Solvability and nilpotency for finite quantum groups

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Abstract

Examples of nilpotent or solvable finite quantum groups are studied. We classify the solvable series of maximal length which show the nilpotency of Kac–Paljutkin's 8-dimensional finite quantum group. We show that Kac–Paljutkin's 8-dimensional finite quantum group and Sekine quantum groups are nilpotent.

1 Introduction

Finite quantum groups are finite dimensional Hopf algebras which are C^*-algebras at the same time. For example, algebras of continuous functions on finite groups and group algebras of finite groups are finite quantum groups. The former are commutative and the latter are cocommutative. In fact commutative or cocommutative ones always arise from genuine finite groups. In 1966, Kac–Paljutkin found a non-commutative and non-cocommutative example ([KP66]), which is called Kac–Paljutkin's (8-dimensional) finite quantum group today. In 1996, Sekine found a family of non-commutative and non-cocommutative finite quantum groups ([Sek96]), which are called Sekine quantum groups today. Many talented mathematicians have studied them from many points of view. For example, McCarthy and Zhang studied the probability theoretical aspects ([McC17, Zha19]), Tambara–Yamagami studied them from categorical points of view ([TY98]) and Suzuki and Wakui revealed the quasitriangular structures ([Suz98, Wak10]).

In 2009, Etingof–Nikshych–Ostrik formulated the nilpotency and the solvability for fusion categories. If we define the nilpotency and the solvability for finite dimensional Hopf algebras via this language, the nilpotency does not imply the solvability in general ([ENO11]). In 2016, Cohen–Westreich proposed intrinsic definitions of the nilpotency and the solvability via integrals ([CW17]). Their definitions are satisfactory in that the nilpotency implies the solvability and the analogue of Burnside's $p^a q^b$ theorem holds.

In 2023, the author of the present notes and his coauthor showed that Kac–Paljutkin's finite quantum group and each of Sekine quantum groups are nilpotent (and hence solvable) and they also studied the quasitriangular structures of Kac–Paljutkin's finite quantum group by giving direct computations of the universal R-matrices. This research gives examles of Cohen–Westreich's general theory, which do not arise from genuine groups.

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2 Preliminaries

We review some items needed to state and prove our results. The best general references here are [CW17], [FG06], [Kas95] and [Mur90].

2.1 Finite quantum groups

Definition 2.1. A vector space A over $\mathbb{C}$ is called an *unital algebra* if it has an associative and bilinear multiplication $A \times A \ni (a, b) \mapsto ab \in A$ and an element 1 satisfying $a1 = 1a = a$ for each $a \in A$. A unital algebra is called a $*$-*algebra* if it has a conjugate linear and involutive operation $A \ni a \mapsto a^* \in A$ satisfying $(ab)^* = b^* a^*$ for all $a, b \in A$, which is called a $*$-*operation.*

Example. Let $\mathbb{M}_n$ denote the algebra of $n \times n$-matrices whose entries are complex numbers. With the usual multiplication of matrices, this vector space is a $*$-algebra.

Example. Let G be a finite group. The algebra $C(G)$ of continuous functions on G is a $*$-algebra. Its multiplication is given by $fg(x) = f(x)g(x)$ $(f, g \in C(G), x \in G)$ its unit is the constant function $1\colon G \ni x \mapsto 1 \in \mathbb{C}$ and the $*$-operation is given by $f^*(x) = \overline{f(x)}$ $(f \in C(G), x \in G)$.

Example. Let G be a finite group. The group algebra $\mathbb{C}[G] = \mathrm{span}\{\delta_x \mid x \in G\}$ is a $*$-algebra. Its multiplication is given by $(\sum_{x \in G} a_x \delta_x)(\sum_{y \in G} b_y \delta_y) = \sum_{x, y \in G} a_x b_y \delta_{xy}$ $(a_x, b_y \in \mathbb{C})$ and its $*$-operation is given by $(\sum_{x \in G} a_x \delta_x)^* = \sum_{x \in G} \overline{a_x} \delta_{x^{-1}}$.

Definition 2.2. Let A be a unital algebra. We say A is a *Hopf algebra* if it has the following algebra homomorphisms $\Delta\colon A \to A \otimes A, \epsilon\colon A \to \mathbb{C}$ (i.e. linear maps preserving multiplication) and anti-multiplicative linear map $S\colon A \to A$ (i.e. $S(xy) = S(y)S(x) \text{where} x, y \in A$) satisfying

$$
\begin{aligned}
&(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta; \\
&(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta; \\
&m(S \otimes \mathrm{id})\Delta = \epsilon(\bullet)1 = m(\mathrm{id} \otimes S)\Delta
\end{aligned}
$$

where m denotes the map $m(x \otimes y) = xy$ $(x, y \in A)$. If moreover Δ and ϵ preserves $*$-operation, we say A is a *Hopf $*$-algebra.* We use Sweedler's sumless notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ $(a \in A)$.

Definition 2.3. A *finite quantum group* is a finite dimensional Hopf $*$-algebra which is a C^*-algebra at the same time.

We do not define C^*-algebras in these notes. Instead of defining the objects, we introduce the following well-known proposition.

Proposition 2.4. *Any finite dimensioanl C^*-algebra is of the form $\bigoplus_{k=1}^N \mathbb{M}_{n_k}$ for some positive integers N and $n_1, \ldots, n_N$.*

For the definition and the proof of the proposition above, see [Mur90] for example. The following statement is nontrivial. See [VD97] for deatails.

Proposition 2.5. *For any finite quantum group A there exists a linear functional $h\colon A \to \mathbb{C}$ such that $h(a^*a) \geqslant 0$ $(a \in A)$, $h(1) = 1$ and $(\mathrm{id} \otimes h)\Delta(\bullet) = h(\bullet)1$. In addition such a state is unique up to a constant multiple and satisfies also the condition $(h \otimes \mathrm{id})\Delta(\bullet) = h(\bullet)1$.*

The linear functional h in the proposition above is called the *Haar state* of A.

Example. The algebra $C(G)$ of continuous functions on G is a finite quantum group G. Its Hopf algebra structure is given by

$$\begin{aligned}
&\Delta\colon C(G) \to C(G) \otimes C(G) \simeq C(G \times G), f \mapsto \Delta(f)(\colon (x,y) \mapsto f(xy));\\
&\epsilon\colon C(G) \to \mathbb{C}, f \mapsto f(1);\\
&S\colon C(G) \to C(G), f \mapsto f(\bullet^{-1}).
\end{aligned}$$

Conversely any commutative finite quantum group is of this form.

Example. The group algebra $\mathbb{C}[G]$ of a finite group G is a finite quantum group. Its Hopf algebra structure is given by

$$\begin{aligned}
&\Delta(\delta_x) = \delta_x \otimes \delta_x;\\
&\epsilon(\delta_x) = 1;\\
&S(\delta_x) = \delta_{x^{-1}}
\end{aligned}$$

where $x \in G$. In this case $\Delta = \Delta^{\mathrm{op}}$, where Δ^{op} denotes the composition of Δ and the flip $\mathbb{C}[G] \otimes \mathbb{C}[G] \ni a \otimes b \mapsto b \otimes a \in \mathbb{C}[G] \otimes \mathbb{C}[G]$.

Conversely, any cocommutative finite quantum group is of this form.

We introduce important examples of finite quantum groups. Notice that they are neither commutative nor cocommutative, which means that they do not arise from genuine groups.

Example. ([KP66]) Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{M}_2$. Its Hopf $*$-algebra structure is given by

$$\begin{aligned}
&\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4\\
&+\frac{1}{2}a_{1,1} \otimes a_{1,1} + \frac{1}{2}a_{1,2} \otimes a_{1,2} + \frac{1}{2}a_{2,1} \otimes a_{2,1} + \frac{1}{2}a_{2,2} \otimes a_{2,2}\\
&\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3\\
&+\frac{1}{2}a_{1,1} \otimes a_{2,2} + \frac{1}{2}a_{2,2} \otimes a_{1,1} - \frac{\sqrt{-1}}{2}a_{1,2} \otimes a_{2,1} + \frac{\sqrt{-1}}{2}a_{2,1} \otimes a_{1,2}\\
&\Delta(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 + e_2 \otimes e_4 + e_4 \otimes e_2\\
&+\frac{1}{2}a_{1,1} \otimes a_{2,2} + \frac{1}{2}a_{2,2} \otimes a_{1,1} + \frac{\sqrt{-1}}{2}a_{1,2} \otimes a_{2,1} - \frac{\sqrt{-1}}{2}a_{2,1} \otimes a_{1,2}\\
&\Delta(e_4) = e_1 \otimes e_4 + e_4 \otimes e_1 + e_2 \otimes e_3 + e_3 \otimes e_2\\
&+\frac{1}{2}a_{1,1} \otimes a_{1,1} + \frac{1}{2}a_{2,2} \otimes a_{2,2} - \frac{1}{2}a_{1,2} \otimes a_{1,2} - \frac{1}{2}a_{2,1} \otimes a_{2,1}\\
&\Delta(a_{1,1}) = e_1 \otimes a_{1,1} + a_{1,1} \otimes e_1 + e_2 \otimes a_{2,2} + a_{2,2} \otimes e_2\\
&+e_3 \otimes a_{2,2} + a_{2,2} \otimes e_3 + e_4 \otimes a_{1,1} + a_{1,1} \otimes e_4\\
&\Delta(a_{1,2}) = e_1 \otimes a_{1,2} + a_{1,2} \otimes e_1 + \sqrt{-1}e_2 \otimes a_{2,1} - \sqrt{-1}a_{2,1} \otimes e_2\\
&-\sqrt{-1}e_3 \otimes a_{2,1} + \sqrt{-1}a_{2,1} \otimes e_3 - e_4 \otimes a_{1,2} - a_{1,2} \otimes e_4\\
&\Delta(a_{2,1}) = e_1 \otimes a_{2,1} + a_{2,1} \otimes e_1 - \sqrt{-1}e_2 \otimes a_{1,2} + \sqrt{-1}a_{1,2} \otimes e_2\\
&+\sqrt{-1}e_3 \otimes a_{1,2} - \sqrt{-1}a_{1,2} \otimes e_3 - e_4 \otimes a_{2,1} - a_{2,1} \otimes e_4\\
&\Delta(a_{2,2}) = e_1 \otimes a_{2,2} + a_{2,2} \otimes e_1 + e_2 \otimes a_{1,1} + a_{1,1} \otimes e_2\\
&+e_3 \otimes a_{1,1} + a_{1,1} \otimes e_3 + e_4 \otimes a_{2,2} + a_{2,2} \otimes e_4\\
&\epsilon(e_1) = 1, \epsilon(e_2) = \epsilon(e_3) = \epsilon(e_4) = 0\\
&S(e_i) = e_i \ (i = 1,2,3,4), S(a_{i,j}) = a_{j,i} \ (i,j = 1,2)
\end{aligned}$$

Here e_i and $a_{i,j}$ denote the following elements:

$$e_1 = 1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = 0 \oplus 1 \oplus 0 \oplus 0 \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$e_3 = 0 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, e_4 = 0 \oplus 0 \oplus 0 \oplus 1 \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$a_{i,j} = 0 \oplus 0 \oplus 0 \oplus 0 \oplus E_{i,j} \quad (i, j = 1, 2)$$

where the $E_{i,j}$'s are matrix units. We call this $\mathcal{A}$ the *Kac–Paljutkin's (8-dimensional) finite quantum group.*

Example. ([Sek96]) Let k be a positive integer. Let $\eta = \exp(2\pi\sqrt{-1}/k)$. Let

$$\mathcal{A}_k = \bigoplus_{i,j \in \mathbb{Z}_k} \mathbb{C} d_{i,j} \oplus \mathbb{M}_k,$$

where the $d_{i,j}$'s are projections (i.e. $d_{i,j}^* = d_{i,j} = d_{i,j}^2$) such that $d_{i,j} d_{k,l} = \delta_{i,k}\delta_{j,l} d_{i,j}$. Its Hopf $*$-algebra structure is given by

$$\Delta(d_{i,j}) = \sum_{m,n \in \mathbb{Z}_k} d_{m,n} \otimes d_{i-m,j-n} + \frac{1}{k} \sum_{m,n \in \mathbb{Z}_k} \eta^{i(m-n)} a_{m,n} \otimes a_{m+j,m+j},$$
$$\Delta(a_{i,j}) = \sum_{m,n \in \mathbb{Z}_k} \eta^{m(i-j)} d_{-m,-n} \otimes a_{i-n,j-n} + \sum_{m,n \in \mathbb{Z}_k} \eta^{m(j-i)} a_{i-n,j-n} \otimes d_{m,n},$$
$$\epsilon(d_{i,j}) = \delta_{i,0}\delta_{j,0}, \quad \epsilon(a_{i,j}) = 0,$$
$$S(d_{i,j}) = d_{-i,-j}, \quad S(a_{i,j}) = a_{j,i}$$

where $i, j \in \mathbb{Z}_k$ and the $a_{i,j}$'s are defined in the same way as the Kac–Paljutkin's finite quantum group. Each algebra $\mathcal{A}_k$ is called a *Sekine qantum group.*

Next we introduce universal R-matrices for Hopf algebras. Simply put, universal R-matrices are solutions of some important equation in Physics, which is called quantum Yang–Baxter equation (QYBE). For details, see [Kas95] and references therein.

Definition 2.6. A Hopf algebra A is called *quasitriangular* if there exists an invertible element $R \in A \otimes A$ such that $R\Delta(\bullet)R^{-1} = \Delta^{\mathrm{op}}(\bullet)$, $(\Delta \otimes \mathrm{id})R = R_{13}R_{23}$ and $(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}$. Here we used the leg-numbering notation: $(a \otimes b)_{13} := a \otimes 1 \otimes b$, $(a \otimes b)_{12} := (a \otimes b \otimes 1)$ and so on. Such an R is called an *universal R-matrix.*

2.2 Nilpotency and solvability for finite quantum groups

In the paper [CW17], Cohen–Westreich introduced intrinsic definitions of solvability and nilpotency for semisimple finite dimendional Hopf algebra. First we define coideals.

Definition 2.7. Let A be a Hopf algebra. An unital subalgebra I is called a *left coideal subalgebra* if $\Delta(I) \subset A \otimes I$. In a similar way, we define *right coideals.* When A is a Hopf $*$-algebra, we say I is a *left (resp. right) coideal $*$-subalgebra* if I is a left (resp. right) coideal subalgebra and $*$-algebra.

Example. Let G be a finite group and H a subgroup. Then the algebra $C(G/H)$ of continuous functions on the homogeneous space G/H is a left coideal $*$-subalgebra.

Example. Let G and H be as above. Then the group algebra $\mathbb{C}[H]$ of H is a left coideal $*$-subalgebra.

Example. (cf. [FG06]) It is not difficult to see that the left coideal $*$-subalgebras of Kac–Paljutkin's finite quantum group $\mathcal{A}$ are the following.

1. $L_1 = \mathcal{A}$
2. $L_2 = \mathrm{span}\{e_1 + e_2, e_3 + e_4, a_{1,1} + a_{2,2}, a_{1,2} - \sqrt{-1}a_{2,1}\}$
3. $L_3 = \mathrm{span}\{e_1 + e_4, e_2 + e_4, a_{1,1} + a_{2,2}, a_{1,2} + \sqrt{-1}a_{2,1}\}$
4. $L_4 = \mathrm{span}\{e_1 + e_4, e_2 + e_3, a_{1,1}, a_{2,2}\}$
5. $L_5 = \mathrm{span}\{e_1 + e_2 + e_3 + e_4, a_{11} + a_{2,2}\}$
6. $L_6 = \mathrm{span}\{e_1 + e_4 + a_{1,1}, e_2 + e_3 + a_{2,2}\}$
7. $L_7 = \mathrm{span}\{e_1 + e_4 + a_{2,2}, e_2 + e_3 + a_{1,1}\}$
8. $L_8 = \mathbb{C}1$

Definition 2.8. An element λ in a Hopf algebra A is called an *integral* if $\lambda a = \epsilon(a)\lambda = a\lambda$ for each $a \in A$.

Example. (cf. [FG06]) Let

1. $p_1 = e_1$,
2. $p_2 = e_1 + e_2$,
3. $p_3 = e_1 + e_3$,
4. $p_4 = e_1 + e_4$,
5. $p_5 = e_1 + e_2 + e_3 + e_4$,
6. $p_6 = e_1 + e_4 + a_{1,1}$,
7. $p_7 = e_1 + e_4 + a_{2,2}$ and
8. $p_8 = e_1 + e_2 + e_3 + e_4 + a_{1,1} + a_{2,2}$.

Then p_i is the integral of the left coideal $*$-subalgebra L_i for each i.

Example. ([Zha19]) We give examples of left coideal $*$-subalgebras of a Sekine quantum group $\mathcal{A}_k$ and their integrals. Let $\Gamma_1 = \{(i,i) \mid i \in \mathbb{Z}_k\}$. Let $q_1 = \sum_{(i,j)\in\Gamma_1} d_{i,j}$ and $q_k = \sum_{(i,j)\in\mathbb{Z}_k\times\mathbb{Z}_k} d_{i,j}$. Then q_1 is the integral of the left coideal $*$-subalgebra $(\mathrm{id} \otimes h_1)\Delta(\mathcal{A}_1)$ and q_k is the integral of the left coideal $*$-subalgebra $(\mathrm{id} \otimes h_k)\Delta(\mathcal{A}_k)$, where $h_1(\bullet) = \dfrac{h(\bullet q_1)}{h(q_1)}$, $h_k(\bullet) = \dfrac{h(\bullet q_k)}{h(q_k)}$ and h denotes the Haar state of $\mathcal{A}_k$.

Definition 2.9. ([CW17, Definition 3.5]) Let A be a semisimple Hopf algebra. A chain of left coideal subalgebras of A

$$I_0 \subset I_1 \subset \cdots \subset I_i$$

is called a *solvable series* if the following conditions are satisfied for all $0 \leqslant j \leqslant i - 1$:

1. $\lambda_j \in Z(I_{j+1})$, where λ_j denotes the integral of I_j and $Z(I_{j+1})$ denotes the center of I_{j+1}.

2. $(a \underset{\text{ad}}{\bullet} b)\lambda_j = \epsilon(a)b\lambda_j$ for all $a, b \in I_{j+1}$.

If there is a solvable series such that $I_0 = \mathbb{C}1$ and $I_i = A$ then the Hopf algebra is called *solvable*. Here $a \underset{\text{ad}}{\bullet} b := a_{(1)}bS(a_{(2)})$ $(a, b \in A)$.

A left coideal subalgebra I of a Hopf algebra A is called *normal* if $a \underset{\text{ad}}{\bullet} x \in I$ for all $a \in A, x \in I$.

Definition 2.10. ([CW17, Proposition 3.8]) A semisimple Hopf algebra A is called *nilpotent* if it has a chain of normal left coideal subalgebras

$$\mathbb{C} = I_0 \subset I_1 \subset \cdots \subset I_i = A$$

satisfying $I_{j+1}\lambda_j \subset Z(A\lambda_j)$ for all $0 \leqslant j \leqslant i-1$.

Remark 2.11. Cohen–Westreich defined the nilpotency in a different way. By [CW17, Proposition 3.8], the definition above is equivalent to the original definition.

The following theorems are surprising. The latter is an analogue of the well-known theorem by Burnside.

Theorem 2.12. *([CW17, Corollary 3.9]) A semisimple Hopf algebra is solvable if it is nilpotent.*

Theorem 2.13. *([CW17, Theorem 3.11]) Let p and q be prime numbers. Let a and b be non-negative integers. Let A be a quasitriangular semisimple Hopf algebra of dimension $p^a q^b$. Then A is solvable. In addition, if I is a left coideal subalgebra, then A has a solvable series containing the left coideal subalgebra I.*

3 Main Results

We state our main results.

Theorem 3.1. *([GHT24, Theorem 3.2]) Any solvable series for Kac–Paljutkin's finite quantum group of length 4 is one of the following.*

1. $\mathbb{C} \subset L_5 \subset L_2 \subset \mathcal{A}$
2. $\mathbb{C} \subset L_5 \subset L_3 \subset \mathcal{A}$
3. $\mathbb{C} \subset L_5 \subset L_4 \subset \mathcal{A}$
4. $\mathbb{C} \subset L_6 \subset L_4 \subset \mathcal{A}$
5. $\mathbb{C} \subset L_7 \subset L_4 \subset \mathcal{A}$

The theorem above can be shown by direct computations.

Theorem 3.2. *([GHT24, Theorem 3.4 and Theorem 3.5]) Kac–Paljutkin's finite quantum group and Sekine quantum groups are nilpotent.*

The series $\mathbb{C} \subset L_5 \subset L_4 \subset \mathcal{A}$ shows the first part of this theorem. We can prove the nilpotency of Sekine quantum groups by observing the series of unital left coideals subalgebras $\mathbb{C} \subset (\mathrm{id} \otimes h_k)\Delta(\mathcal{A}_k) \subset (\mathrm{id} \otimes h_1)\Delta(\mathcal{A}_k) \subset \mathcal{A}_k$.

Remark 3.3. It is easy to show that Kac–Paljutkin's finite quantum since its dimension is $8 = 2^3$, it is quasitriangular and we have Cohen–Westreich's Burnside theorem. For the quasitriangular structure see [Suz98, Wak10] and see also the proposition below and [GHT24, Appendix A].

Remark 3.4. Note that Theorem 3.2 shows that the converse of Cohen–Westreich's Burnside theorem has a counter-example $\mathcal{A}_{15}$.

Proposition 3.5. *([GHT24, Appendix A] cf. [Suz98, Wak10]) The universal R-matrices of Kac–Paljutkin's finite quantum group $\mathcal{A}$ are of the form*

$$\begin{aligned}
R =& A_{11} e_1 \otimes e_1 + A_{12} e_1 \otimes e_2 + A_{13} e_1 \otimes e_3 + A_{14} e_1 \otimes e_4 \\
&+ A_{21} e_2 \otimes e_1 + A_{22} e_2 \otimes e_2 + A_{23} e_2 \otimes e_3 + A_{24} e_2 \otimes e_4 \\
&+ A_{31} e_3 \otimes e_1 + A_{32} e_3 \otimes e_2 + A_{33} e_3 \otimes e_3 + A_{34} e_3 \otimes e_4 \\
&+ A_{41} e_4 \otimes e_1 + A_{42} e_4 \otimes e_2 + A_{43} e_4 \otimes e_3 + A_{44} e_4 \otimes e_4 \\
&+ B_1 e_1 \otimes a_{11} + B_1 e_1 \otimes a_{22} + B_2 e_2 \otimes a_{11} - B_2 e_2 \otimes a_{22} \\
&+ B_3 e_3 \otimes a_{11} - B_3 e_3 \otimes a_{22} + B_4 e_4 \otimes a_{11} + B_4 e_4 \otimes a_{22} \\
&+ C_1 a_{11} \otimes e_1 + C_1 a_{22} \otimes e_1 + C_2 a_{11} \otimes e_2 - C_2 a_{22} \otimes e_2 \\
&+ C_3 a_{11} \otimes e_3 - C_3 a_{22} \otimes e_3 + C_4 a_{11} \otimes e_4 + C_4 a_{22} \otimes e_4 \\
&+ D_{1111} a_{11} \otimes a_{11} + D_{1122} a_{11} \otimes a_{22} + D_{1212} a_{12} \otimes a_{12} + D_{1221} a_{12} \otimes a_{21} \\
&+ D_{1221} a_{21} \otimes a_{12} + D_{1212} a_{21} \otimes a_{21} - D_{1122} a_{22} \otimes a_{11} + D_{1111} a_{22} \otimes a_{22},
\end{aligned}$$

where

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and other coefficients are given by one of the following four cases:

- $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \pm 1 \\ 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \mp 1 \\ 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \\ -\sqrt{-1}\lambda \end{bmatrix}$
  (λ is a square root of $\sqrt{-1}$.)
- $\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} D_{1111} \\ D_{1122} \\ D_{1212} \\ D_{1221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \\ \sqrt{-1}\lambda \end{bmatrix}$
  (λ is a square root of $\sqrt{-1}$.)

For detailed proofs of our main results and other results including direct computations of the universal R-matrices of Kac–Paljutkin's finite quantum group, see [GHT24].

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