Connection's Completeness on Statistical Manifolds

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Abstract

On a statistical manifold, we observe the completeness of statistical connections. We observe new results by assuming that either the statistical connection or the Riemannian metric is complete.

1 Introduction

In Riemannian geometry, the completeness of the Levi-Civita connection is usually assumed for most claims. In comparison, for affine connections appearing in other geometries, the consequences of assuming its completeness are not well known. For example, in recent years, the study of statistical manifolds has become popular due to information geometry. Statistical manifolds have appeared in affine differential geometry which have been studied for over 100 years, yet there are not many results led by the completeness of its affine connections.

Definition 1.1. A triplet (M, g, ∇) is called a **statistical manifold** if M is a smooth manifold, g a Riemannian metric, and ∇ a torsion-free affine connection such that ∇g is totally symmetric. The pair (g, ∇) is called the **statistical structure**, and ∇ is called the **statistical connection**.

Example 1.2. Let (M, g) be a Riemannian manifold and ∇^g its Levi-Civita connection. Then, since $\nabla^g g = 0$ holds, (M, g, ∇^g) is a statistical manifold.

Example 1.3. Let $M = (-\frac{\pi}{6}, \frac{\pi}{6}) \times \mathbb{R}$, and g the Euclidean metric restricted on M. Let (x^1, x^2) be a coordinate system on M. Define a torsion-free affine connection on M by

$$\nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^1} = \tan 3x^1 \frac{\partial}{\partial x^1}, \quad \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^2} = 0, \quad \nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^2} = 0.$$

Then, (M, g, ∇) is a statistical manifold.

For a statistical manifold (M, g, ∇), the **difference tensor** K is a $(1, 2)$-tensor field on M, defined by

$$K = \nabla - \nabla^g. \tag{1}$$

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The **cubic form** C of (g, ∇) is defined as a $(0,3)$-tensor field on M, by $C = \nabla g$. The tensor fields K and C are related by

$$C(X, Y, Z) = -2g(K_X Y, Z), \quad X, Y, Z \in \Gamma(TM). \tag{2}$$

Conversely, if we have a totally symmetric $(0,3)$-tensor field on a Riemannian manifold (M, g), we obtain a statistical manifold by (1) and (2). It is clear that $K = 0$ and $C = 0$ are equivalent. In this case, we will say (g, ∇) is **Riemannian**.

Statistical manifolds have appeared in the study of affine differential geometry[4]. Let $f : M \to \mathbb{R}^{n+1}$ be an immersed hypersurface, and D the standard flat connection of $\mathbb{R}^{n+1}$. If we choose a transversal vector field ξ of f, we obtain a torsion-free affine connection ∇ and a symmetric $(0,2)$-tensor field h on M by the Gauss formula

$$D_X f_* Y = f_* \nabla_X Y + g(X, Y)\xi, \quad X, Y \in \Gamma(TM). \tag{3}$$

The definiteness of g does not depend on the choice of ξ. If g is positive definite, we say f is locally strongly convex. We also obtain an $(1,1)$-tensor field S and an 1-form τ on M by the Weingarten formula

$$D_X \xi = -f_* S\xi + \tau(X)\xi, \quad X \in \Gamma(TM). \tag{4}$$

If $\tau = 0$, the transversal vector field ξ is called equiaffine. This is equivalent to the total symmetry of ∇g, since the following Codazzi equation holds for g, ∇, and τ.

$$(\nabla_X g)(Y, Z) + \tau(X) g(Y, Z) = (\nabla_Y g)(X, Z) + \tau(Y) g(X, Z), \quad X, Y, Z \in \Gamma(TM). \tag{5}$$

An equiaffine hypersurface is an immersed hypersurface equipped with an equiaffine transversal vector field. Therefore, the structure (g, ∇) on M induced by a locally strongly convex equiaffine immersion is a statistical structure on M.

Example 1.4. Let $M = \{(x^1, x^2) \in \mathbb{R}^2 | \quad x^1 > 0, x^2 > 0\}$. By distinguishing M as

$$\left\{ \left. \begin{pmatrix} x^1 & 0 \\ 0 & x^2 \end{pmatrix} \right| \quad x^1, x^2 \in \mathbb{R} \quad \text{and} \quad x^1 > 0, x^2 > 0 \right\},$$

the manifold M becomes a Lie group with the multiplication of matrices. Define an immersed hypersurface $f : M \to \mathbb{R}^3$ by

$$f(x^1, x^2) = \begin{pmatrix} x^1 \\ x^2 \\ \frac{1}{x^1 x^2} \end{pmatrix} \in \mathbb{R}^3, \quad (x^1, x^2) \in M.$$

Take $\xi = {}^t(0, 0, 1)$ as a transversal vector field of f. Then, by the Gauss formula (3) we obtain the following torsion-free affine connection ∇ and a symmetric $(0,2)$-tensor field h on M.

$$\nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^1} = \frac{1}{3}\frac{\partial}{\partial x^1} - \frac{2}{3}\frac{\partial}{\partial x^2}, \quad \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^2} = -\frac{1}{3}\frac{\partial}{\partial x^1} - \frac{1}{3}\frac{\partial}{\partial x^2}, \quad \nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^2} = -\frac{2}{3}\frac{\partial}{\partial x^1} + \frac{1}{3}\frac{\partial}{\partial x^2},$$

$$g = \frac{2}{3}(dx^1)^2 + \frac{1}{3} dx^1 \cdot dx^2 + \frac{2}{3}(dx^2)^2.$$

Since ξ is a constant vector field, ∇g is totally symmetric by the Weingarten formula (4) and the Codazzi equation (5). Therefore, we obtain a statistical manifold (M, g, ∇) by the equiaffine immersion.

The most fundamental immersed hypersurface is the hyperquadric. A hyperquadric in $\mathbb{R}^{n+1}$ of coordinate system $y^1, ..., y^{n+1}$ is the hypersurface defined by the following equation

$$ {}^t yQy + Py + R = 0 $$

where $y = {}^t(y^1, ..., y^{n+1})$, Q is a $(n+1) \times (n+1)$ matrix, $P \in \mathbb{R}^{n+1}$, and $R \in \mathbb{R}$.

Theorem 1.5 (Pick, Berward [4]). *Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex equiaffine hypersurface. If the statistical structure on M induced by f is Riemannian, then $f(M)$ is contained in a hyperquadric of $\mathbb{R}^{n+1}$.*

From this theorem, in affine differential geometry, it is crucial to know when the induced statistical structure is Riemannian. For example, we have the following propostion.

Proposition 1.6 ([2]). *Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex equiaffine hypersurface, where M is compact. If the statistical structure (g, ∇) on M induced by f satisfies $\nabla^g C = 0$, then $C = 0$.*

The term "statistical" was born when statistical structures appeared in 1982 when Shun-ichi Amari and Hiroshi Nagaoka introduced them in the field of information theory, which gave birth to information geometry [1].

Example 1.7 (Normal distribution). Consider the upper half plane, $\mathbb{H}^2 = \{(x^1, x^2) \in \mathbb{R}^2 | \quad x^2 > 0\}$. By distinguishing $\mathbb{H}^2$ as

$$ \left\{ \begin{pmatrix} x^1 & x^2 \\ 0 & 1 \end{pmatrix} \middle| \quad x^1, x^2 \in \mathbb{R} \quad \text{and} \quad x^2 > 0 \right\}, $$

the manifold $\mathbb{H}^2$ becomes a Lie group with the multiplication of matrices. Set a Riemannian metric on $\mathbb{H}^2$ by

$$ g = \frac{(dx^1)^2 + 2(dx^2)^2}{(x^2)^2}, $$

and a torsion-free affine connection ∇ by

$$ \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^1} = 0, \quad \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^2} = -\frac{2}{x^2} \frac{\partial}{\partial x^1}, \quad \nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^2} = -\frac{3}{x^2} \frac{\partial}{\partial x^2}. $$

The triplet $(\mathbb{H}^2, g, \nabla)$ is a statistical manifold, and it is called the statistical manifold of normal distributions.

In Example 1.4 and Example 1.7, it is easy to check that g and ∇ are both left-invariant. In such a case, we call the statistical manifold a **statistical Lie group**.

Many aspects of statistical manifolds are still not studied enough, and the completeness of statistical connections is one of them.

Definition 1.8. Let ∇ be an affine connection on a smooth manifold M. A smooth curve $c \colon I \to M$ is called a ∇-**geodesic** if it satisfies the following differential equation:

$$ \nabla_{\dot{c}} \dot{c} = 0 \quad \text{on} \quad I. $$

Here, $\dot{c}$ is the velocity vector of c. If each ∇-geodesic can extend its domain on $\mathbb{R}$ as a ∇-geodesic, ∇ is said to be **complete**.

In Example 1.7, the Levi-Civita connection of g is complete while ∇ is not. In terms of amount, the study on the completeness of statistical connections might be insignificant. In the study of hypersurfaces, "affine completeness" always meant the completeness of the Levi-Civita connection of the affine metric, not the induced statistical connection. In information geometry, to the knowledge of the presenter, there are no studies of the completeness of any affine connections.

In the geometry of statistical manifolds, the only papers that have results on the completeness of statistical connections are those of Mitsunori Noguchi [3] and Barbara Opozda [5]. The first paper was published in 1992. In this paper, M. Noguchi observed the geodesics of an affine connection not necessarily of Levi-Civita, on a Riemannian manifold.

Theorem 1.9 (M. Noguchi [3]). *Let (M, g, ∇) be a statistical manifold. Suppose that ∇^g is complete and that the cubic form C has the shape of*

$$C(X, Y, Z) = d\sigma(X)g(Y, Z) + d\sigma(Y)g(Z, X) + d\sigma(Z)g(X, Y)$$

for some $\sigma \in C^\infty(M)$ and for any $X, Y, Z \in \Gamma(TM)$. If σ is bounded from above, then ∇ is complete.

There were no study of the completeness of statistical connections for almost 30 years until the second paper [5] was presented in 2021 by B. Opozda. In this paper, there is another theorem that determines the completeness of statistical connections.

Theorem 1.10 (B. Opozda [5]). *Let (M, g, ∇) be a statistical manifold. Suppose that M is diffeomorphic to the Euclidean sphere.*

- *(i) If the Ricci tensor of ∇ is symmetric, non-degenerate, and if ∇ is projectively flat, then ∇ is complete on M.*
- *(ii) If $\nabla^g K$ is totally symmetric and if ∇ is projectively flat, then ∇ is complete on M.*

B. Opozda focused on the completeness of the induced statistical connection on a hypersurface. In the study of hypersurfaces, the case of when $\nabla^g C = 0$ is important, and it is completely classified in [2]. B. Opozda observed the completeness of the statistical connection in the case of $\nabla^g C = 0$. As a result, she found the next theorem and concluded that the statistical connection is not complete unless the statistical structure is Riemannian.

Theorem 1.11 (B. Opozda [5]). *Let (M, g, ∇) be a statistical manifold. Suppose that*

$$(\nabla^g_X C)_p(X, X, X) \geq 0$$

for any $(p, X) \in TM$. If ∇ is complete, then (g, ∇) must be Riemannian.

2 Main Results

To begin with, the presenter obtained a corollary of Theorem 1.11.

Corollary 2.1. *Let* (G, g, ∇) *be a statistical Lie group with a bi-invariant metric. Then, if* ∇ *is complete, then* (g, ∇) *must be Riemannian.*

From this corollary, we see that the compactness of the statistical manifold does not imply the completeness of its statistical connection.

Example 2.2. Let (T^n, g_0) be a flat torus. With any left-invariant totally symmetric $(0,3)$-tensor field on T^n that is not 0, we obtain a statistical manifold by (1) and (2). In this case, the statistical connection is not complete since g_0 is bi-invariant.

The main theorem of this presentation consists of a similar inequality to the previous theorem.

Theorem 2.3. *Let* (M, g, ∇) *be a statistical manifold. Suppose that*

$$(\nabla_X C)_p(X, X, X) \leq 0 \tag{6}$$

for any $(p, X) \in TM$. *If* ∇ *is complete, then* (g, ∇) *must be Riemannian. Moreover, the same holds if* ∇^g *is complete.*

Here are some applications of this theorem. In Example 1.3, the statistical structure satisfies (6), therefore the statistical connection is not complete. The next theorem is immediate from Theorem 1.5 and Theorem 2.3.

Theorem 2.4. *Let* $f : M \to \mathbb{R}^{n+1}$ *be a locally strongly convex equiaffine hypersurface with the induced statistical structure* (g, ∇) *that satisfies (6). If* ∇ *or* ∇^g *is complete, then the hypersurface is contained in a hyperquadric.*

Remark 2.5. The definition of statistical structure is often extended by considering a pseudo-Riemannian metric in place of the Riemannian metric. However, Theorem 2.3 does not hold when the metric is not Riemannian. For example, the induced structure (g, ∇) on the Cayley surface [4] has a complete connection ∇ and a non-degenerate metric g, while $C = \nabla g \neq 0$ and $\nabla C = 0$ on M.

References

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