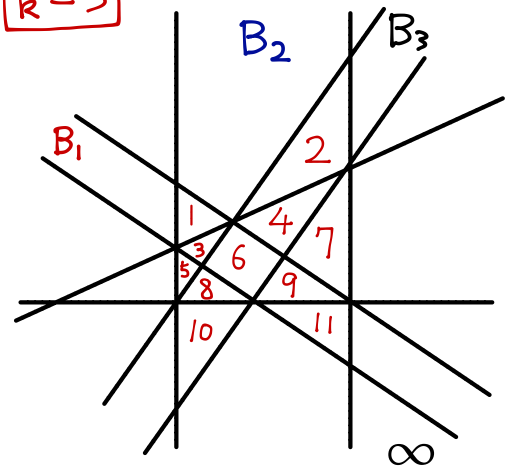
Example (Pappus arr.)

$\boxed{k=3}$



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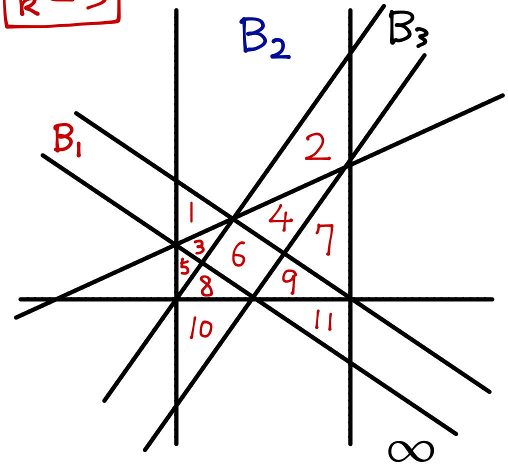
Example (Pappus arr.) $\frac{2}{\sqrt{3}}\nabla(B_1) = [C_1] + [C_3] - [C_9] - [C_{11}]$.

$\boxed{k=3}$



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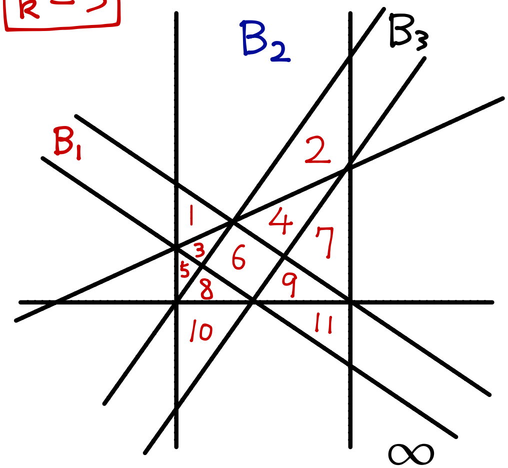

$$\frac{2}{\sqrt{3}}\nabla(B_1) = [C_1] + [C_3] - [C_9] - [C_{11}].$$

$$\frac{2}{\sqrt{3}}\nabla(B_3) = [C_2] + [C_4] - [C_8] - [C_{10}]$$

2. Resonant band algorithm

<u>Theorem.</u> $\operatorname{Ker}\left(\nabla : \mathbb{C}[RB_k(\mathcal{A})] \to \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda$.

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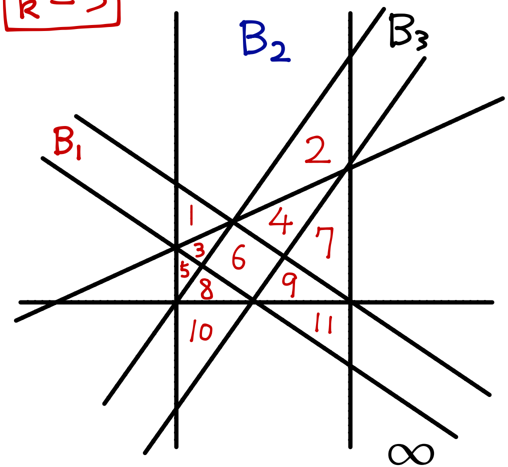

$$\frac{2}{\sqrt{3}}\nabla(B_1) = [C_1]+[C_3]-[C_9]-[C_{11}].$$

$$\frac{2}{\sqrt{3}}\nabla(B_3) = [C_2]+[C_4]-[C_8]-[C_{10}]$$

$$\frac{2}{\sqrt{3}}\nabla(B_2) = [C_1]+[C_2]+[C_3]+[C_4]$$

$$-[C_8]-[C_9]-[C_{10}]-[C_{11}]$$

2. Resonant band algorithm

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(\mathcal{A})] \longrightarrow \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda$.

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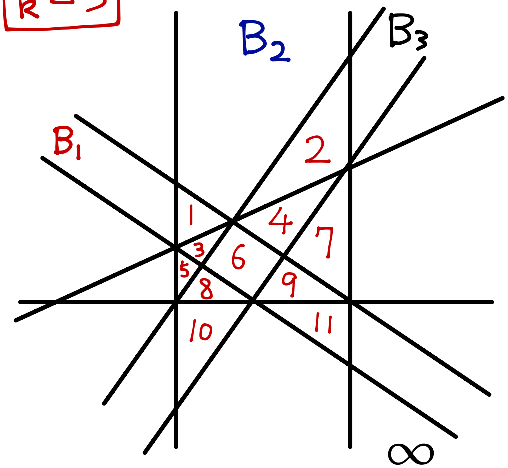

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$$\frac{2}{\sqrt{3}}\nabla(B_2) = [C_1] + [C_2] + [C_3] + [C_4]$$

$$- [C_8] - [C_9] - [C_{10}] - [C_{11}]$$

$$= \frac{2}{\sqrt{3}}\nabla(B_1) + \frac{2}{\sqrt{3}}\nabla(B_3).$$

2. Resonant band algorithm

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(A)] \to \mathbb{C}[ch(A)]\right) \cong H^1(F)_\lambda$.

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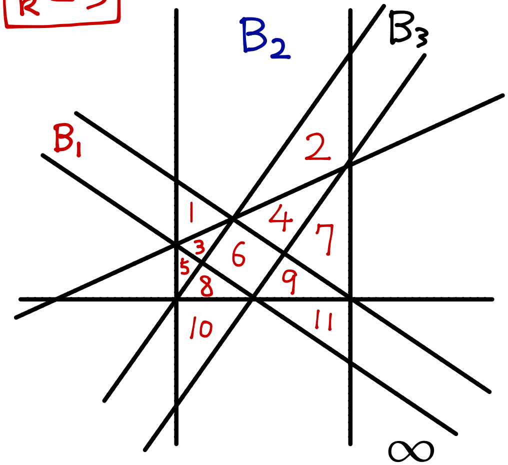

$$\frac{2}{\sqrt{3}}\nabla(B_1) = [C_1]+[C_3]-[C_9]-[C_{11}].$$

$$\frac{2}{\sqrt{3}}\nabla(B_3) = [C_2]+[C_4]-[C_8]-[C_{10}]$$

$$\frac{2}{\sqrt{3}}\nabla(B_2) = [C_1]+[C_2]+[C_3]+[C_4]$$

$$-[C_8]-[C_9]-[C_{10}]-[C_{11}]$$

$$= \frac{2}{\sqrt{3}}\nabla(B_1) + \frac{2}{\sqrt{3}}\nabla(B_3).$$

$$\therefore \dim H^1(F)_{e^{2\pi i/3}} = 1.$$

2. Resonant band algorithm

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(\mathcal{A})] \longrightarrow \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda$.

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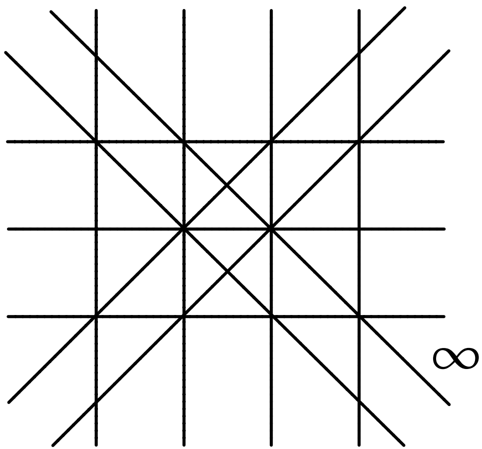
Example ($\mathcal{A}(12, 2)$, Grünbaum's Catalogue)



2. Resonant band algorithm

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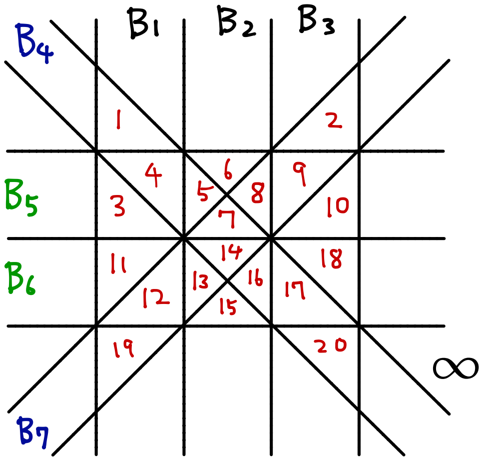

2. Resonant band algorithm

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(\mathcal{A})] \to \mathbb{C}[ch(\mathcal{A})]\right) \simeq H^1(F)_\lambda$.

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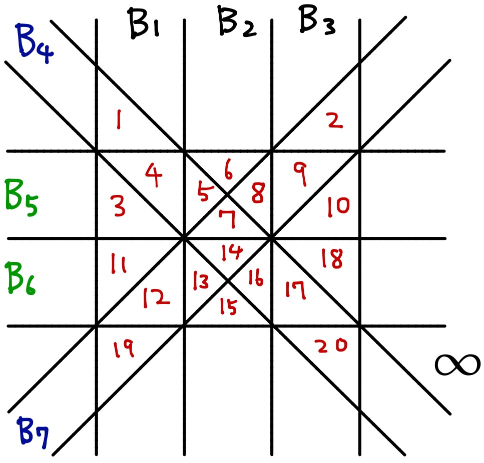

$n+1=12$. $k=2,3,4,6,12$.

2. Resonant band algorithm

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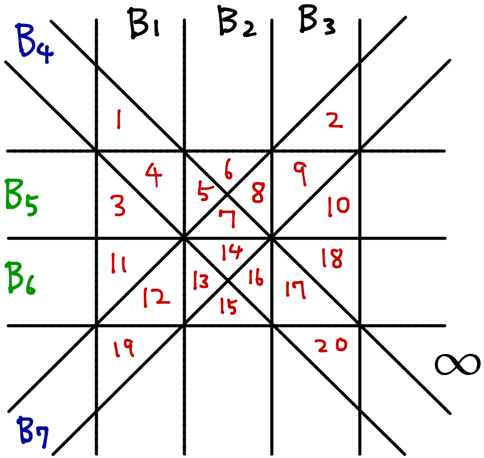

$n+1=12. \quad k=2,3,4,6,12.$

Bands	Length
B_1, B_2, B_3	7
B_4, B_7	9
B_5, B_6	8

2. Resonant band algorithm

<u>Theorem</u>. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(\mathcal{A})] \longrightarrow \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda$.

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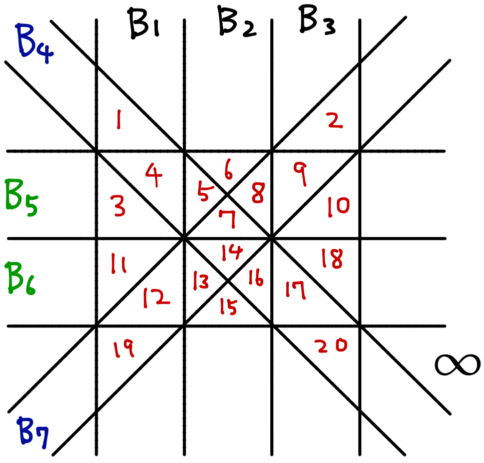

$n+1=12$. $k = 2, 3, 4, 6, 12$.

Bands	Length
B_1, B_2, B_3	7 ← never k-resonant
B_4, B_7	9
B_5, B_6	8

2. Resonant band algorithm

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(\mathcal{A})] \to \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda$.

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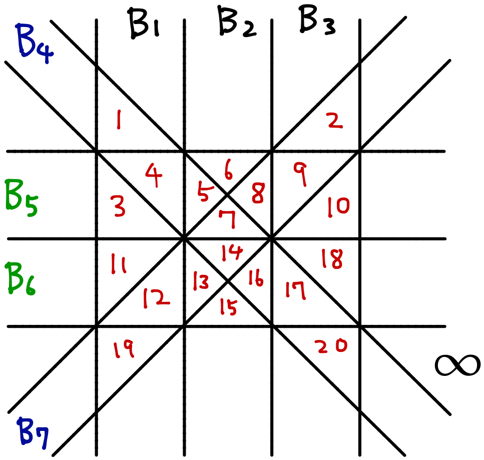

$n+1=12.\quad k=2,3,4,6,12.$

Bands	Length
B_1, B_2, B_3	7 ← never k-resonant
B_4, B_7	9 ← 3-resonant
B_5, B_6	8

2. Resonant band algorithm

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_k(\mathcal{A})] \longrightarrow \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda.$

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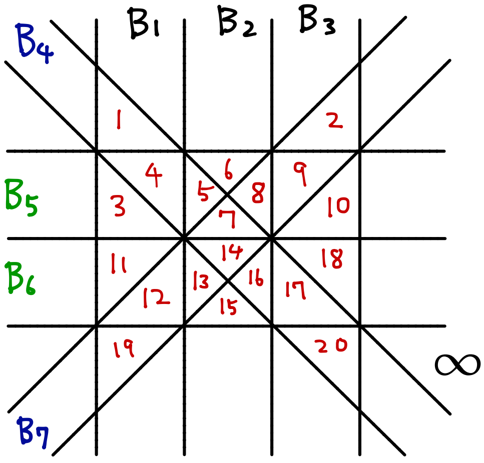
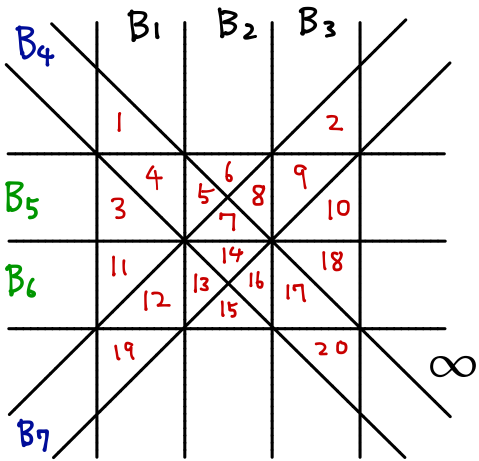


$n+1=12$. $k=2,3,4,6,12$.

Bands	Length	
B_1, B_2, B_3	7	← never k-resonant
B_4, B_7	9	← 3-resonant
B_5, B_6	8	← 2 & 4-resonant.

2. Resonant band algorithm

Example (A(12,2), Grünbaum's Catalogue)



$n+1=12.\quad k=2,3,4,6,12.$

Bands	Length
B_4, B_7	9
B_5, B_6	8

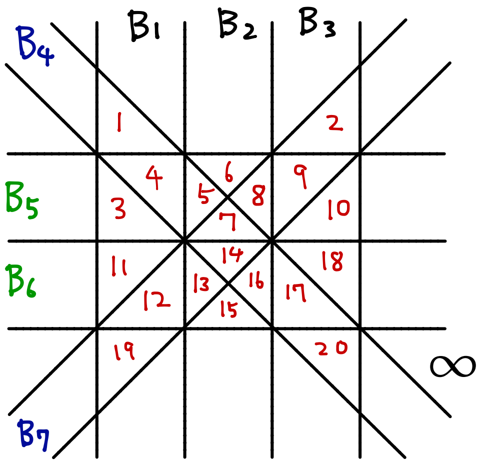
← 3-resonant

← 2 & 4-resonant.

$$\boxed{k=3,\quad RB_3(A)=\{B_4,B_7\}}.$$

2. Resonant band algorithm

Example ($\mathcal{A}(12,2)$, Grünbaum's Catalogue)



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Bands	Length
B_4, B_7	9
B_5, B_6	8

← 3-resonant

← 2 & 4-resonant.

$\boxed{k=3,\quad RB_3(\mathcal{A})=\{B_4,B_7\}},$

$\nabla(B_4)=\frac{\sqrt{+3}}{2}[C_1]+\cdots,$

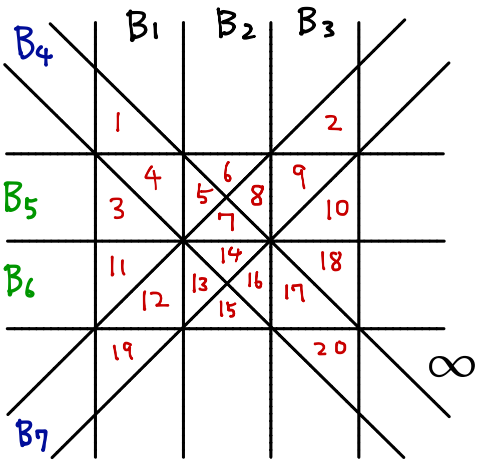
$\nabla(B_7)=\frac{\sqrt{+3}}{2}[C_{19}]+\cdots.$

[C_1] does not appear.

Hence $\nabla(B_4)$ and $\nabla(B_7)$ are linearly indep. Hence $H^1(F)_{e^{2\pi i/3}}=0.$

2. Resonant band algorithm

<u>Example</u> ($\mathcal{A}(12,2)$, Grünbaum's Catalogue)



$n+1=12$. $k=2,3,4,6,12$.

Bands	Length
B_4, B_7	9
B_5, B_6	8

← 3-resonant

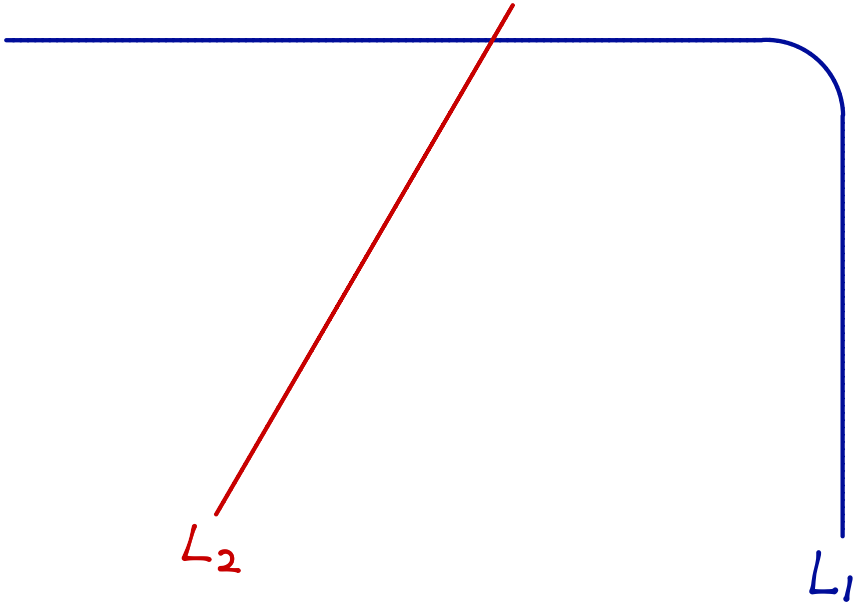
← 2 & 4-resonant.

parallel

Similarly, $RB_2 = RB_4 = \{B_5, B_6\}$, $\nabla(B_5)$ and $\nabla(B_6)$ are linearly indep. (No overlaps!), hence $H^1(F)_{\neq 1} = 0$.

2. Resonant band algorithm

Recall that two lines L_1 and L_2 divides $\mathbb{RP}^2$ into two regions.



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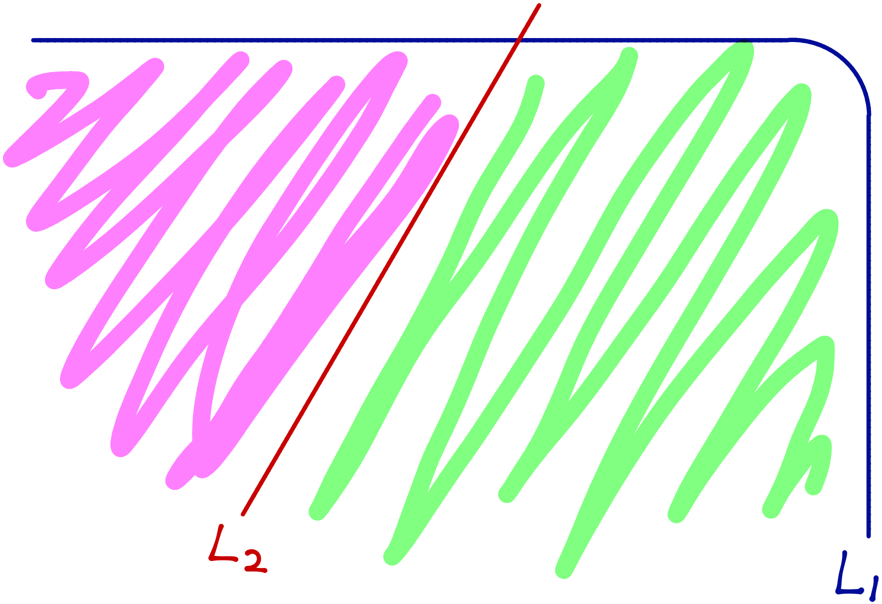
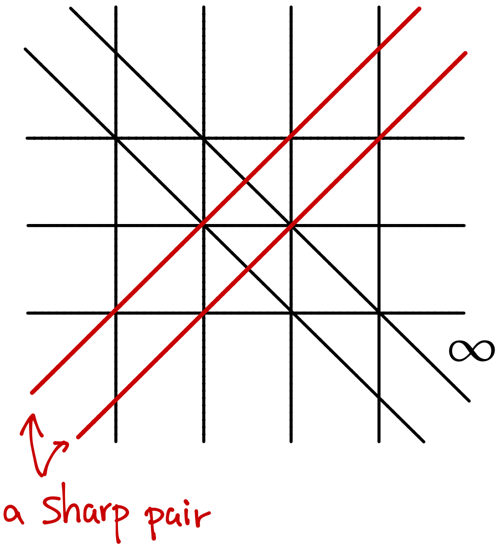
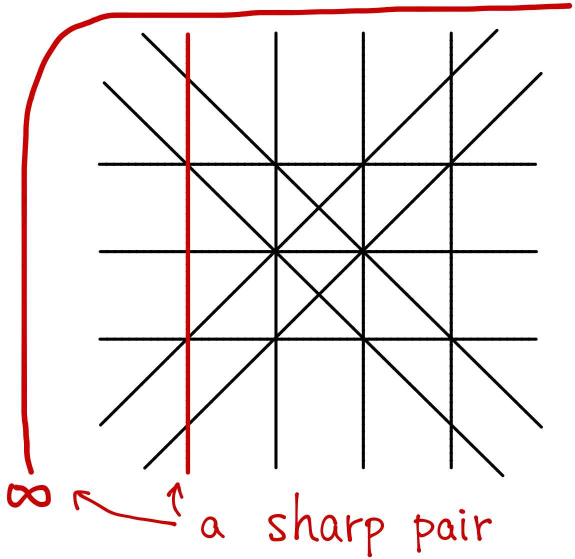


2. Resonant band algorithm

Recall that two lines L_1 and L_2 divides $\mathbb{RP}^2$ into two regions.

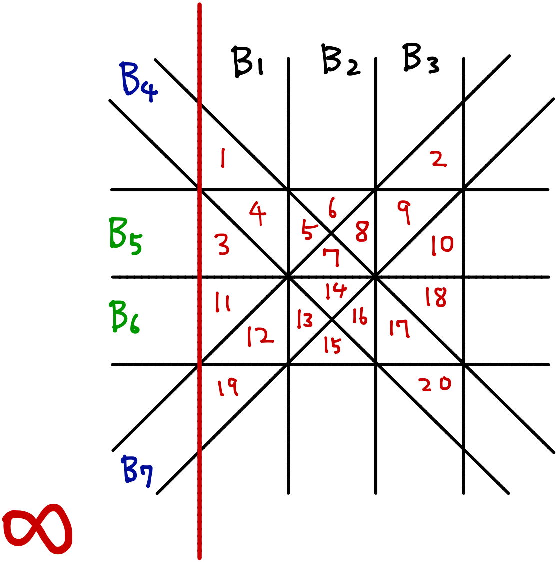
Def. The pair $H_1, H_2 \in \mathcal{A}$ is called "sharp pair" if one of two regions does not contain intersections (in its interior).





2. Resonant band algorithm

<u>Def.</u> The pair $H_1, H_2 \in c\mathcal{A}$ is called "sharp pair" if one of two regions does not contain intersections (in its interior).



<u>Thm.</u> (Y.)

If $c\mathcal{A}$ has a sharp pair, then

$$\dim H^1(F)_\lambda \leqq 1 \quad \text{for } {}^\forall \lambda \neq 1.$$

Idea of proof: Suppose $\sum_i a_i \cdot \nabla(B_i) = 0.$

(In the left figure) a_1 determines a_4, a_5, a_6, a_7.

Then a_2, a_3 are also uniquely determined. (Q.E.D.)

2. Resonant band algorithm

Thm. (Libgober)

If $\exists H_i \in \mathcal{A}$ s.t. all intersections on H_i has multiplicity 2 (i.e. No higher multiple points exist on H_i)

$$\Longrightarrow H^1(F)_\lambda = 0 \quad \text{for } \lambda \neq 1.$$

(proof)

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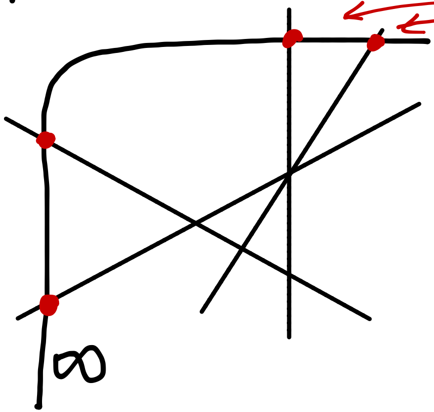
(proof) Put H_i at ∞.



All intersections on H_∞ have multiplicity $=2$

$\Longleftrightarrow$

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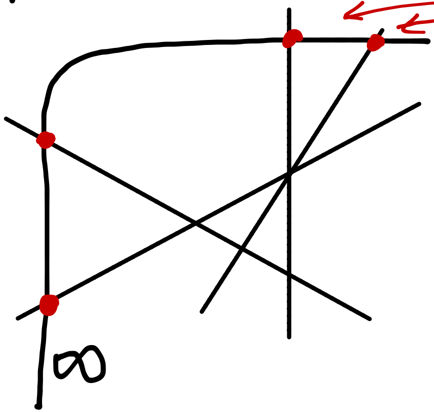

All intersections on H_∞ have multiplicity $=2$

$\Longleftrightarrow$ There are no bands.

$RB_\ell(\mathcal{A}) = \phi$. Hence $\mathrm{Ker}(\mathbb{C}[RB_\ell] \to \mathbb{C}[ch])$

$= 0$ (Q.E.D.)

2. Resonant band algorithm

<u>Thm.</u> (Libgober)

If $\exists H_i \in \mathcal{A}$ s.t. all intersections on H_i has multiplicity 2 (i.e. No higher multiple points exist on H_i)

$$\Longrightarrow H^1(F)_\lambda = 0 \quad \text{for } \lambda \neq 1.$$

<u>Rem.</u> Libgober proved the above result for any complex line arrangements. Our new proof works only for those defined $/\mathbb{R}$. However ⋯

2. Resonant band algorithm

We can strengthen for real arrangements

<u>Prop.</u> Suppose $\mathcal{A}$ is not a pencil. If $\exists H_i \in \mathcal{A}$ s.t.

there is at most one higher mult. pt. on H_i

$$\Longrightarrow H^1(F)_\lambda = 0 \text{ for } \lambda \neq 1$$

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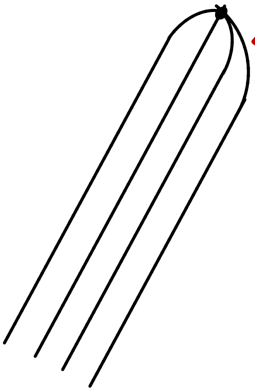

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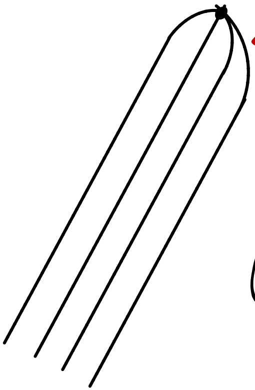
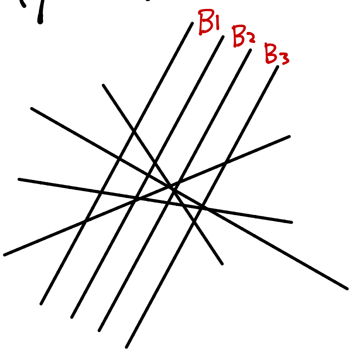


There is at most one higher mult. pt. on H_i,

$$\Longrightarrow H^1(F)_\lambda = 0 \text{ for } \lambda \neq 1$$

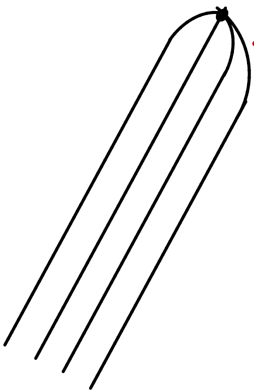
(proof) In this case, all bands are parallel.



Since parallel bands have no overlaps, $\nabla(B_1), \cdots$ are linearly indep. So $H^1(F)_{\neq 1} = 0$ (Q.E.D.)

2. Resonant band algorithm

Thm. (Y.) Suppose $\mathcal{A}$ is not a pencil. If $\exists H_i \in \mathcal{A}$ s.t.



There are exactly two higher mult. pt. on H_i,

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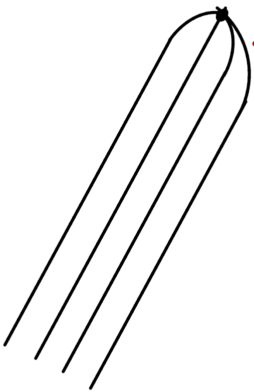

There are exactly two higher mult. pt. on H_i, and $H^1(F)_{\neq 1} \neq 0$, then

$\Longrightarrow$

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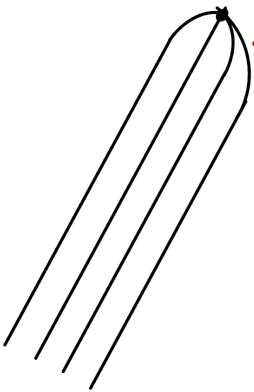
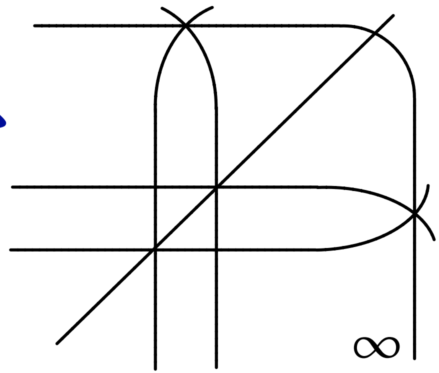


There are exactly two higher mult. pt. on H_i, and $H^1(F)_{\neq 1} \neq 0$, then

$$\Longrightarrow$$

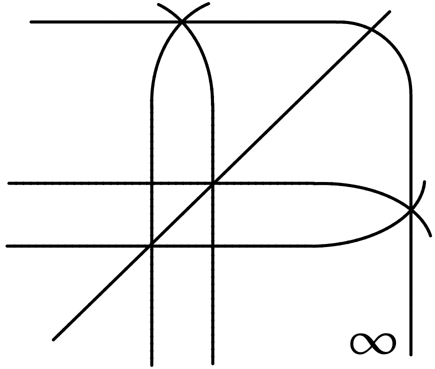
$\mathcal{A}$ is A_3-arr

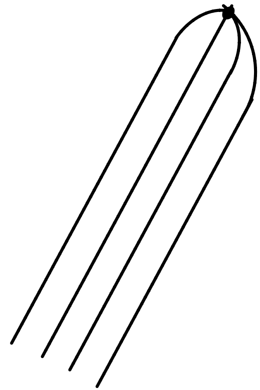


proof: Laborious combinatorial arguments.

2. Resonant band algorithm

<u>Cor.</u> Except for A_3-arr. and pencils,





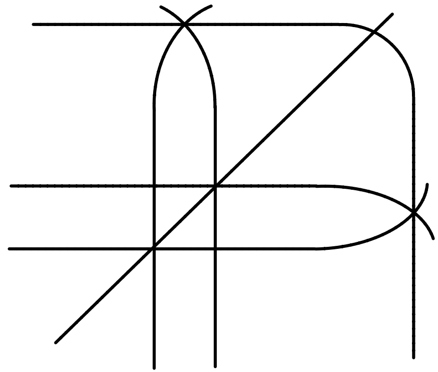
If $\exists$ line $H_i \in \mathcal{A}$

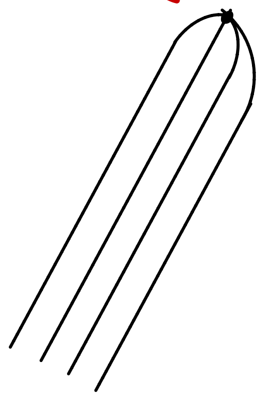
s.t. There are at most two points of multiplicity ≥ 3 on H_i,

then $H^1(F)_{\neq 1} = 0$.

2. Resonant band algorithm

Cor. Except for A_3-arr. and pencils,





$H^1(F)_{\neq 1} \neq 0$ implies that each line $H \in c\mathcal{A}$ has at least 3 multiple intersections on it.

(Roughly, $H^1(F)_{\neq 1} \neq 0$ implies $c\mathcal{A}$ is far from generic.)

2. Resonant band algorithm

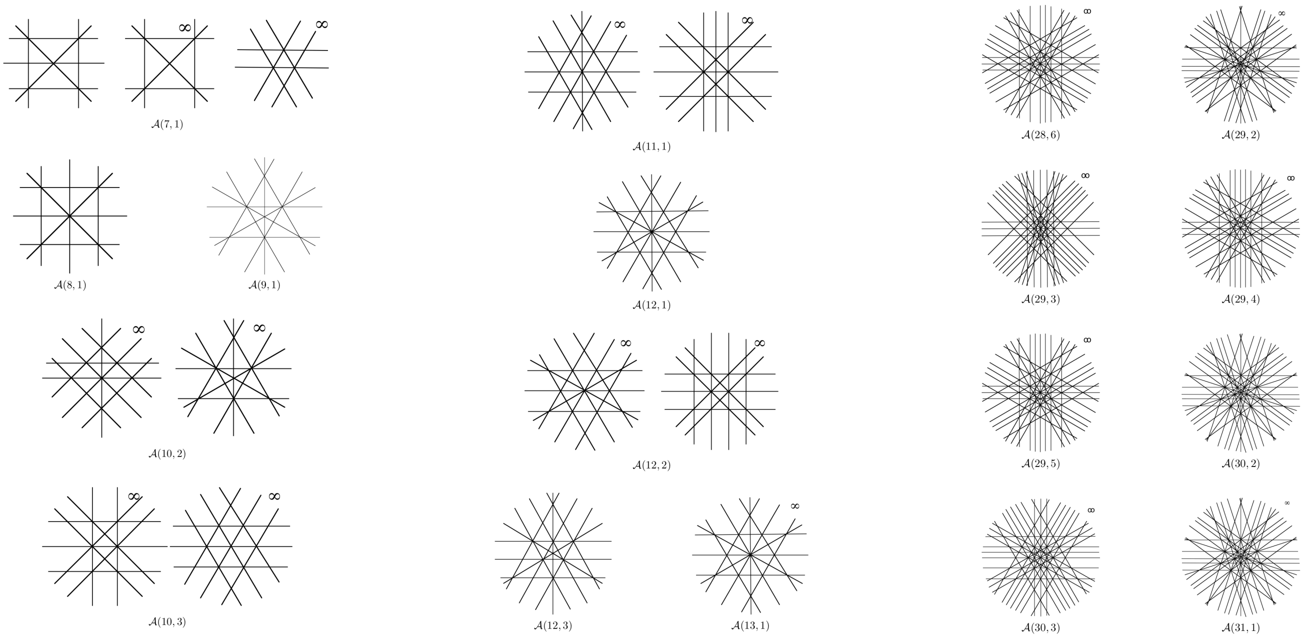
We have computed non trivial eigen space $H^1(F)_{\neq 1}$ for arrangements which are found in

- B. Grünbaum, A catalogue of simplicial arrangements in the real projective plane.
  (and some additional information by Michael Cuntz)

- B. Grünbaum, Configurations of points and lines.
  (Book, AMS)

2. Resonant band algorithm

<u>Def.</u> $\mathcal{A}$ is called simplicial if it gives a triangulation of $\mathbb{RP}^2$.



From Grunbaum's paper "A catalogue of simplicial arrangements in the projective plane"

2. Resonant band algorithm

<u>Def.</u> $\mathcal{A}$ is called simplicial if it gives a triangulation of $\mathbb{RP}^2$.

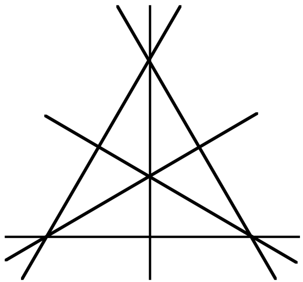
Notes:

- There are some infinite series.
- It is generally believed that there exists only finitely many sporadic simplicial arr's. (Not yet classified.)
- $M(\mathcal{A})$, $F_{\mathcal{A}}$ are $K(\pi, 1)$-spaces. (Deligne)
- Theory of free arrangements started with these examples. (Terao.)

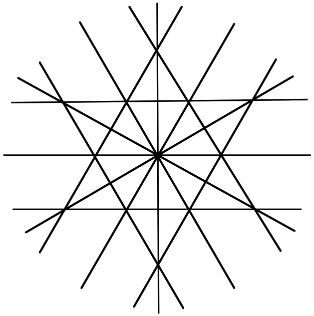
2. Resonant band algorithm

<u>Def.</u> $\mathcal{A}$ is called simplicial if it gives a triangulation of $\mathbb{RP}^2$.

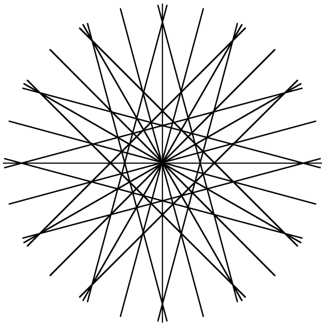
<u>Def.</u> For $m \geq 1$, $A(6m,1) =$ 3m edges of regular 3m-gon + 3m lines of symmetry.



A(6,1)



$\mathcal{A}(12,1)$



$\mathcal{A}(24,1)$

2. Resonant band algorithm

<u>Conjecture 1.</u> For real line arrangements (except for pencils)

(i) $\dim H^1(F)_\lambda \leqq 1$ if $\lambda \neq 1$.

(ii) $H^1(F)_\lambda = 0$ if $\lambda^3 \neq 1$.

<u>Conjecture 2.</u> Suppose $\mathcal{A}$ is simplicial. Then

$H^1(F)_{\neq 1} \neq 0$ iff $\mathcal{A}$ is of type $A(6m,1)$.

(Conj 2 is verified for all known examples. In particular, true for $|\mathcal{A}| \leq 28$.)

3. Milnor fiber and 3-mutinets

3. Milnor fiber and 3-mutinets

<u>Def.</u> A projective line arrangement $\mathcal{A}$ is said to have 3-multinets structure (of multiplicity 1) if $\exists$ decomposition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3$ s.t.

(i) $|\mathcal{A}_1| = |\mathcal{A}_2| = |\mathcal{A}_3|$, and

(ii) For any intersection p, $(\mathcal{A})_p = \{H \in \mathcal{A} \mid p \in H\}$ is either

- consisting one color, or
- $|(\mathcal{A}_1)_p| = |(\mathcal{A}_2)_p| = |(\mathcal{A}_3)_p|$.

<u>Rem.</u> Having a 3-multinet str. is a combinatorial property.

3. Milnor fiber and 3-mutinets

(i) $|\mathcal{A}_1| = |\mathcal{A}_2| = |\mathcal{A}_3|$, and

(ii) For any intersection p, $(c\mathcal{A})_p = \{H \in c\mathcal{A} \mid p \in H\}$ is either

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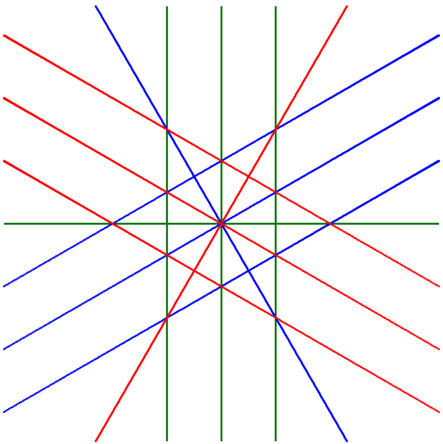
Ex. $\mathcal{A}(12,1)$.



3. Milnor fiber and 3-mutinets

<u>Thm.</u> (Libgober, again for any complex arr. $\mathcal{A}$)

Suppose $\mathcal{A}$ has only double and triple points.

Then $H^1(F)_{\neq 1} \neq 0$ iff $\mathcal{A}$ has 3-multinet str.

(and then nontrivial eigen values are $e^{\pm 2\pi i/3}$)

<u>Rem.</u> (0) "if" part has been known (Falk-Yuzvinsky).

(1) Hence for double and triple points arrangements,

$H^1(F)_{\neq 1} = 0$ or $\neq 0$ is combinatorial.

(2) An example of $\mathcal{A}$ with no 3-multinet str. and

$H^1(F)_{\sqrt{-1}} \neq 0$ is known. (defined $/\mathbb{C}$)

3. Milnor fiber and 3-mutinets

If we restrict to the real cases, all examples (from Grünbaum's articles) supports the following.

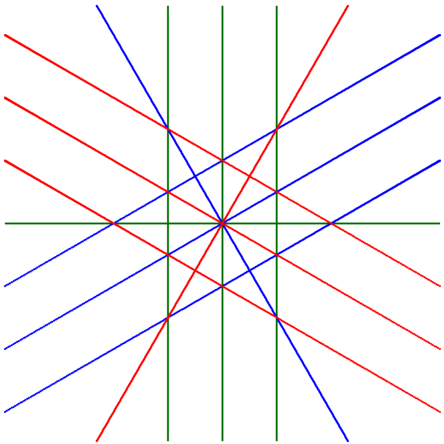
Conjecture 3. For any real arrangement $\mathcal{A}$, $H^1(F)_{\neq 1} \neq 0$ iff $\mathcal{A}$ has 3-multinet str.

(i.e. Libgober's resut holds for all real arr's without the assumption "at most triple points".)

3. Milnor fiber and 3-mutinets

<u>Conjecture 4</u>. Suppose $\mathcal{A}$ is a simplicial arrangement. Then $\mathcal{A}$ has 3-multinet iff $\mathcal{A} = A(6m, 1)$ for some $m \geq 1$.

(This is purely combinatorial problem)



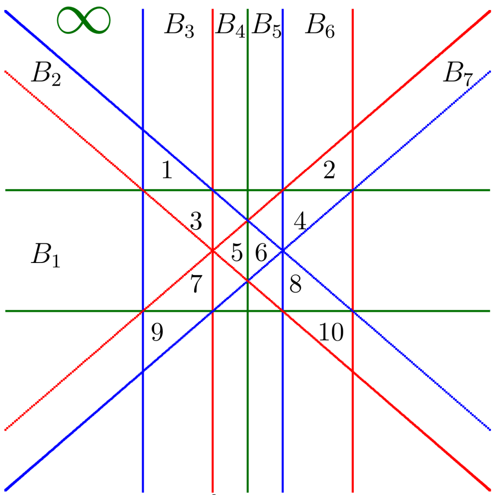
$A(12,1)$

3. Milnor fiber and 3-mutinets

(Work in progress with Michele Torielli)

Towards "Conj.3 : $H^1(F)_{\neq 1} \neq 0 \iff$ 3-nets".

Observation:



A(12,1)

In A(12,1) (left), unique lin. rel. $(k=3)$

$$\nabla(B_2)-\nabla(B_3)+\nabla(B_6)-\nabla(B_7)=0$$

Only the bands with blue and red walls appear.

(Bands with green walls do not appear.)

↑ ∞.

$RB_3=\{B_1,B_2,\cdots,B_7\}$

4. Idea of the proof.

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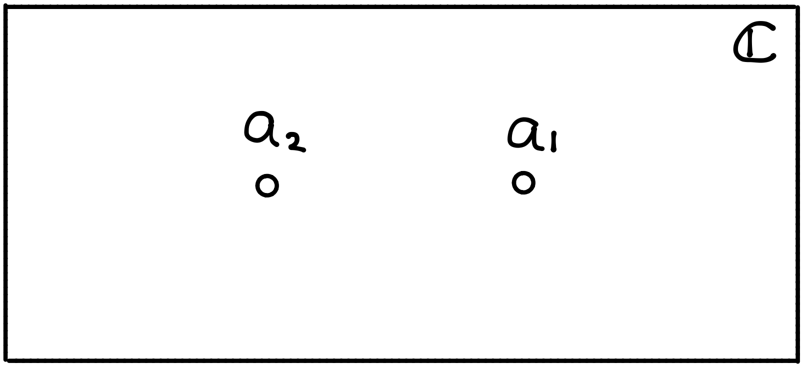
— Construct a good stratification for $M(\mathcal{A})$.
("minimal stratification")

— Computing $H^1(M(\mathcal{A}), \mathcal{L}_\lambda)$ by using the stratification.

4. Idea of the proof.

1-dimensional example: $A = \{a_1, a_2\} \subset \mathbb{R}$.



Semi-algebraic subset defined by "ratio of consective linear forms $\in \mathbb{R}_{<0}$"

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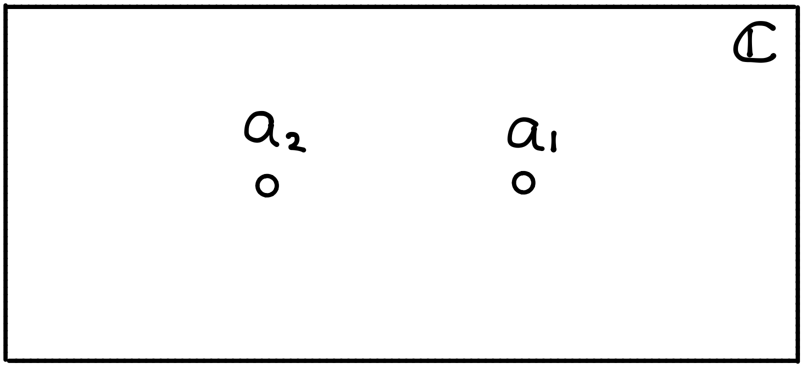

Semi-algebraic subset defined by "ratio of consective linear forms $\in \mathbb{R}_{<0}$"

$$S_1 = \left\{ z \in \mathbb{C} \;\middle|\; \frac{z - a_1}{z - a_2} \in \mathbb{R}_{<0} \right\}$$

$$S_2 = \left\{ z \in \mathbb{C} \;\middle|\; \frac{z - a_2}{1} \in \mathbb{R}_{<0} \right\}$$

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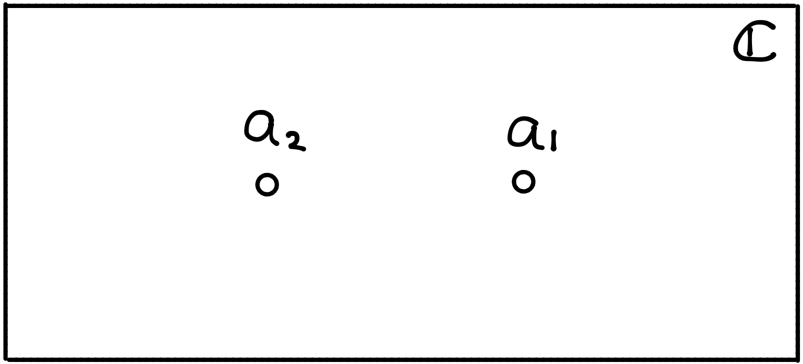

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"1" can be considered as defining linear form of "$-\infty$"

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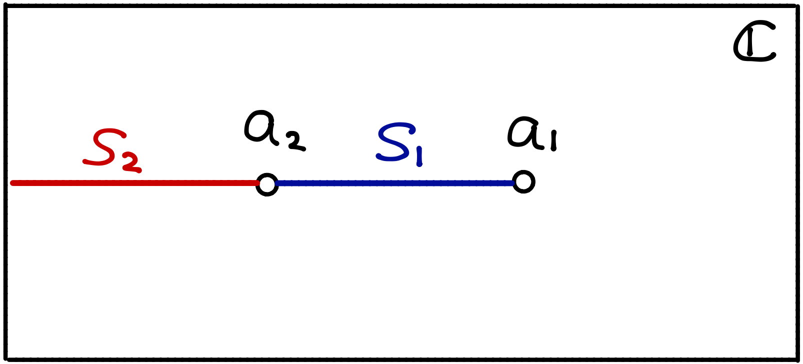

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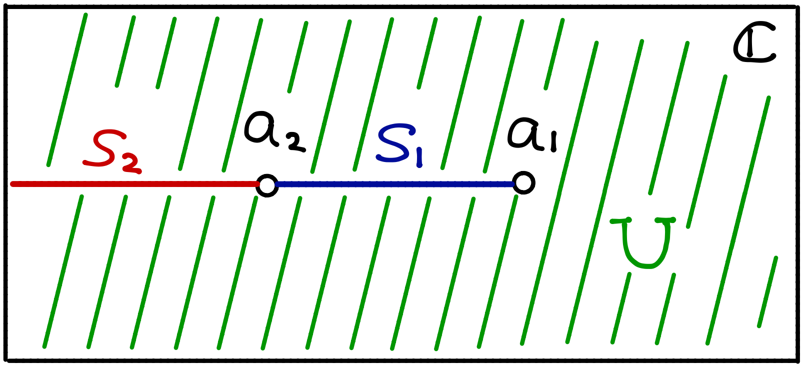

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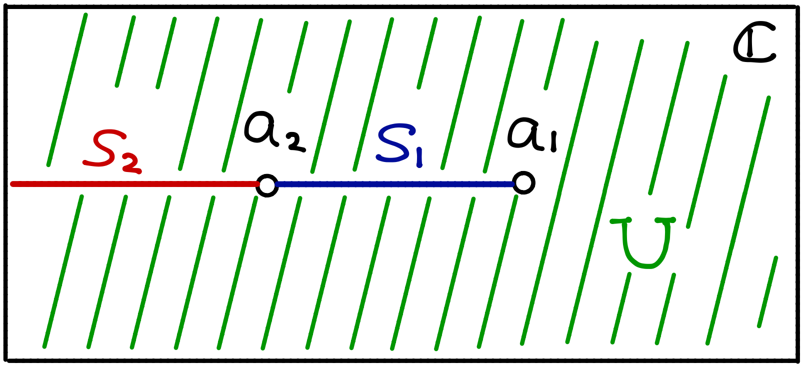
$U := M(A) \setminus S_1 \sqcup S_2$.

4. Idea of the proof.

1-dimensional example: $\mathcal{A} = \{a_1, a_2\} \subset \mathbb{R}$.



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"1" can be considered as defining linear form of "$-\infty$"

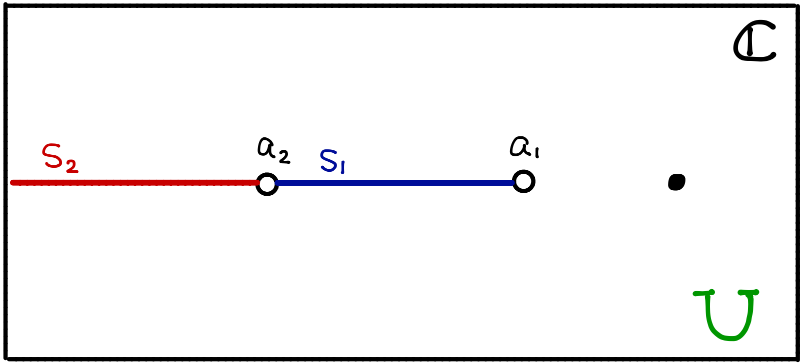
$U := M(\mathcal{A}) \setminus S_1 \sqcup S_2$.

$M(\mathcal{A}) = U \sqcup S_1 \sqcup S_2$

Partition by contractible pieces!!

4. Idea of the proof.

Consider the local system $\mathcal{L}_\lambda$ (monodromy around a_i is $\lambda \in \mathbb{C}^*$)



$M(A) = U \sqcup S_1 \sqcup S_2$

Set $\lambda = e^{2\pi i/k}$.

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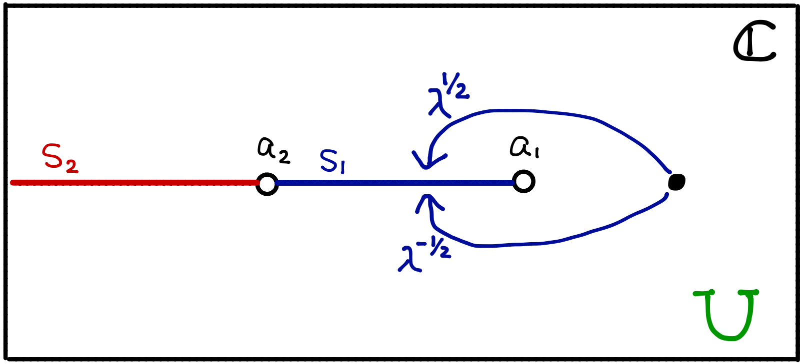

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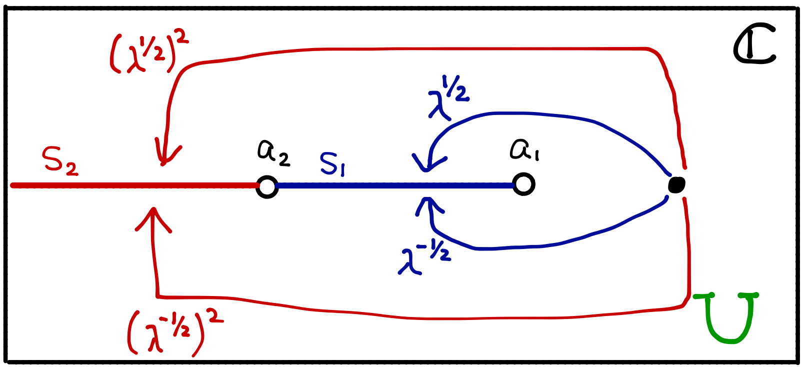

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Twisted boundary map:

$$\nabla([U]) = (\lambda^{1/2} - \lambda^{-1/2})[S_1] + ((\lambda^{1/2})^2 - (\lambda^{-1/2})^2)[S_2]$$

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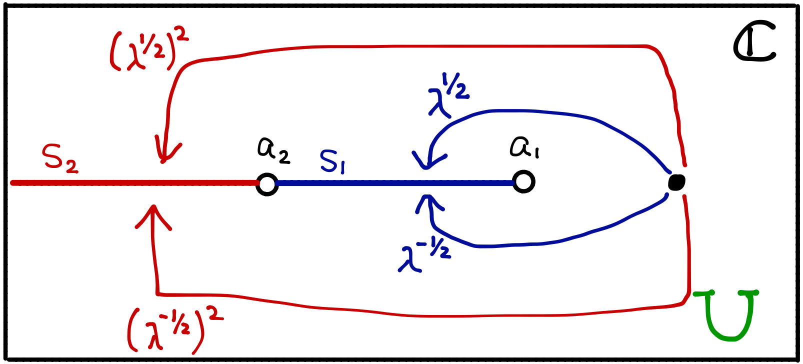

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Twisted boundary map:

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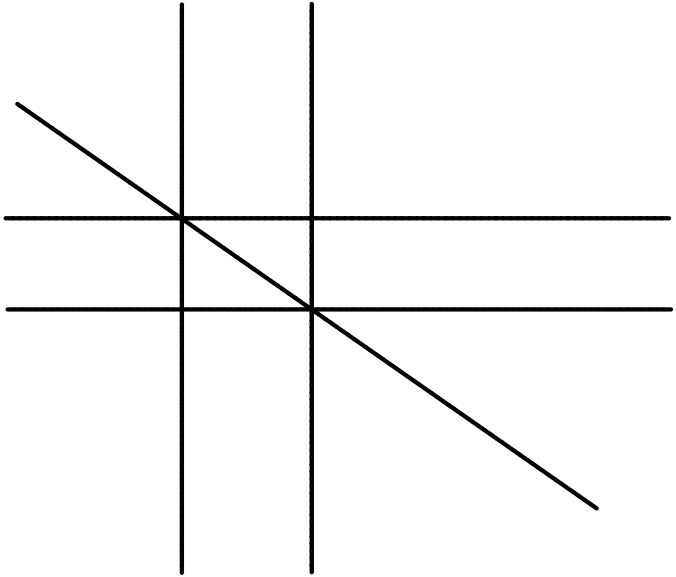
$$= 2\cdot\sqrt{-1}\cdot\left\{ \sin\left(\frac{\pi}{k}\cdot 1\right)\cdot[S_1] + \sin\left(\frac{\pi}{k}\cdot 2\right)\cdot[S_2] \right\}$$

Similar construction works for 2-dim case.

4. Idea of the proof.

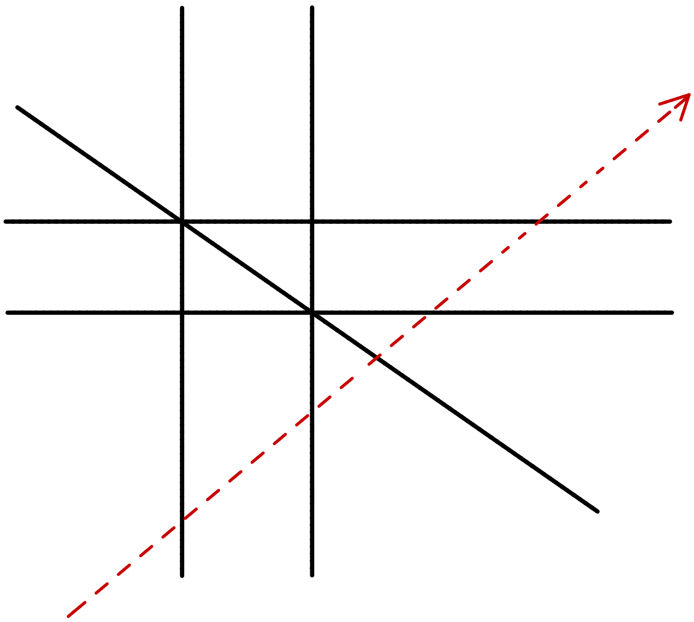
Key Construction



4. Idea of the proof.

Key Construction

(i) Fix a generic line (w. ori.)

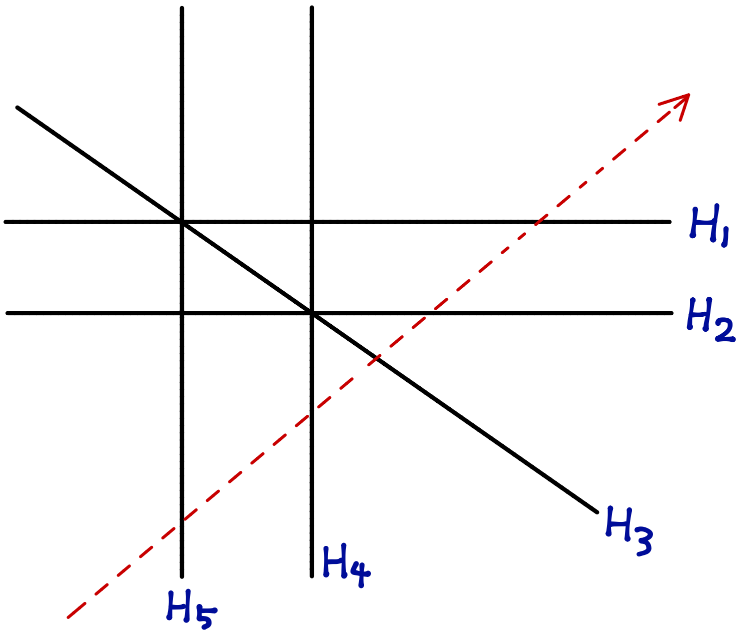


4. Idea of the proof.

$\boxed{\text{Key Construction}}$

(i) Fix a generic line (w. ori.)

(ii) Choose numbering accordingly



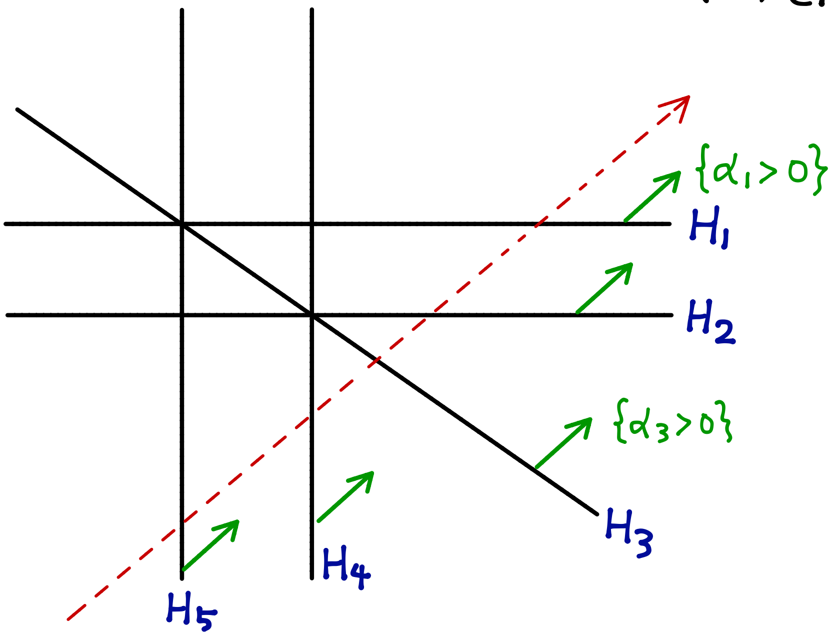
4. Idea of the proof.

$\boxed{\text{Key Construction}}$

(i) Fix a generic line (w. ori.)

(ii) Choose numbering accordingly.

(iii) Choose the sign of the defining linear eq. α_i as in the figure.



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Def.

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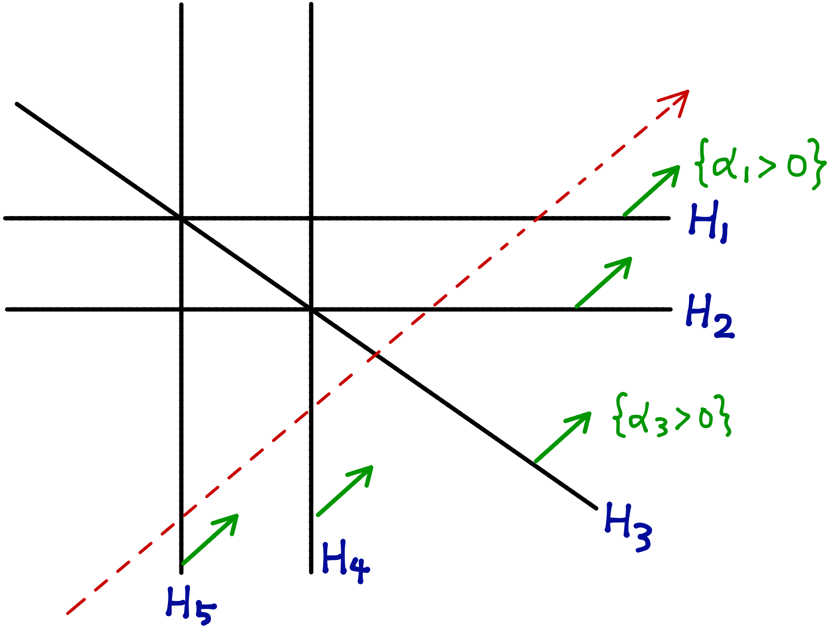

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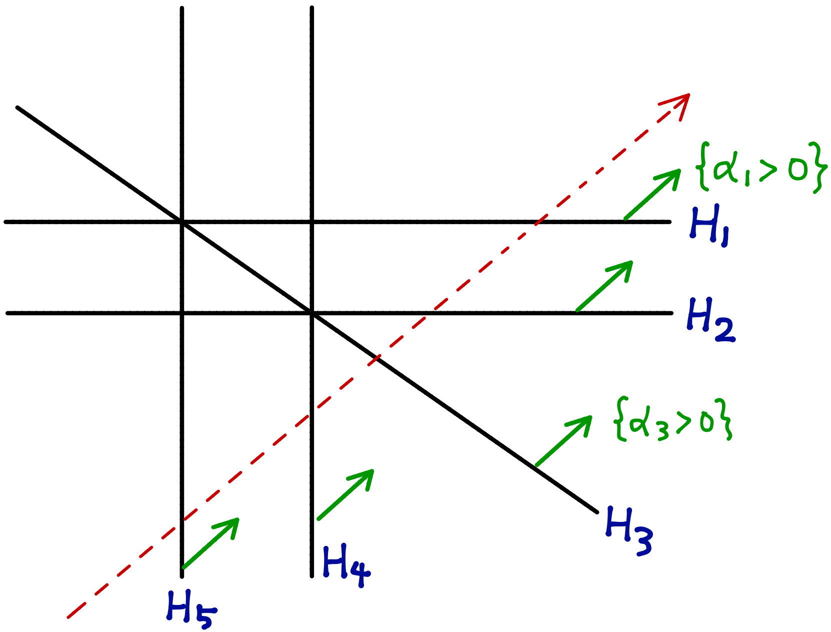

Def.

$$S_i := \left\{ z \in M(A) \,\middle|\, \frac{\alpha_i(z)}{\alpha_{i+1}(z)} \in \mathbb{R}_{<0} \right\}$$

(We set $\alpha_{n+1} := 1$)

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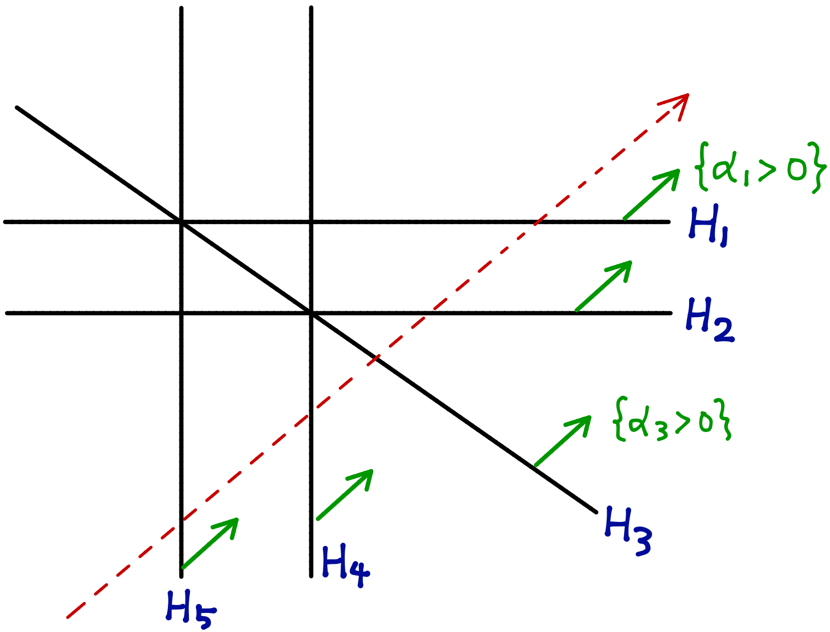

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Intersecting nicely, among others.

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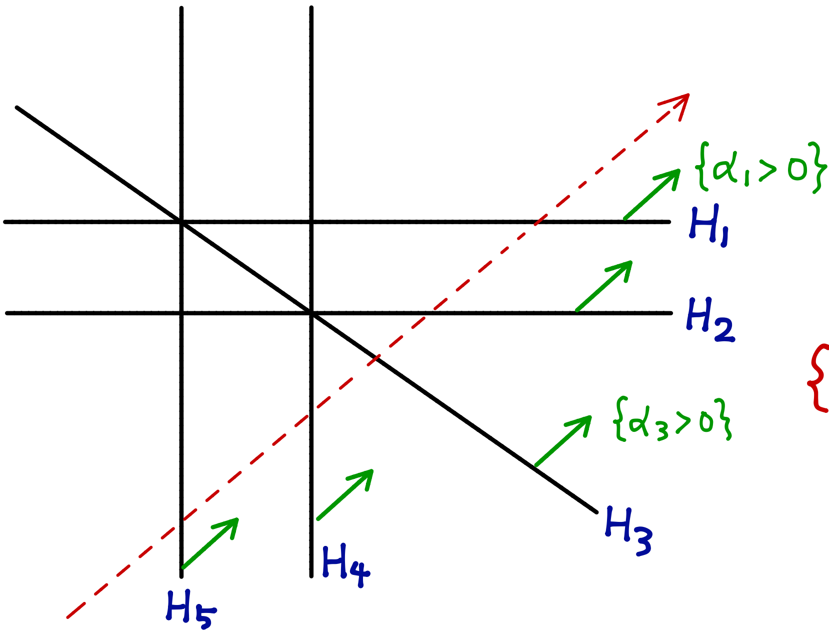

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$\{S_i\}$ induces a partition of $M(A)$ into contractible pieces.

4. Idea of the proof.

Thm. (Y.)

$\{S_i\}$ induces a partition of $M(\mathcal{A})$ into contractible pieces.

- Compute $H^1(M(\mathcal{A}), \mathcal{L}_\lambda)$ using this partition.
- Simplify the cochain complex (purely algebraically).

$\downarrow$

Cor. $H^1(F)_\lambda = \mathrm{Ker}\left(\nabla : \mathbb{C}[\mathrm{RB}_k(\mathcal{A})] \longrightarrow \mathbb{C}[\mathrm{ch}(\mathcal{A})]\right)$.

Rem. The above partition is a "dual" description of Dimca-Papadima's minimal structure.

Summary

- Some topological problems on $\mathcal{A}$ are interpreted by using the notion of "resonant bands".
- The study of minimal structure is crucial.

Reference:

Y. Milnor fibers of real line arrangements. J. of Singularities. Vol. 7 (2013) 220–237.

Y. Minimal stratification for line arrangements and⋯