# ODA'S PROBLEM FOR $G$-SPECIAL LOCI

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## 1. The $\ell$-monodromy outer actions and Oda's problem

Let $\mathcal{M}_{g,m}$ be the moduli space of curves over $\mathbf{Q}$ of genus $g$ with $m$ marked points satisfying the hyperbolicity condition $2g-2+m \geq 1$. On one hand we have the étale homotopy exact sequence

$$1 \longrightarrow \pi_1(\mathcal{M}_{g,m\overline{\mathbf{Q}}}) \longrightarrow \pi_1(\mathcal{M}_{g,m}) \xrightarrow{\ p\ } G_{\mathbf{Q}}$$

on the other hand Oda showed in [Oda97] that there is an exact sequence of étale fundamental groups induced by the point forgetting map

$$1 \longrightarrow \pi_1(C_{\overline{\mathbf{Q}}}) \longrightarrow \pi_1(\mathcal{M}_{g,m+1}) \longrightarrow \pi_1(\mathcal{M}_{g,m}) \longrightarrow 1$$

where $C/\mathbf{Q}$ is a hyperbolic curve represented on $\mathcal{M}_{g,m}$. From this last exact sequence, by considering conjugation in the middle and by denoting $\pi_1^{\ell}(C_{\overline{\mathbf{Q}}})$ the pro-$\ell$ completion of $\pi_1(C_{\overline{\mathbf{Q}}})$ we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(C_{\overline{\mathbf{Q}}}) & \longrightarrow & \pi_1(\mathcal{M}_{g,m+1}) & \longrightarrow & \pi_1(\mathcal{M}_{g,m}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Inn}\pi_1(C_{\overline{\mathbf{Q}}}) & \longrightarrow & \operatorname{Aut}\pi_1(C_{\overline{\mathbf{Q}}}) & \longrightarrow & \operatorname{Out}\pi_1(C_{\overline{\mathbf{Q}}}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Inn}\pi_1^{\ell}(C_{\overline{\mathbf{Q}}}) & \longrightarrow & \operatorname{Aut}\pi_1^{\ell}(C_{\overline{\mathbf{Q}}}) & \longrightarrow & \operatorname{Out}\pi_1^{\ell}(C_{\overline{\mathbf{Q}}}) & \longrightarrow & 1
\end{array}$$

The long vertical arrow obtained from the right hand column gives a universal $\ell$-monodromy outer action $\Phi_{g,m}^{\ell}\colon \pi_1(\mathcal{M}_{g,m}) \to \operatorname{Out}\pi_1^{\ell}(C_{\overline{\mathbf{Q}}})$ which is independent of the choice of $C$. Moreover, applying the same construction starting from the homotopy exact sequence of $C$

$$1 \longrightarrow \pi_1(C_{\overline{\mathbf{Q}}}) \longrightarrow \pi_1(C) \longrightarrow G_{\mathbf{Q}} \longrightarrow 1$$

we obtain the usual $\ell$-monodromy outer action $\varphi_C^{\ell}\colon G_{\mathbf{Q}} \to \operatorname{Out}\pi_1^{\ell}(C_{\overline{\mathbf{Q}}})$ canonically associated with $C$. Now, considering $s_C$ the section of the canonical projection $p\colon \pi_1(\mathcal{M}_{g,m}) \to G_{\mathbf{Q}}$ induced

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by the rational point representing $C$ on $\mathcal{M}_{g,m}$ we obtain a triangle

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\varphi^{\ell}_{C}} & \operatorname{Out}\pi_1^{\ell}(C_{\overline{\mathbf{Q}}}) \\ {\scriptstyle s_C}\big\downarrow & \nearrow_{\Phi^{\ell}_{g,m}} & \\ \pi_1(\mathcal{M}_{g,m}) & & \end{array}$$

which is commutative. The fact that every such triangle commutes for any choice of curve $C/\mathbf{Q}$ represented on the moduli space is the universality property of $\Phi^{\ell}_{g,m}$.
By construction the groups $\operatorname{Ker}\varphi^{\ell}_{C}$ and $p(\operatorname{Ker}\Phi^{\ell}_{g,m})$ are closed subgroups of $G_{\mathbf{Q}}$. The corresponding fixed fields $\mathbf{Q}^{\ell}_{g,m} := \overline{\mathbf{Q}}^{p(\operatorname{Ker}\Phi^{\ell}_{g,m})}$ and $\mathbf{Q}^{\ell}_{C} := \overline{\mathbf{Q}}^{\operatorname{Ker}\varphi^{\ell}_{C}}$ are called $\ell$-monodromy fixed fields. It follows from the universality property of $\Phi^{\ell}_{g,m}$ represented by the equality $\Phi^{\ell}_{g,m} \circ s_C = \varphi^{\ell}_{C}$ that we have the inclusion $\mathbf{Q}^{\ell}_{g,m} \subset \mathbf{Q}^{\ell}_{C}$.

In the particular example of $(g,m) = (0,3)$ we have $\Phi^{\ell}_{g,m} = \varphi^{\ell}_{\mathbf{P}^1_{\mathbf{Q}}\setminus\{0,1,\infty\}}$ as $\mathcal{M}_{0,4}$ is isomorphic to $\mathbf{P}^1_{\mathbf{Q}} \setminus \{0,1,\infty\}$ as a scheme and $\mathcal{M}_{0,3}$ to $\operatorname{Spec}\mathbf{Q}$. It should be noted that the standard monodromy outer action $\varphi_{\mathbf{P}^1_{\mathbf{Q}}\setminus\{0,1,\infty\}} \colon G_{\mathbf{Q}} \to \widehat{F}_2$ is injective by Belyi's theorem while its pro-$\ell$ variant is not.

Understanding how the $\ell$-monodromy fixed fields $\mathbf{Q}^{\ell}_{g,m}$ $(g,m)$ vary according to $g$ and $m$ is the key question of Oda's problem. The result, which has been obtained through the work of many mathematicians, is as follows. The complete proof of independence has been obtained through different techniques such as computations with Lie algebras [Ma96], consideration of divisorial stratifications of the moduli spaces [IN97] and combinatorial anabelian geometry [HM11], we refer to [Tak14] for a survey.

**Theorem 1.1** (Hoshi, Ihara, Matsumoto, Mochizuki, Nakamura, Oda, Takao, Ueno)**.** *For all* $(g,m)$ *of hyperbolic type we have*

$$\mathbf{Q}^{\ell}_{g,m} = \mathbf{Q}^{\ell}_{0,3}.$$

For a survey of the different techniques involved in the proof we refer to [Tak14].

## 2. The introduction of cyclic special loci

To introduce the special loci, which are closed substacks of $\mathcal{M}_{g,[m]}$ we first have to consider the Hurwitz stacks for a finite group $G$. Consider the moduli space $\mathcal{M}_{g,[m]}[G]$ which for every scheme $S$ classifies the triples

$$(C/S, D, \iota \colon G \hookrightarrow \operatorname{Aut}_S C)$$

with $C$ a proper smooth curve of genus $g$ and $D$ a divisor on $C$ of degree $m$. By considering the maps given by forgetting the $G$-action

$$(C/S, D, \iota \colon G \hookrightarrow \operatorname{Aut}_S C) \longrightarrow (C/S, D)$$

functorially in $S$ we get a map of stacks $\mathcal{M}_{g,[m]}[G] \to \mathcal{M}_{g,[m]}$ which we call the forgetful functor. The image of this map is defined to be $G$-special loci $\mathcal{M}_{g,[m]}(G)$. These spaces have been studied by Collas and Maugeais in [CM15] in the case of cyclic groups $G$. They prove that the irreducible components of $\mathcal{M}_{g,[m]}(G)$ are geometrically irreducible and classified by a so called abstract

Hurwitz data noted $\underline{kr}$. We denote those components by $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$.

In [CP23] we show that there is a universal $\ell$-monodromy outer action $\Phi^{\ell}_{g,m}(G)_{\underline{kr}}$ as in the classical situation of Oda's problem. We thus define accordingly the $\ell$-monodromy fixed fields $\mathbf{Q}^{\ell}_{g,m}(G)_{\underline{kr}}$. We then ask, what we call Oda's problem for cyclic special loci, how do the $\ell$-monodromy fixed fields vary according to $G$, $g$, $m$ and the data $\underline{kr}$. We provide an answer in the case of cyclic groups of prime order.

**Theorem 2.1** ([CP23]). *For $G = \mathbf{Z}/\ell\mathbf{Z}$, $(g, m)$ of hyperbolic type and associated abstract Hurwitz data $\underline{kr}$ such that $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ is non-empty, we have*

$$\mathbf{Q}^{\ell}_{g,m}(G)_{\underline{kr}} = \mathbf{Q}^{\ell}_{0,3}.$$

The proof is done in two parts. First we consider the relations between the special loci and the spaces involved in the classical version of Oda's problem. The functors involved are the forgetful functor giving the inclusion $\mathcal{M}_{g,[m]}(G)_{\underline{kr}} \subset \mathcal{M}_{g,[m]}$ and the quotient functor $\mathcal{M}_{g,[m]}[G]/\operatorname{Aut} G \to \mathcal{M}_{g',[m']}$ for which we have to consider the intermediate space $\mathcal{M}_{g,[m]}[G]/\operatorname{Aut} G$ where the action of $\operatorname{Aut} G$ is made by composition with $\iota$ on the right on $\mathcal{M}_{g,[m]}[G]$. Studying both functors and the relations to the $\ell$-monodromy outer actions we obtain a diagram of field inclusions

$$\begin{array}{ccccc} \mathbf{Q}^{\ell}_{g,[m]} & \hookrightarrow & \mathbf{Q}^{\ell}_{g,[m]}(G)_{\underline{kr}} & \hookrightarrow & \mathbf{Q}^{\ell}_{g,[m]}[G]_{\underline{kr}} \\ \uparrow & & & \nearrow & \\ \mathbf{Q}^{\ell}_{0,[3]} & \hookrightarrow & \mathbf{Q}^{\ell}_{g',[m']} & & \end{array}$$

The second part of the proof consist of producing a curve $C$ represented on the moduli stack $\mathcal{M}_{g,[m]}(G)_{\underline{kr}}$ such that $\varphi^{\ell}_{C} = \mathbf{Q}^{\ell}_{0,[3]}$. By the universality this leads to the inclusion $\mathbf{Q}^{\ell}_{g,[m]}[G]_{\underline{kr}} \subset \mathbf{Q}^{\ell}_{0,[3]}$ which closes our diagram. This is done by the method of maximal degeneration, introduced by Ihara and Nakamura in [IN97], adapted to our situation.

## References


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