NCTS (South) Geometry Conference
Mathematics New Goals
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Milnor fibers and semi-algebraic stratification of real line arrangements

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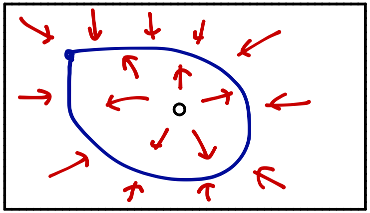
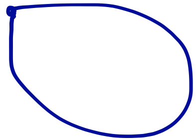
Contents

1. Cell decompostion of affine varieties

Examples of cell decompositions:

- $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \{pt\}$: CW-decomposition.

- $\mathbb{C}^* =$  $\approx$  : Homotopic to a 1-$\dim_{\mathbb{R}}$ CW-cpx.

  ($\dim_{\mathbb{R}} \mathbb{C}^* = 2.$)

More generally, every complex smooth affine variety (i.e. closed subvariety of $\mathbb{C}^N$ e.g. $\mathbb{C}^* \cong \{xy=1\}$) is homotopic to a finite CW-complex of half dimension.

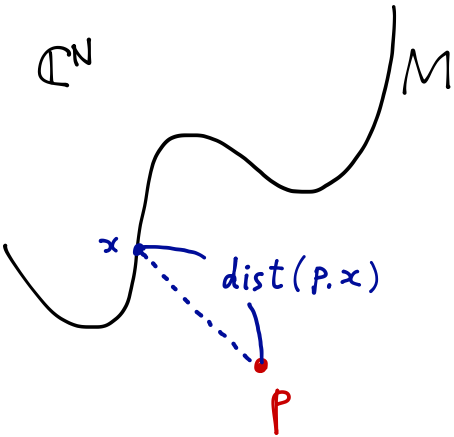
1. Cell decompostion of affine varieties

Theorem (Lefschetz?)

Let M be a smooth affine variety $(/\mathbb{C})$ of $\dim_{\mathbb{C}} M = n$.

Then M is homotopic to a finite CW cpx of $\dim \leqq n$.

(Proof) Suppose $M \subset \mathbb{C}^N$: closed holomorphic embedding.

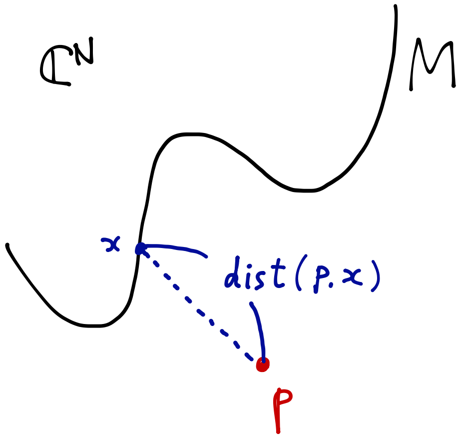


Choose $p \in \mathbb{C}^N \setminus M$ generically, and consider the distance function:

$$f : M \longrightarrow \mathbb{R}_{\geq 0}, \quad x \longmapsto \mathrm{dist}(p, x)$$

Then prove that f is Morse with Morse index $\leq n$. (Q.E.D.)

1. Cell decompostion of affine varieties



Morse function

$$\begin{array}{ccc} f : M & \longrightarrow & \mathbb{R}_{\geq 0} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & \mathrm{dist}(p, x) \end{array}$$

$\rightsquigarrow$ Existence of cell decomposition.

However, the cells obtained from the Morse function is

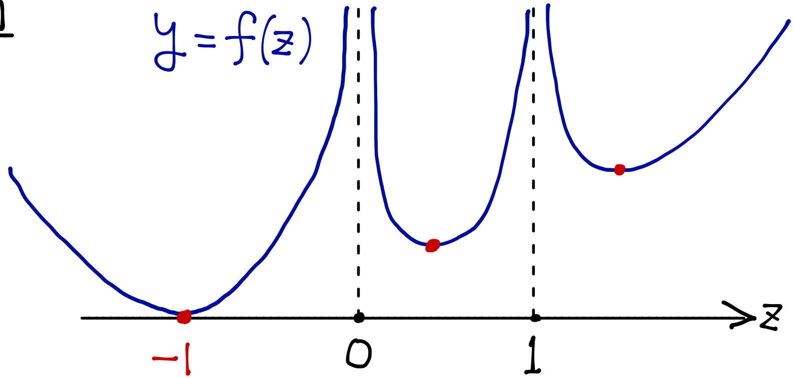
HIGHLY TRANSCENDENTAL!

1. Cell decompostion of affine varieties

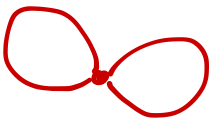
An Example of Morse function

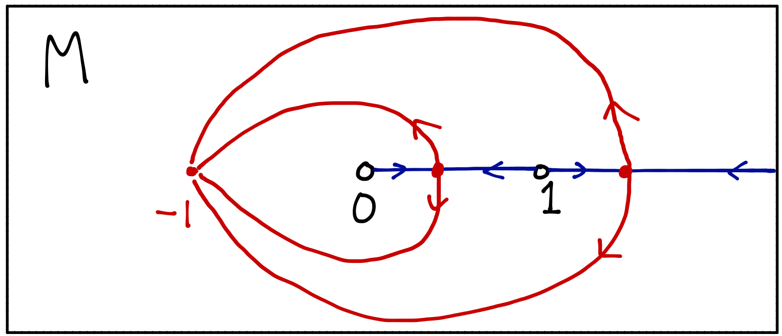
Let $M := \mathbb{C} \setminus \{0, 1\}$.

$$f : M \longrightarrow \mathbb{R}_{\geq 0}$$
$$z \longmapsto \left| \frac{(z+1)^2}{\sqrt{z(z-1)}} \right|$$



The grad flow $-\mathrm{grad}(f)$ gives a homotopy equiv.

$M \approx$ 



1. Cell decompostion of affine varieties

An Example of Morse function

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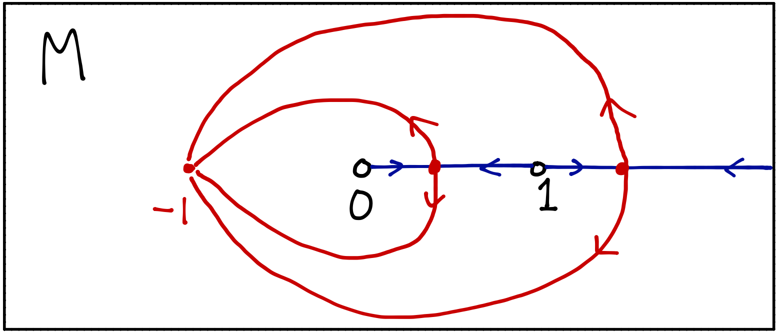


Observations.

- The unstable cells are transcendental.
  But, the stable cells are "Semi-algebraic".

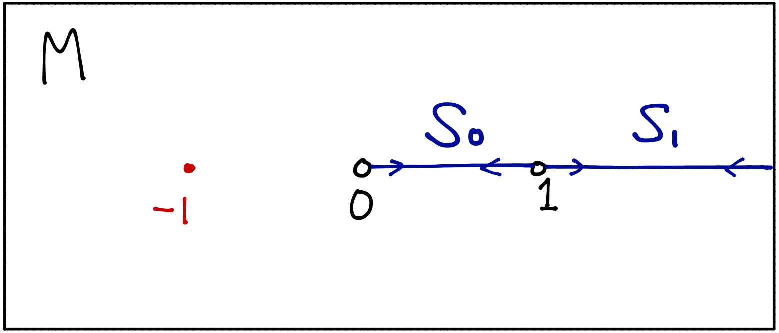
1. Cell decompostion of affine varieties

An Example of Morse function

Observation

- The unstable cells are transcendental.

But, the stable cells are "semi-algebraic"



- Furthermore, stable cells have the following presentation:

$$S_0 = (0,1) = \left\{ z \in M \;\middle|\; \frac{z-1}{z} \in \mathbb{R}_{<0} \right\}$$

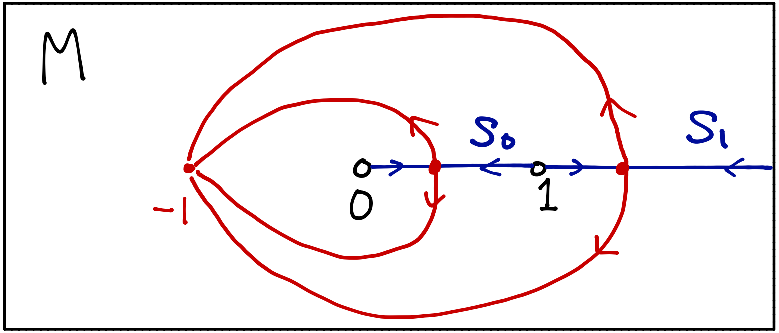
open interval

"ratio of linear forms"

$$S_1 = (1,+\infty) = \left\{ z \in M \;\middle|\; \frac{z-1}{-1} \in \mathbb{R}_{<0} \right\}$$

defining linear form of $+\infty$ (?)

1. Cell decompostion of affine varieties



$U := M \setminus S_0 \cup S_1$

CW-complex

stratification (or Partition)

$M = U \sqcup S_0 \sqcup S_1$

One 0-cell	dual	One 0-codim stratum
Two 1-cells	description	Two 1-codim strata
Cell decomp.		Strata are contractible
Transcendental (?)		Semi-algebraic.

1. Cell decompostion of affine varieties

General setting $f_1, f_2, \cdots, f_k \in \mathbb{C}[z_1, z_2, \cdots, z_n]$

$M := \mathbb{C}^n \setminus \{ f_1 f_2 \cdots f_k = 0 \}$ complement of hypersurfaces.

Question. Can one describe a good stratification semi-algebraically? ("dual" to the CW cpx.)

Partial answer

$$\text{If} \left.\begin{array}{l} (1)\ n=2 \\ (2)\ \deg f_i = 1 \\ (3)\ f_i \in \mathbb{R}[z_1, z_2] \end{array}\right\} \Longrightarrow$$

Then the above idea works. (We can construct a good semi-algebraic stratification.)

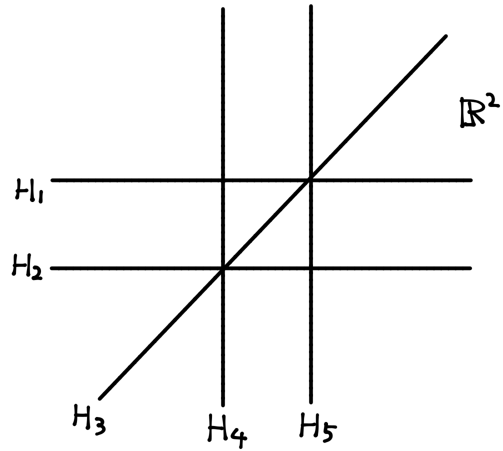
2. Minimal Stratification

<u>Notations</u> Let $\mathcal{A}$ be a real line arrangement, i.e.

$\mathcal{A}=\{H_1, H_2, \cdots, H_n\}$, $H_i \subset \mathbb{R}^2$: a line.

$H_i = \{\alpha_i = 0\}$, α_i : defining equation.

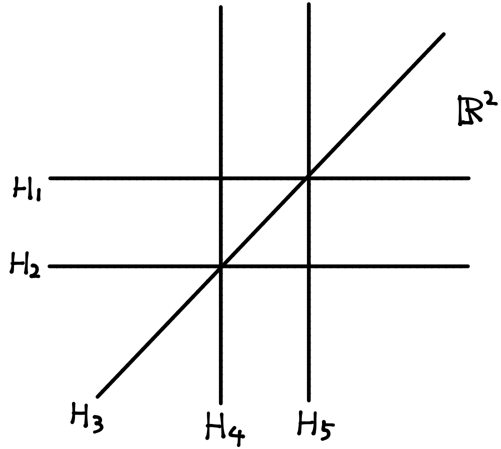
$(\alpha_i \in \mathbb{R}[z_1, z_2],\ \deg \alpha_i = 1)$



$$M = M(\mathcal{A}) := \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i \otimes \mathbb{C}$$

: the complexified complement.

2. Minimal Stratification



Notations:

$\mathcal{A} = \{H_1, H_2, \cdots, H_n\}$, $H_i = \{\alpha_i = 0\}$

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$M = M(\mathcal{A}) := \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i \otimes \mathbb{C}$

Consider the subset defined by

$$\left\{ (z_1, z_2) \in M(\mathcal{A}) \;\middle|\; \frac{\alpha_i(z)}{\alpha_j(z)} \in \mathbb{R}_{<0} \right\} \subset M(\mathcal{A}).$$

2. Minimal Stratification

Consider the subset defined by

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Ratio of linear forms:

$$\frac{z_1}{z_2} : (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^*, \quad (z_1, z_2) \longmapsto \frac{z_1}{z_2}$$

is a trivial fibration with fiber $\cong \mathbb{C}^*$

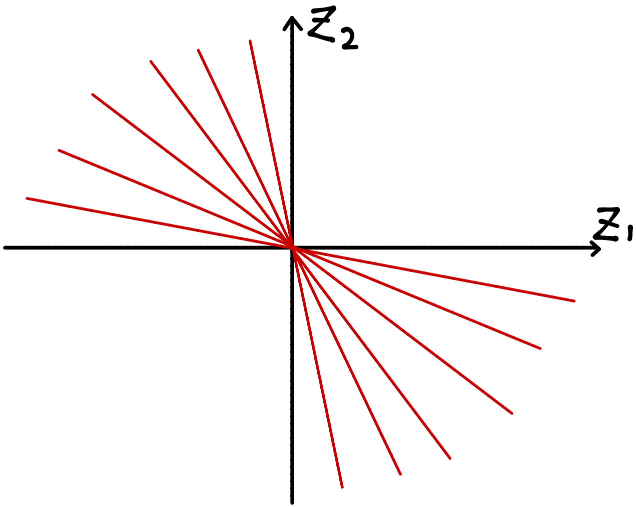
Therefore, $\left(\frac{z_1}{z_2}\right)^{-1} \mathbb{R}_{<0} \cong \mathbb{R}_{<0} \times \mathbb{C}^*$.

2. Minimal Stratification

$$\frac{z_1}{z_2} : (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^*, \quad (z_1, z_2) \longmapsto \frac{z_1}{z_2}$$

is a trivial fibration with fiber $\cong \mathbb{C}^*$, and

$$\left(\frac{z_1}{z_2}\right)^{-1} \mathbb{R}_{<0} \cong \mathbb{R}_{<0} \times \mathbb{C}^*.$$



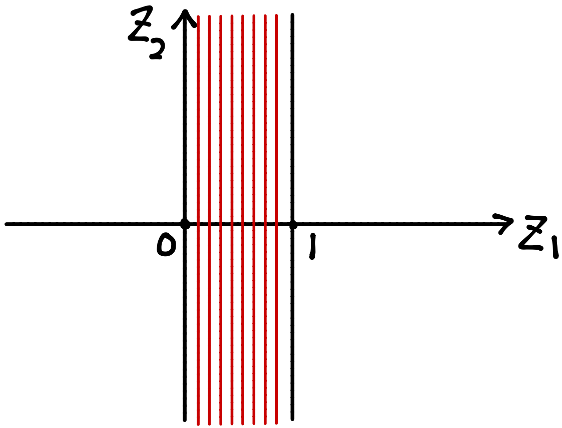
The real part

$$\left(\left(\frac{z_1}{z_2}\right)^{-1} \mathbb{R}_{<0} \right)_{\mathbb{R}} \cong \mathbb{R}_{<0} \times \mathbb{R}^*$$

2. Minimal Stratification

Ratio of linear forms (another case)

$$\frac{z_1-1}{z_1} : (\mathbb{C}\setminus\{0,1\})\times\mathbb{C} \longrightarrow \mathbb{C}\setminus\{0,1\},\ (z_1,z_2)\mapsto \frac{z_1-1}{z_1}$$



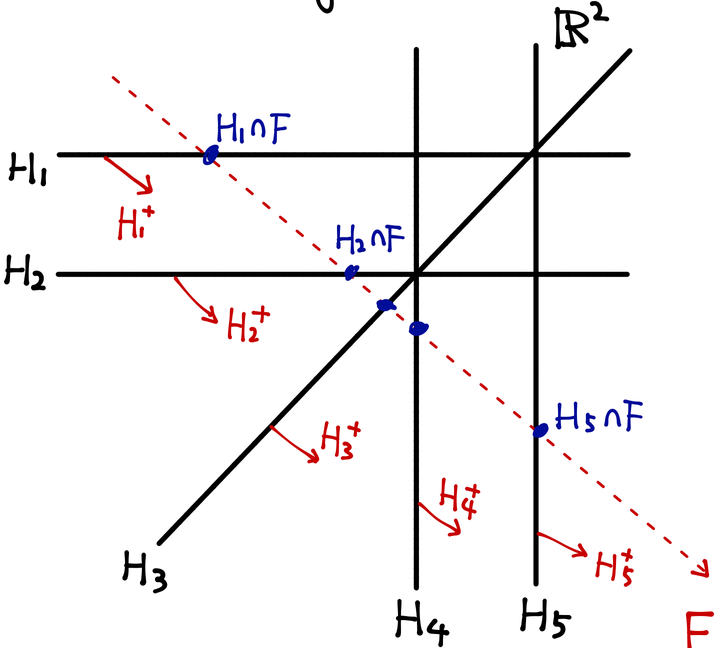
$$\left(\frac{z_1-1}{z_1}\right)^{-1}(\mathbb{R}_{<0}) = (0,1)\times\mathbb{C}.$$

The real part is

$$\left(\left(\frac{z_1-1}{z_1}\right)^{-1}\mathbb{R}_{<0}\right)_{\mathbb{R}} = (0,1)\times\mathbb{R}$$

2. Minimal Stratification

More setting:



F : a generic line (oriented)

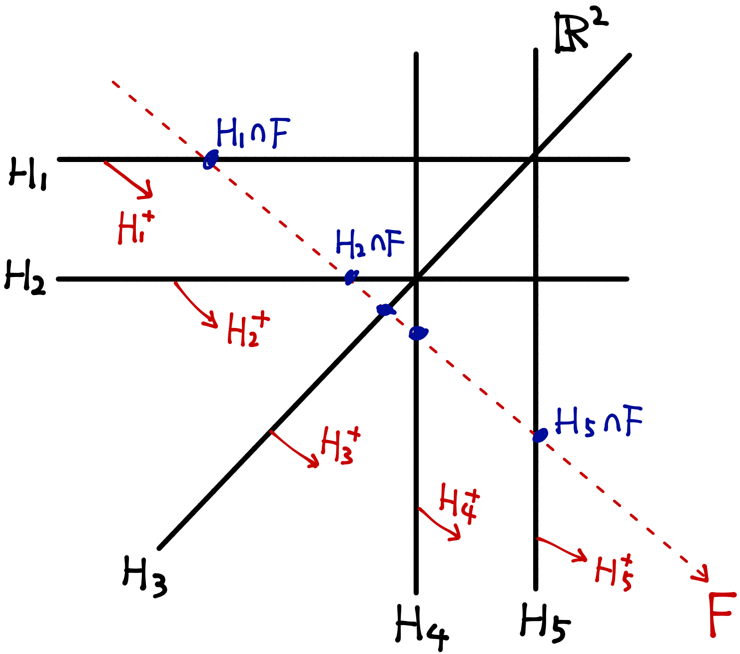
① Fix a generic line F (oriented)

② Re-numbering as

$$H_1 \cap F < H_2 \cap F < \cdots < H_n \cap F$$

③ Fix the sign of α_i so that the half space $H_i^+ := \{\alpha_i > 0\}$ covers positive side of F.

2. Minimal Stratification



Def.

$$S_i := \left\{ z \in M(\mathcal{A}) \;\middle|\; \frac{\alpha_{i+1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\},$$

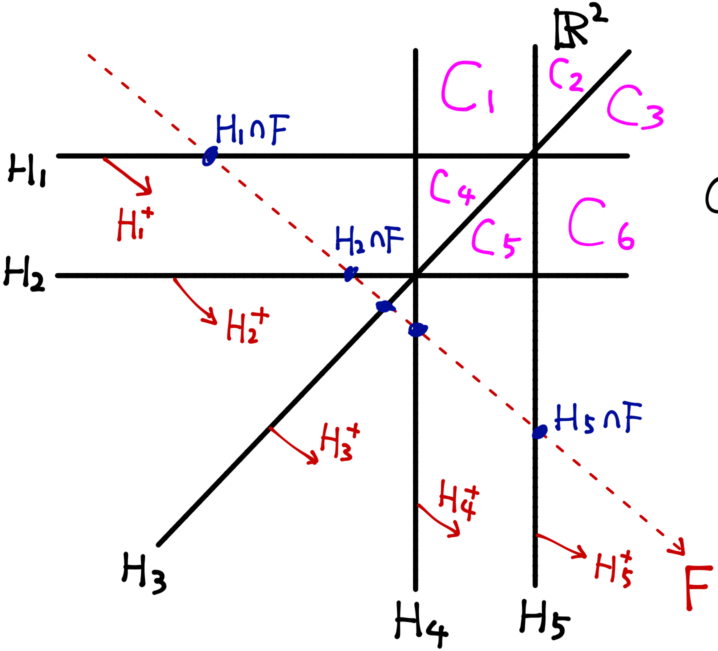
where $\alpha_{n+1} := -1$

Then

- $\dim_{\mathbb{R}} S_i = 3$,
- $S_i \pitchfork S_j$

(hence $\dim_{\mathbb{R}} S_i \cap S_j = 2$)

2. Minimal Stratification



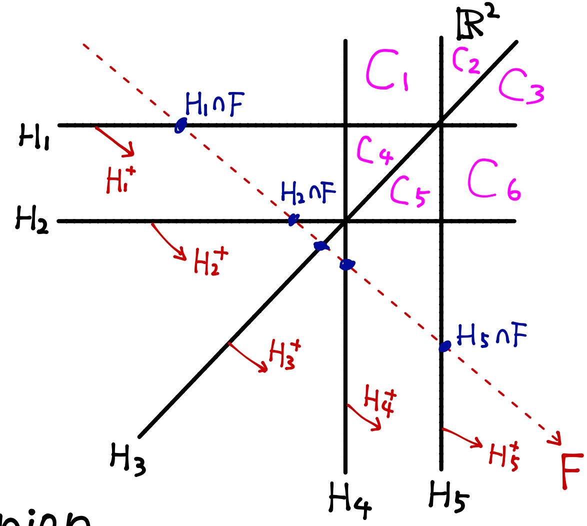
<u>Def.</u>

$$ch_F(\mathcal{A}) := \{ C : \text{chamber} \,;\; C \cap F = \phi \}$$

$$ch_F(\mathcal{A}) = \{ C_1, C_2, C_3, C_4, C_5, C_6 \}$$

<u>Rem.</u> $\# ch_F(\mathcal{A}) = b_2(M(\mathcal{A}))$

2. Minimal Stratification



<u>Def.</u> $\cdot\ S_i := \left\{ z \in M(\mathcal{A}) \;\middle|\; \dfrac{\alpha_{i+1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\}$,

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<u>Theorem</u> (Y. 2012)

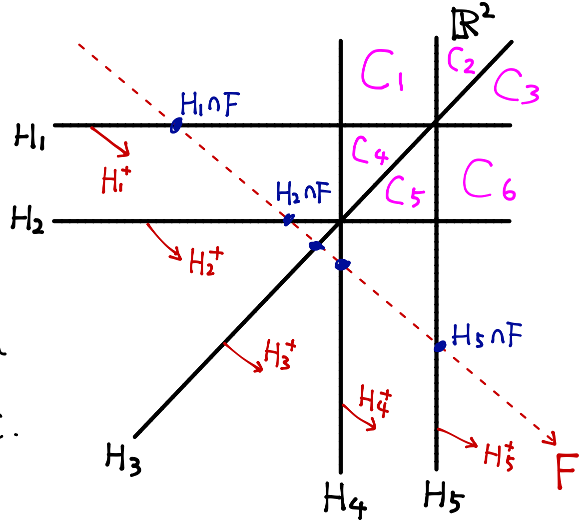
(i) $S_i \pitchfork S_j$, and $S_i \cap S_j$ is a disjoint union of some chambers $C \in ch_F(\mathcal{A})$.

(ii) Denote $S_i^\circ := S_i \setminus \bigcup_{C \in ch_F(\mathcal{A})} C$, $\mho := M(\mathcal{A}) \setminus \bigcup_{i=1}^{n} S_i$. Then

S_i, $\mho$ are contractible, and $M(\mathcal{A}) = \underbrace{\mho}_{\text{codim}=0} \sqcup \underbrace{\bigsqcup_{i=1}^{n} S_i}_{\text{codim}=1} \sqcup \underbrace{\bigsqcup_{C \in ch_F(\mathcal{A})} C}_{\text{codim}=2}$.

(iii) It is minimal. $\left(b_1 = n,\ b_2 = \# ch_F(\mathcal{A}) \right)$

2. Minimal Stratification



Theorem (Y. 2012)

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(ii) Denote $S_i^\circ := S_i \setminus \bigcup_{C \in ch_F(\mathcal{A})} C$, $U := M(\mathcal{A}) \setminus \bigcup_{i=1}^{n} S_i$. Then S_i, U are contractible, and $M(\mathcal{A}) = U \sqcup \bigsqcup_{i=1}^{n} S_i \sqcup \bigsqcup_{C \in ch_F(\mathcal{A})} C$.

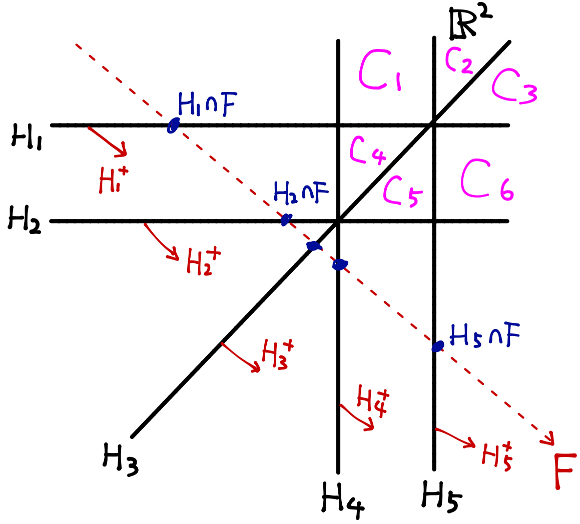
Proof: Elementary.

(Only "contractibility of U" is complicated.)

Conjecture

The same holds for hyperplane arrangements in any dimension.

2. Minimal Stratification



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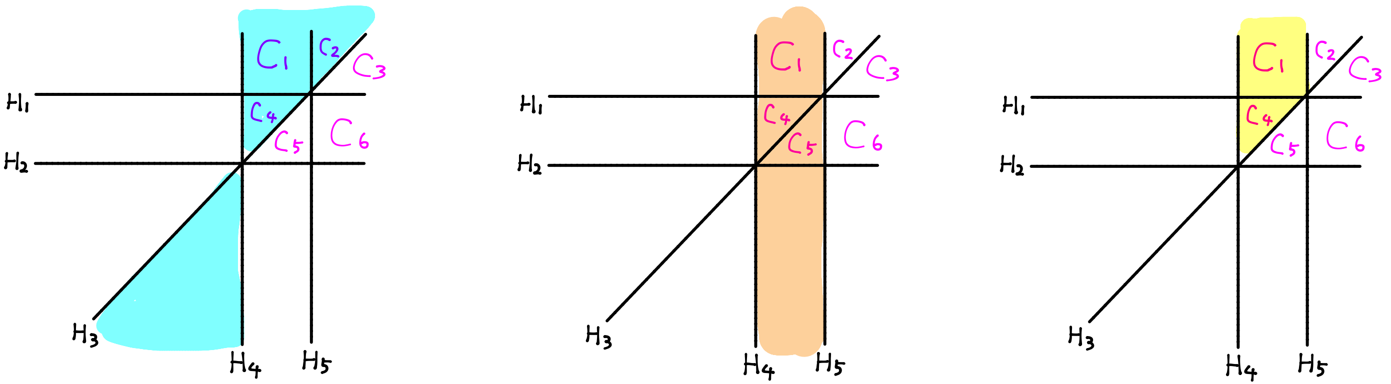
S_i, $\mathcal{U}$ are contractible, and $M(\mathcal{A}) = \mathcal{U} \sqcup \bigsqcup_{i=1}^{n} S_i \sqcup \bigsqcup_{C \in ch_F(\mathcal{A})} C$.

S_i are 3-$\dim_{\mathbb{R}}$ submanifolds which are intersecting in $\mathbb{R}^2$.
So we can deduce topological information from "real picture".

2. Minimal Stratification

An Example:



$S_3 = \left\{ \frac{\alpha_4}{\alpha_3} \in \mathbb{R}_{<0} \right\}$ $\quad$ $S_4 = \left\{ \frac{\alpha_5}{\alpha_4} \in \mathbb{R}_{<0} \right\}$ $\quad$ $S_3 \cap S_4 = C_1 \sqcup C_4$

3. Application to Milnor fibers.

Aim :

$Q(x,y,z) = \prod_{i=0}^{n} \alpha_i(x,y,z)$

A product of homogeneous linear forms

$\rightsquigarrow$

$Q^{-1}(1) \subset \mathbb{C}^3$

The Milnor fiber.

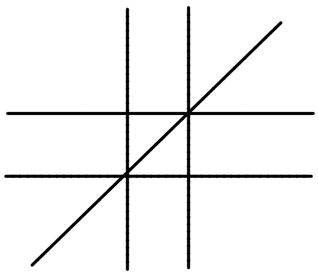
$\wr$

"The real picture"

??

3. Application to Milnor fibers.

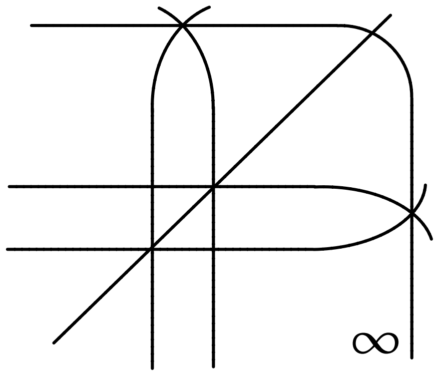
Given datum $\rightsquigarrow$ Its Cone



$x(x-1)y(y-1)(x-y) = 0$

a line arrangement

$\mathcal{A} = \{H_1, \cdots, H_n\}$



$Q := x(x-z)y(y-z)(x-y)\cdot z = 0.$

$c\mathcal{A} := \{H_1, \cdots, H_n, H_\infty\}$

$M(\mathcal{A}) := \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i = \mathbb{P}^2_{\mathbb{C}} \setminus Q^{-1}(0)$

"Milnor Fiber"

$F = F_{\mathcal{A}}$

$:= Q^{-1}(1)$

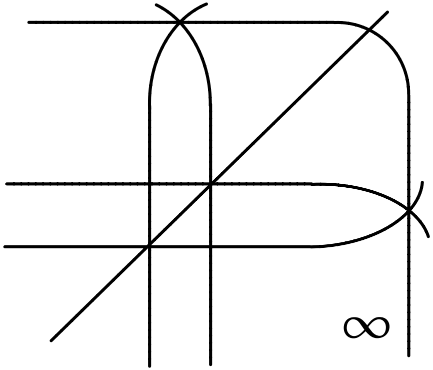
$= \{(x,y,z) \in \mathbb{C}^3 \mid Q(x,y,z) = 1\}$

$\subset \mathbb{C}^3$

$\boxed{\text{Problem: } b_1(F) = ?}$

3. Application to Milnor fibers.

Remark: $Q(x,y,z)$ is homogeneous of $\deg = n+1$.



$$F = \{(x,y,z) \in \mathbb{C}^3 \mid Q(x,y,z) = 1\}$$
$$= Q^{-1}(1)$$

$$Q := x(x-z)y(y-z)(x-y)\cdot z = 0.$$

Monodromy action

$$\begin{array}{cccc} \rho : & F & \longrightarrow & F \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (x,y,z) & \longmapsto & (\zeta x, \zeta y, \zeta z), \end{array}$$

where $\zeta = e^{2\pi i/n+1}$

3. Application to Milnor fibers.

$$\rho : F \longrightarrow F : (x,y,z) \longmapsto (\zeta x, \zeta y, \zeta z),$$

where $\zeta = e^{2\pi i/n+1}$

induces a linear automorphism

$$\rho^* : H^1(F) \longrightarrow H^1(F).$$

We have eigendecomposition

λ-eigen space

$$H^1(F) = \bigoplus_{\lambda^{n+1}=1} H^1(F)_\lambda$$

Since $\rho^{n+1} = id.$

3. Application to Milnor fibers.

Easy part: $\lambda = 1$,

ρ-invariant part

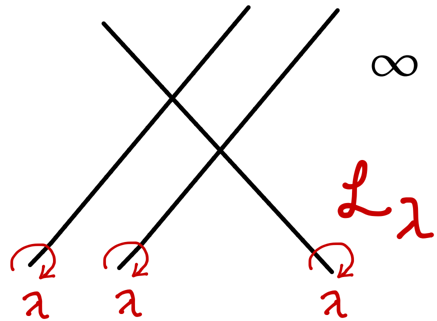
$$H^1(F)_1 = H^1(F)^{\rho} \cong H^1(F/\langle\rho\rangle) \cong H^1(M(\mathcal{A})) \cong \mathbb{C}^n.$$

(Recall: $F/\langle\rho\rangle = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i$)

Nontrivial part: $\lambda \neq 1$

$$H^1(F)_\lambda \cong H^1(M(\mathcal{A}), \mathcal{L}_\lambda)$$

rank 1 local system on $M(\mathcal{A})$, s.t. each monodromy around H_i is λ.

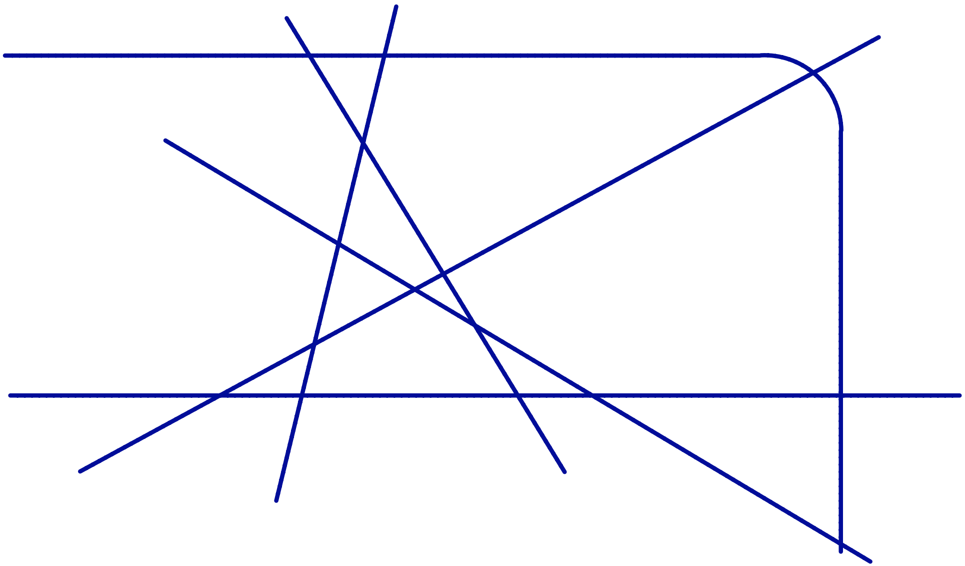


$$H^1(F)_{\neq 1} := \bigoplus_{\lambda \neq 1} H^1(F)_\lambda \quad : \text{non-trivial eigenspace.}$$

3. Application to Milnor fibers.

Known Fact and examples

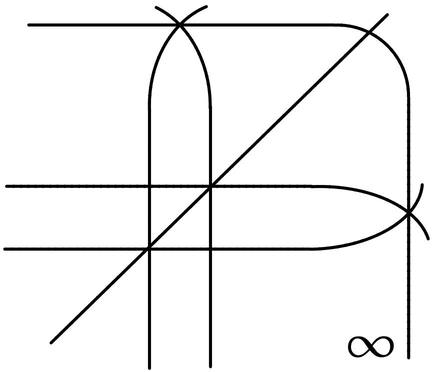
Thm. (Orlik-Randell, Hattori) If $\mathcal{A} = \{H_1, \cdots, H_n, H_\infty\}$ is generic (= only double points), then $H^1(F)_{\neq 1} = 0$. (i.e. $b_1(F) = n$)



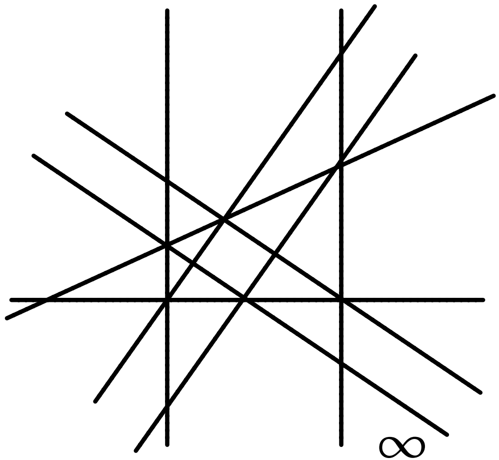
3. Application to Milnor fibers.

Non-generic cases:

$$\zeta = e^{2\pi i/3}$$



A_3-arr.



Pappus arr.

$$H^1(F) = \underset{\mathbb{C}^5}{\underset{\wr\wr}{H^1(F)_1}} \oplus \underset{\mathbb{C}}{\underset{\wr\wr}{H^1(F)_\zeta}} \oplus \underset{\mathbb{C}}{\underset{\wr\wr}{H^1(F)_{\zeta^2}}}$$

$$H^1(F) = \underset{\mathbb{C}^8}{\underset{\wr\wr}{H^1(F)_1}} \oplus \underset{\mathbb{C}}{\underset{\wr\wr}{H^1(F)_\zeta}} \oplus \underset{\mathbb{C}}{\underset{\wr\wr}{H^1(F)_{\zeta^2}}}$$

3. Application to Milnor fibers.

$\boxed{\text{Problem}}$ Compute $\dim H^1(F)_\lambda = \dim H^1(M, \mathcal{L}_\lambda)$

for $\lambda \neq 1$, by using "real structure".

<u>Rem.</u> $\dim H^1(F)_\lambda$ is related to many other things.

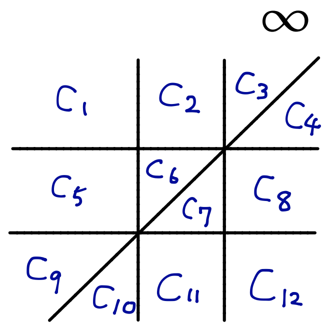
- Betti numbers of certain covering spaces of $M(\mathcal{A})$.
- Alexander polynomial of $\pi_1(M)$.
- Counting certain plane curves.
- Hodge str. of $H^1(F, \mathbb{C})$.

3. Application to Milnor fibers.

Now we assume $\mathcal{A}=\{H_1,\cdots,H_n\}$ is defined $/\mathbb{R}$.

Notation:

$ch(\mathcal{A})$: the set of chambers.



Adjacency distance:

For $C, C' \in ch(\mathcal{A})$,

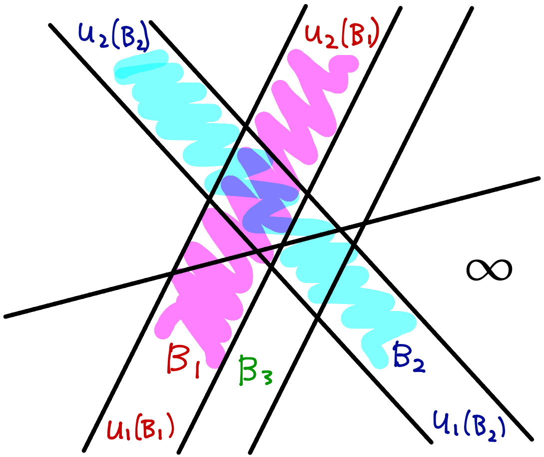
$d(C,C') :=$ # of lines which separates C & C'.

$ch(\mathcal{A}) = \{C_1, \cdots, C_{12}\}$

e.g. $d(C_5, C_{12}) = 4$

3. Application to Milnor fibers.

Fix $\lambda \in \mathbb{C}^*$ $(\lambda \neq 1)$ with order k. ($k > 1$, $k \mid (n+1)$)

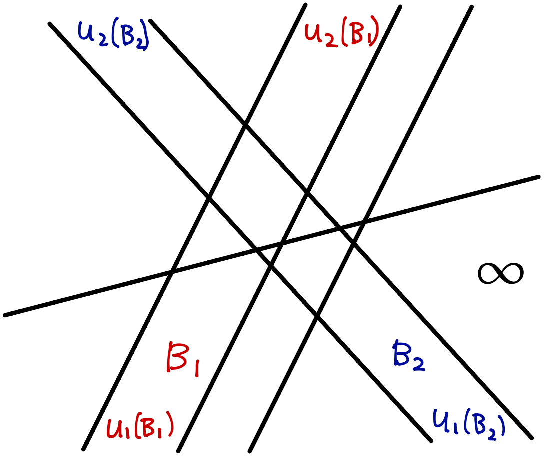


A band is a region bounded by a pair of consecutive parallel lines.

In a band B, there are exactly two unbounded chambers, set $U_1(B)$ and $U_2(B)$.

Next we define k-resonance.

3. Application to Milnor fibers.



A **band** is a region bounded by a pair of consecutive parallel lines.

In a band B, there are exactly two unbounded chambers, set $u_1(B)$ and $u_2(B)$.

A band B is **k-resonant** $\overset{\text{def}}{\iff}$ $k \mid d(u_1(B), u_2(B))$.

E.g. $d(u_1(B_1), u_2(B_1)) = 3 \rightsquigarrow B_1$ is 3-resonant.

$d(u_1(B_2), u_2(B_2)) = 4 \rightsquigarrow B_2$ is not 3-resonant.

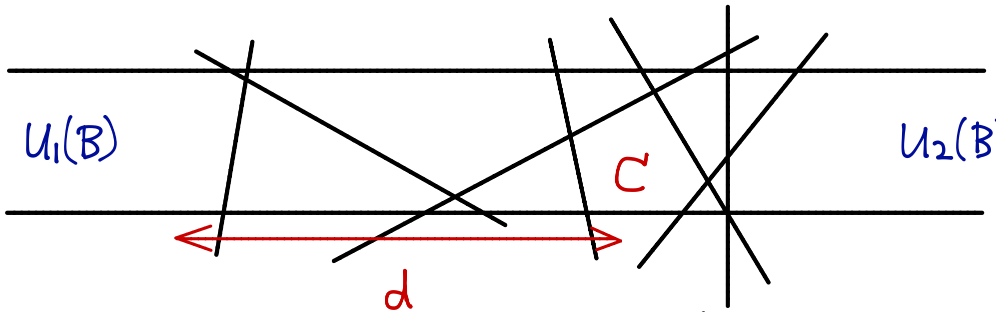
3. Application to Milnor fibers.

Suppose that $\mathcal{B}$ is $\mathbb{k}$-resonant. $\left(\Longleftrightarrow \sin\left(\dfrac{\pi \cdot d(u_1, u_2)}{\mathbb{k}} \right) = 0 \right)$

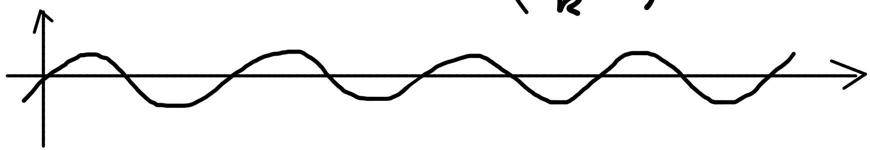
<u>Def.</u>

$$\nabla(\mathcal{B}) := \sum_{C \subseteq \mathcal{B}} \sin\left(\frac{\pi \cdot d(u_1(\mathcal{B}), C)}{\mathbb{k}} \right) \cdot [C] \in \mathbb{C}[\mathrm{ch}(\mathcal{A})]$$

vector space spanned by chambers.



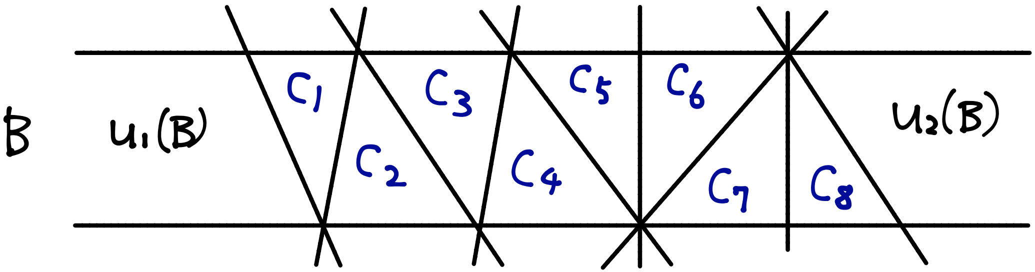
$$\nabla(\mathcal{B}) = \cdots + \sin\left(\frac{\pi}{\mathbb{k}} \cdot d \right) [C] + \cdots$$



3. Application to Milnor fibers.

$$\nabla(B) := \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(u_1(B), C)}{k}\right) \cdot [C] \in \mathbb{C}[ch(\mathcal{A})]$$

Ex. $d(u_1(B), u_2(B)) = 9, \quad k = 3.$



$$\nabla(B) = \sum_{p=1}^{8} \sin\left(\frac{\pi}{3} \cdot p\right) \cdot [C_p]$$

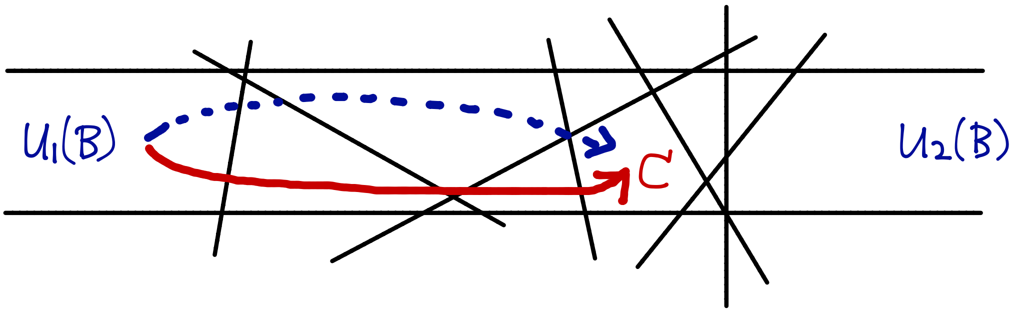
$d(u_1(B), C_p) = p.$

$$= \frac{\sqrt{3}}{2}[C_1] + \frac{\sqrt{3}}{2}[C_2] + 0 \cdot [C_3] - \frac{\sqrt{3}}{2}[C_4] - \frac{\sqrt{3}}{2}[C_5] + 0\,[C_6] + \frac{\sqrt{3}}{2}[C_7] + \frac{\sqrt{3}}{2}[C_8].$$

3. Application to Milnor fibers.

Geometric idea behind

$$\nabla(B) := \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(U_1(B), C)}{k}\right) \cdot [C] \in \mathbb{C}[ch(A)].$$



$$\sin\left(\frac{\pi \cdot d(U_1(B), C)}{k}\right) = \frac{1}{2i}\left(e^{\frac{\pi i d}{k}} - e^{-\frac{\pi i d}{k}}\right)$$

This measures the monodromy of certain local system.

3. Application to Milnor fibers.

$$\nabla(B) = \sum_{C \subseteq B} \sin\left(\frac{\pi \cdot d(U_1(B), C)}{k}\right) \cdot [C]$$

Notation: $RB_k(A)$ denotes the set of k-resonant bands.

<u>Theorem</u> (Y.) Let $A = \{H_1, \cdots, H_n\}$ be a line arr. in $\mathbb{R}^2$, $\lambda \in \mathbb{C}^*$ ($\lambda \neq 1$) with order k s.t. $1 < k$, $k \mid (n+1)$. Then

$$\operatorname{Ker}\left(\nabla : \mathbb{C}[RB_k(A)] \to \mathbb{C}[ch(A)]\right) \cong H^1(F)_\lambda .$$

In particular, $\dim H^1(F)_\lambda$ is equal to the number of linear relations among $\nabla(B)$, $B \in RB_k(A)$.

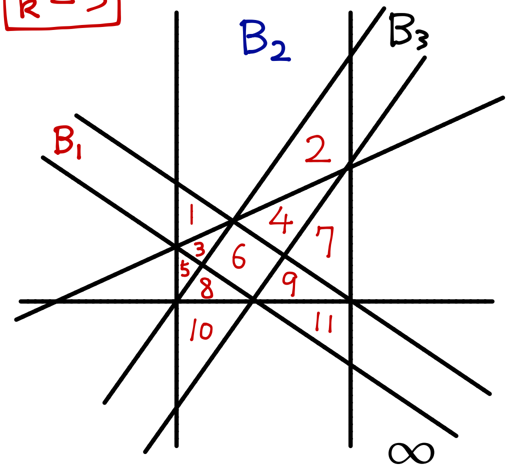
3. Application to Milnor fibers.

Theorem. $\mathrm{Ker}\left(\nabla : \mathbb{C}[RB_{\mathfrak{k}}(\mathcal{A})] \to \mathbb{C}[ch(\mathcal{A})]\right) \cong H^1(F)_\lambda$.

(Hence $\dim H^1(F)_\lambda$ is equal to # of linear rel's among $\nabla(B)$'s.)

Example (Pappus arr.)

$\boxed{\mathfrak{k} = 3}$



$$\frac{2}{\sqrt{3}}\nabla(B_1) = [C_1] + [C_3] - [C_9] - [C_{11}].$$

$$\frac{2}{\sqrt{3}}\nabla(B_3) = [C_2] + [C_4] - [C_8] - [C_{10}]$$

$$\frac{2}{\sqrt{3}}\nabla(B_2) = [C_1] + [C_2] + [C_3] + [C_4]$$

$$- [C_8] - [C_9] - [C_{10}] - [C_{11}]$$

$$= \frac{2}{\sqrt{3}}\nabla(B_1) + \frac{2}{\sqrt{3}}\nabla(B_3).$$

$$\therefore \dim H^1(F)_{e^{2\pi i/3}} = 1.$$

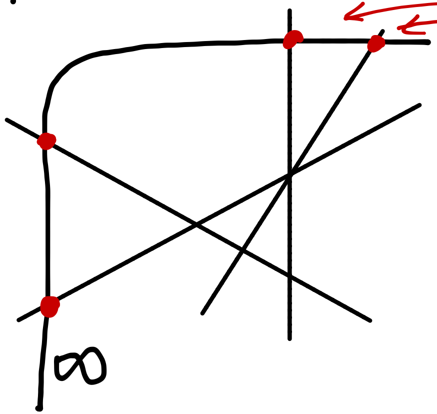
3. Application to Milnor fibers.

Thm. (Libgober)

If $\exists H_i \in \mathcal{A}$ s.t. all intersections on H_i has multiplicity 2 (i.e. No higher multiple points exist on H_i)

$$\Longrightarrow H^1(F)_\lambda = 0 \quad \text{for } \lambda \neq 1.$$

(proof) Put H_i at ∞.



All intersections on H_∞ have multiplicity $= 2$

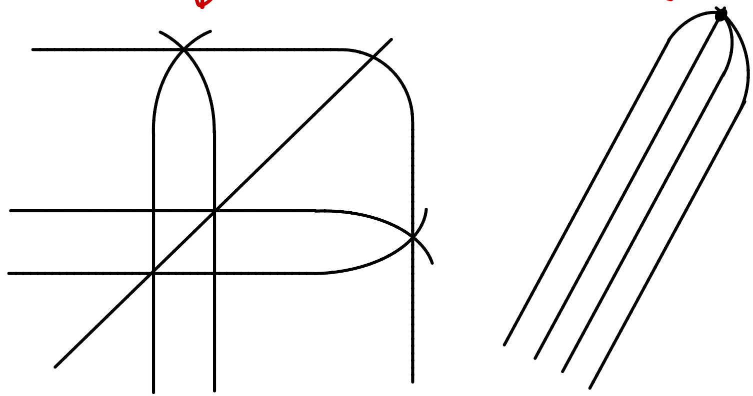
$\Longleftrightarrow$ There are no bands.

$RB_\ell(\mathcal{A}) = \phi$. Hence $\mathrm{Ker}(\mathbb{C}[RB_\ell] \to \mathbb{C}[ch])$

$= 0$ (Q.E.D.)

3. Application to Milnor fibers.

<u>Thm (Y.)</u> Except for A_3-arr. and pencils,



$H^1(F)_{\neq 1} \neq 0$ implies

that each line $H \in \mathcal{A}$ has at least 3 multiple intersections on it.

(Roughly, $H^1(F)_{\neq 1} \neq 0$ implies $\mathcal{A}$ is far from generic.)

3. Application to Milnor fibers.

"Thm": $H^1(F)_{\neq 1} \neq 0$ implies $\forall H \in \mathcal{A}$ has at least 3 higher multiple points.

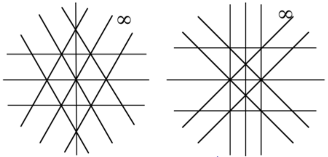
Such line arrangements are rare, however discrete geometers have found many interesting examples.

Two Sources:

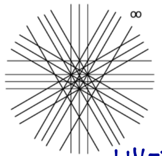
- B. Grünbaum, A catalogue of simplicial arrangements in the real projective plane.
- B. Grünbaum, Configurations of points and lines (Book, AMS)

3. Application to Milnor fibers.

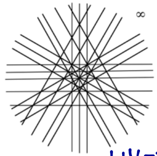
Examples (From Grünbaum's paper "A Catalogue of Simplicial arrangements")



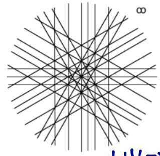
$\mathcal{A}(11,1)$ $H^1(F)_{\neq 1}=0$



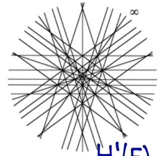
$\mathcal{A}(22,4)$ $H^1(F)_{\neq 1}=0$



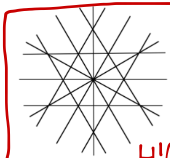
$\mathcal{A}(23,1)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(28,6)$ $H^1(F)_{\neq 1}=0$

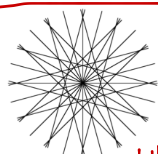


$\mathcal{A}(29,2)$ $H^1(F)_{\neq 1}=0$

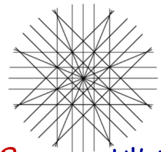


$\mathcal{A}(12,1)$ $H^1(F)_\zeta \cong \mathbb{C}$

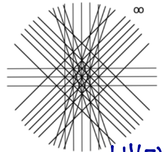
$\zeta = e^{2\pi i/3}$



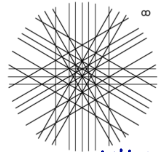
$\mathcal{A}(24,1)$ $H^1(F)_\zeta \cong \mathbb{C}$



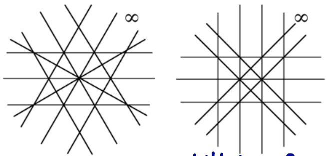
$\mathcal{A}(24,2)$ $H^1(F)_{\neq 1}=0$



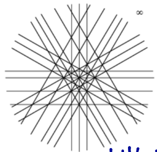
$\mathcal{A}(29,3)$ $H^1(F)_{\neq 1}=0$



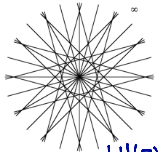
$\mathcal{A}(29,4)$ $H^1(F)_{\neq 1}=0$



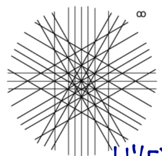
$\mathcal{A}(12,2)$ $H^1(F)_{\neq 1}=0$



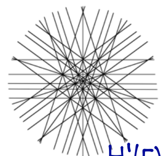
$\mathcal{A}(24,3)$ $H^1(F)_{\neq 1}=0$



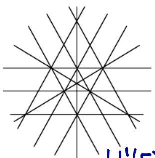
$\mathcal{A}(25,1)$ $H^1(F)_{\neq 1}=0$



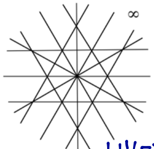
$\mathcal{A}(29,5)$ $H^1(F)_{\neq 1}=0$



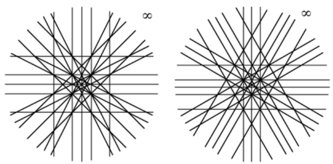
$\mathcal{A}(30,2)$ $H^1(F)_{\neq 1}=0$



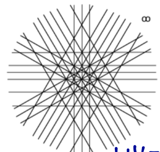
$\mathcal{A}(12,3)$ $H^1(F)_{\neq 1}=0$



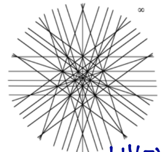
$\mathcal{A}(13,1)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(25,2)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(30,3)$ $H^1(F)_{\neq 1}=0$



$\mathcal{A}(31,1)$ $H^1(F)_{\neq 1}=0$

3. Application to Milnor fibers.

<u>Conjecture</u> Assume $\mathcal{A} = \{H_1, H_2, \cdots, H_n, H_\infty\}$ is defined $/\mathbb{R}$

① $H^1(F)_\lambda = 0$ if $\lambda^3 \neq 1$.

② $\dim H^1(F)_\lambda \leq 1$ if $\lambda \neq 1$.

③ $H^1(F)_{\neq 1} \neq 0$ iff $\mathcal{A}$ has 3-multinet str.

(Coloring of lines satisfying certain combinatorial conditions.)

References:

- Milnor fibers of real line arrangements. J. of Singularities (2013)
- Minimal stratifications for line arrangements and .... (2012).