A Novel Approach to Proving Conserved Quantities in the Particle System of m neighbours and Its Application to Generate New Systems

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Abstract

This paper presents an approach to analytically and numerically examine the second conserved quantity in particle systems with m-neighbours, where the preservation of the first conserved quantity is ensured by the evolution equation. Leveraging this method, we introduce three new particle systems with 5-neighbours, akin to the one proposed by Endo and Takahashi. Additionally, we briefly discuss the symmetry of these systems, exploring its potential role in generating new systems.

1 Introduction

Endo and Takahashi introduced a novel particle system[1] with five neighbours, characterized by two conserved quantities in [2]. This particle system, as defined in this paper, was explored to determine the system's momentum. Endo and Takahashi rigorously proved the second conserved quantity through detailed analysis of the specific particle system.

In this work, we present an alternative and more generalized method for proving conserved quantities within similar particle systems. Our novel approach is capable of generating proofs for conserved quantities in various particle systems numerically, offering a versatile tool for computationally exploring systems with desired conserved quantities.

The evolution equation governing this system under periodic boundary conditions[2] is given by

$$u_j^{n+1} = u_j^n + q\left(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n\right) - q\left(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n\right), \tag{1}$$

where q is the flux [3] and all $u_j^n \in \{0,1\}$. Equation (1) ensures the preservation of the first conserved quantity, which is the total number of 1s in the system at time n (labeled $(\#1)^n$), as demonstrated by the following calculation:

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[1] Here, the term "particle system" refers to a system that both dependent variables u_j^n and independent variables (n,j) are discretized, where u_j^n takes binary values (0 or 1).

[2] If a system with periodic boundary conditions has a period of L, then we have $u_{L+1}^n = u_1^n$, $u_{L+2}^n = u_2^n$, and so on. Similarly, in the opposite direction, the following relationships hold: $u_{L-1}^n = u_{-1}^n$, $u_{L-2}^n = u_{-2}^n$, and so forth. All figures in this paper will only depict cells with indices 1 to L rather than the entire infinitely repeating system.

[3] In equation (1), $+q\left(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n\right)$ represents the flux into the cell with index j during the time n to $n+1$. Similarly, $-q\left(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n\right)$ denotes the flux going out of the cell.

$$
\begin{aligned}
(\#1)^{n+1} = \sum_{j=1}^{L} u_j^{n+1} &= \sum_{j=1}^{L} \left[u_j^n + q\left(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n\right) - q\left(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n\right)\right] \\
&= \left(\sum_{j=1}^{L} u_j^n\right) \\
&\quad + q\left(u_{-1}^n, u_0^n, u_1^n, u_2^n\right) - q\left(u_0^n, u_1^n, u_2^n, u_3^n\right) \\
&\quad + q\left(u_0^n, u_1^n, u_2^n, u_3^n\right) - q\left(u_1^n, u_2^n, u_3^n, u_4^n\right) \\
&\quad \cdots \\
&\quad + q\left(u_{L-2}^n, u_{L-1}^n, u_L^n, u_{L+1}^n\right) - q\left(u_{L-1}^n, u_L^n, u_{L+1}^n, u_{L+2}^n\right) \\
&= \left(\sum_{j=1}^{L} u_j^n\right) + q\left(u_{-1}^n, u_0^n, u_1^n, u_2^n\right) - q\left(u_{L-1}^n, u_L^n, u_{L+1}^n, u_{L+2}^n\right) \\
&= \sum_{j=1}^{L} u_j^n = (\#1)^n,
\end{aligned}
$$

where L represents the size or period of the system. The periodic boundary condition, denoted as $u_{-1}^n = u_{L-1}^n, u_0^n = u_L^n, \cdots$, ensures the validity of the last row in the above computation.

In their paper [2], Endo and Takahashi introduced a rule table for the flux variable q, as shown in Table 1.

(a,b,c,d)	1111	1110	1101	1100	1011	1010	1001	1000
$q(a,b,c,d)$	1	1	1	1	0	0	0	0
(a,b,c,d)	0111	0110	0101	0100	0011	0010	0001	0000
$q(a,b,c,d)$	0	1	0	0	0	0	0	0

Table 1: Rule set for flux q in [2]

If we initialize the system with a random condition of density 0.5 as in [2], signifying that half of the system is set to 0 and the other half to 1, the resulting time evolution is depicted in Figure 1.

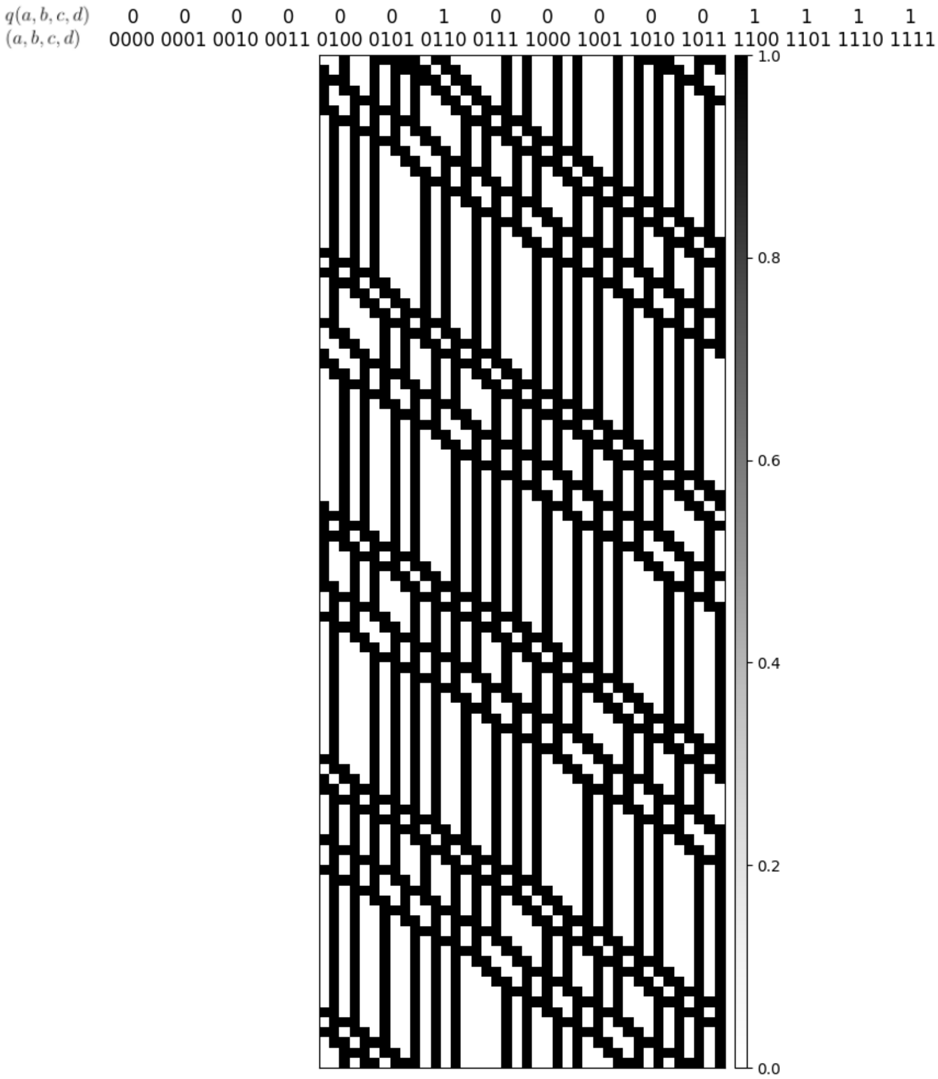

Figure 1: The Time evolution of the particle system described in [2] with period $L = 40$

In the subsequent section, we will provide a comprehensive proof that the system, governed by the evolution equation (1) and the specific flux q proposed by Endo et al., possesses a second conserved quantity.

2 Proof of the second conserved quantity #011

In the study by Endo and Takahashi [2], the preservation of the pattern 011 is demonstrated through an analysis of pattern reproduction within a specific particle system. We, alternatively, establish the conservation of the quantity #011 through arithmetic calculations. Importantly, this method can be generally applied to systems with different rule tables for flux q.

To accomplish our goal, we begin by defining a mathematical expression for the total number of the pattern 011 in a system at time n, denoted as $(\#011)^n$. We propose the expression:

$$(\#011)^n = \sum_{j=1}^{L} (1 - u_j^n) u_{j+1}^n u_{j+2}^n, \tag{2}$$

for a system with a period of L. The term $(1-u_j^n)u_{j+1}^n u_{j+2}^n$ evaluates to 1 when $(u_j^n, u_{j+1}^n, u_{j+2}^n) = (0, 1, 1)$, indicating the presence of the pattern 011. Otherwise, it evaluates to 0. By applying this test to each position j from 1 to L and summing the results, we obtain the total count of #011 at time n.

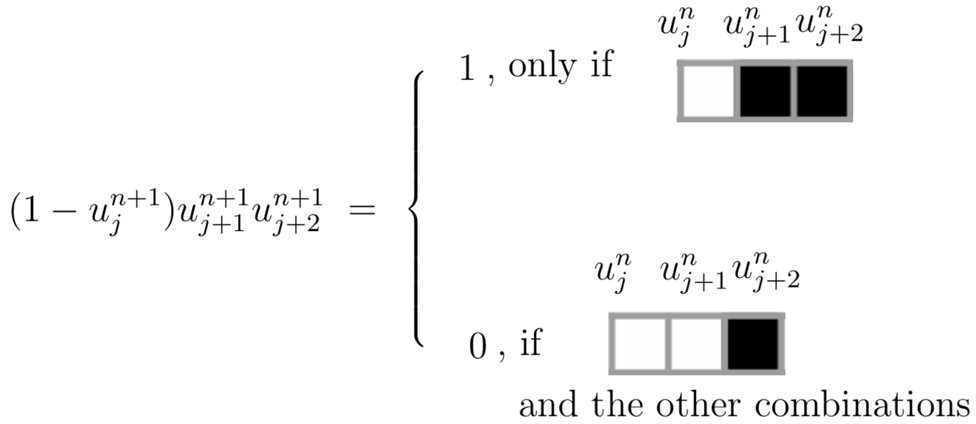

Figure 2: $(1 - u_j^n) u_{j+1}^n u_{j+2}^n$ will give 1 if $(u_j^n, u_{j+1}^n, u_{j+2}^n) = (0, 1, 1)$. Otherwise it will give 0.

We will establish the conservation of #011 by demonstrating that $(\#011)^n = (\#011)^{n+1}$, representing the count of the pattern 011 at time n. To begin, we substitute the time evolution equation into $(\#011)^{n+1}$.

$$\begin{aligned}
(\#011)^{n+1} &= \sum_{j=1}^{L} (1 - u_j^{n+1}) u_{j+1}^{n+1} u_{j+2}^{n+1} \\
&= \sum_{j=1}^{L} \left[1 - u_j^n - q\left(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n\right) + q\left(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n\right)\right] \\
&\qquad \cdot \left[u_{j+1}^n + q\left(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n\right) - q\left(u_j^n, u_{j+1}^n, u_{j+2}^n, u_{j+3}^n\right)\right] \\
&\qquad \cdot \left[u_{j+2}^n + q\left(u_j^n, u_{j+1}^n, u_{j+2}^n, u_{j+3}^n\right) - q\left(u_{j+1}^n, u_{j+2}^n, u_{j+3}^n, u_{j+4}^n\right)\right] \\
&= \sum_{j=1}^{L} \left[1 - u_j^n - q\left(u_{j-2}^n, \cdot\right) + q\left(u_{j-1}^n, \cdot\right)\right] \left[u_{j+1}^n + q\left(u_{j-1}^n, \cdot\right) - q\left(u_j^n, \cdot\right)\right], \qquad (3)
\end{aligned}$$

where $q\left(u_{j-2}^n, \cdot\right)$ is the abbreviation of $q\left(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n\right)$.

The expansion of (3) can be intricate, but we can simplify it by categorizing the terms based on the following propositions. By applying these propositions, we can eliminate other terms, leaving only $\sum_{j=1}^{L} (1 - u_j^n) u_{j+1}^n u_{j+2}^n = (\#011)^n$.

Proposition 1 *For any given* $q(u_j^n, \cdot) \in \{0, 1\}$, $q^2(u_j^n, \cdot) = q(u_j^n, \cdot)$.

Proof: Trivial. $\square$

Proposition 2 *The following three identities holds:*

1. $\sum_{j=1}^{L} q(u_{j-1}^n,\cdot)q(u_j^n,\cdot)q(u_{j+1}^n,\cdot) = \sum_{j=1}^{L} q(u_{j-2}^n,\cdot)q(u_{j-1}^n,\cdot)q(u_j^n,\cdot)$,
2. $\sum_{j=1}^{L} q(u_{j-2}^n,\cdot)q(u_j^n,\cdot)q(u_{j+1}^n,\cdot) = \sum_{j=1}^{L} q(u_{j-2}^n,\cdot)q(u_{j-1}^n,\cdot)q(u_{j+1}^n,\cdot)$,
3. $\sum_{j=1}^{L} q^2\left(u_j^n,\cdot\right) q\left(u_{j-2}^n,\cdot\right) = \sum_{j=1}^{L} q\left(u_{j-1}^n,\cdot\right) q\left(u_{j+1}^n\right)$.

Proof: We will now prove the first identity

We can express the LHS in terms of the RHS as follows:

$$\begin{aligned}\sum_{j=1}^{L} q\left(u_{j-1}^n,\cdot\right) q\left(u_j^n,\cdot\right) q\left(u_{j+1}^n,\cdot\right) &= -q\left(u_{-1}^n,\cdot\right) q\left(u_0^n,\cdot\right) q\left(u_1^n,\cdot\right) + \sum_{j=1}^{L} q\left(u_{j-2}^n,\cdot\right) q\left(u_{j-1}^n,\cdot\right) q\left(u_j^n,\cdot\right) \\ &\quad + q\left(u_{L-1}^n,\cdot\right) q\left(u_L^n,\cdot\right) q\left(u_{L+1}^n,\cdot\right) \\ &= \sum_{j=1}^{L} q\left(u_{j-2}^n,\cdot\right) q\left(u_{j-1}^n,\cdot\right) q\left(u_j^n,\cdot\right),\end{aligned}$$

where the periodic boundary condition ensures $q\left(u_{-1}^n,\cdot\right) = q\left(u_{L-1}^n,\cdot\right), q\left(u_0^n,\cdot\right) = q\left(u_L^n,\cdot\right)$ and $q\left(u_1^n,\cdot\right) = q\left(u_{L+1}^n,\cdot\right)$.

The second and third identities can be proven in a similar manner. $\square$

Proposition 3 *In this system, the following identity holds:* $q(u_{j-1}^n,\cdot)[1-q(u_j^n,\cdot)-u_j^n+u_{j+1}^n] = 0$.

Proof: We begin by assuming $q(u_{j-1}^n,\cdot) = 1$, as it is trivially true when $q(u_{j-1}^n,\cdot) = 0$. To satisfy $q(u_{j-1}^n,\cdot) = 1$, the values of $(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n)$ must be one of the following: $(1,1,1,1)$, $(1,1,1,0)$, $(1,1,0,1)$, $(1,1,0,0)$, or $(0,1,1,0)$. We will discuss each case separately:

- Case 1: For $(1,1,1,1)$, we have $[1-q(u_j^n,\cdot)-u_j^n+u_{j+1}^n] = [1-q(1,1,1,u_{j+3}^n)-1+1]$. Since $q(1,1,1,u_{j+3}^n) = 0$ according to the flux $q(a,b,c,d)$ table, we have proved that $q(u_{j-1}^n,\cdot)[1-q(u_j^n,\cdot)-u_j^n+u_{j+1}^n] = 0$.

- Case 2: Since $(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n) = (1,1,1,0)$, we will have $[1-q(u_j^n,\cdot)-u_j^n+u_{j+1}^n] = [1-q(1,1,0,u_{j+3}^n)-1+1]$. The following proof is similar to Case 1.

- Case 3: For $(1,1,0,1)$, we have $[1-q(u_j^n,\cdot)-u_j^n+u_{j+1}^n] = [1-q(1,0,0,u_{j+3}^n)-1+0] = -q(1,0,0,u_{j+3}^n)$. From the table, we know $q(1,0,0,u_{j+3}^n) = 0$. Thus, we have completed the proof under this condition.

- Case 4: For $(1,1,0,0)$, the proof is the same as Case 3.

- Case 5: For $(0,1,1,0)$, the proof is similar to Case 2.

$\square$

Proposition 4 *In this system, the following identity holds:* $q(u_{j-1}^n,\cdot)(u_j^n-1) = 0$.

Proof: We consider only the case where $q(u^n_{j-1}, \cdot) = 1$, as the statement is trivially true otherwise.

From the table of q, when $q(u^n_{j-1}, u^n_j, u^n_{j+1}, u^n_{j+2}) = 1$, several possibilities exist: $(1, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 1)$, $(1, 1, 0, 0)$, or $(0, 1, 1, 0)$. In each of these cases, $u^n_j = 1$. Consequently, $u^n_j - 1 = 0$. □

Proposition 5 *In this specific particle system, the expression for the flux q can be written as $q(a, b, c, d) = ab + (1 - a)bc(1 - d)$ when $a, b, c, d \in \{0, 1\}$.*

Proof: The term ab encompasses all possible scenarios of $q(1, 1, x, x)$, while $(1 - a)bc(1 - d)$ specifically accounts for the situation represented by $q(0, 1, 1, 0)$. □

Proposition 6 *In this system, the following identity holds:* $\sum_j q(u^n_{j-1}, \cdot)[q(u^n_{j+1}, \cdot)(q(u^n_{j+2}, \cdot) - 1) - u^n_{j+2}(q(u^n_{j-2}, \cdot) - 1)] = \sum_j u^n_{j-1}u^n_j u^n_{j+1}u^n_{j+2}(u^n_{j+3} - u^n_{j-2}) = 0$ [4].

Proof: We begin by applying Proposition 5 to $LHS = \sum_j q(u^n_{j-1}, \cdot)[q(u^n_{j+1}, \cdot)(q(u^n_{j+2}, \cdot) - 1) - u^n_{j+2}(q(u^n_{j-2}, \cdot) - 1)] - \sum_j u^n_{j-1}u^n_j u^n_{j+1}u^n_{j+2}(u^n_{j+3} - u^n_{j-2})$. This allows us to express it without q, resulting in the expanded form:

$$
\begin{aligned}
LHS = &\\
&(u^n_{j-1}u^n_j + (1 - u^n_{j-1})u^n_j u^n_{j+1}(1 - u^n_{j+2}))\cdot \\
&\big[(u^n_{j+1}u^n_{j+2} + (1 - u^n_{j+1})u^n_{j+2}u^n_{j+3}(1 - u^n_{j+4}))(u^n_{j+2}u^n_{j+3} + \\
&(1 - u^n_{j+2})u^n_{j+3}u^n_{j+4}(1 - u^n_{j+5}) - 1) - u^n_{j+2}(u^n_{j-2}u^n_{j-1} + (1 - u^n_{j-2})u^n_{j-1}u^n_j(1 - u^n_{j+1}) - 1)\big].
\end{aligned}
$$

Upon expansion, most terms cancel each other, leaving only $u^n_{j-1}u^n_j u^n_{j+1}u^n_{j+2}u^n_{j+3}$ $-u^n_{j-2}u^n_{j-1}u^n_j u^n_{j+1}u^n_{j+2}$. Due to the boundary condition, $\sum_j u^n_{j-1}u^n_j u^n_{j+1}u^n_{j+2}u^n_{j+3}$ $-u^n_{j-2}u^n_{j-1}u^n_j u^n_{j+1}u^n_{j+2} = 0$ by Proposition 2. □

Proposition 7 *In this system, the following identity holds:* $\sum_j u^n_{j+1}[q(u^n_{j-2}, \cdot)(q(u^n_{j+1}, \cdot) - u^n_{j+2}) + q(u^n_{j+1}, \cdot)(u^n_j - q(u^n_{j-1}, \cdot) - 1 + q(u^n_j, \cdot))] = 0$.

Proof: Similar to Proposition 6. □

This method can be extended to prove other conserved quantities in similar systems. To do this, one needs to identify a similar expression for the desired conserved quantity (e.g., $\sum_{j=1}^{L} u^n_j u^n_{j+1}u^n_{j+2}$ for #111, $\sum_{j=1}^{L}(1 - u^n_j)u^n_{j+1}u^n_{j+2}(1 - u^n_{j+3})$ for #1001, etc.). Then, by either discovering identities or directly substituting the polynomial expression of flux q into the calculation of $(\#011)^{n+1}$, one can establish the conserved quantity in the system.

3 New systems with second conserved quantity #011

In this section, our objective is to generate additional systems that exhibit similar conserved quantities. To achieve this, we establish specific conditions for the systems under consideration.

1. The values of u^n_j in the system must satisfy $u^n_j \in \{0, 1\}$.
2. The system should possess the first conserved quantity, denoted as #1.

[4] Meanwhile, $q(u^n_{j-1}, \cdot)[q(u^n_{j+1}, \cdot)(q(u^n_{j+2}, \cdot) - 1) - u^n_{j+2}(q(u^n_{j-2}, \cdot) - 1)] = 0$, the case where the LHS lacks summation, is not guaranteed.

3. The system should also exhibit the second conserved quantity, represented by #011.

The satisfaction of the second condition has been demonstrated in Section 1, due to the evolution equation. However, determining how to fulfil the first condition remains an open question. In our computational approach, we exclude systems that numerically violate the first requirement.

To meet the third condition, we apply the methodology outlined in Section 2 to numerically generate additional systems featuring the conserved quantity #011. Following the approach introduced by Endo and Takahashi (2022) [2], we initiate the process with a non-trivial initial condition, ensuring that #011 is non-zero. Subsequently, we calculate the occurrence of the pattern 011 at each time step. Our focus lies in identifying systems where #011 remains invariant throughout the evolution. Due to numerical limitations, we cannot conduct infinite evolutions and checks for #011. Therefore, for each system, we perform checks over 100 time evolutions.

Our computational analysis reveals the existence of 22 systems, out of a total of 2^{16} possibilities, that exhibit the conserved quantity #011. This section will present four non-trivial cases, with the complete list provided in our future paper.

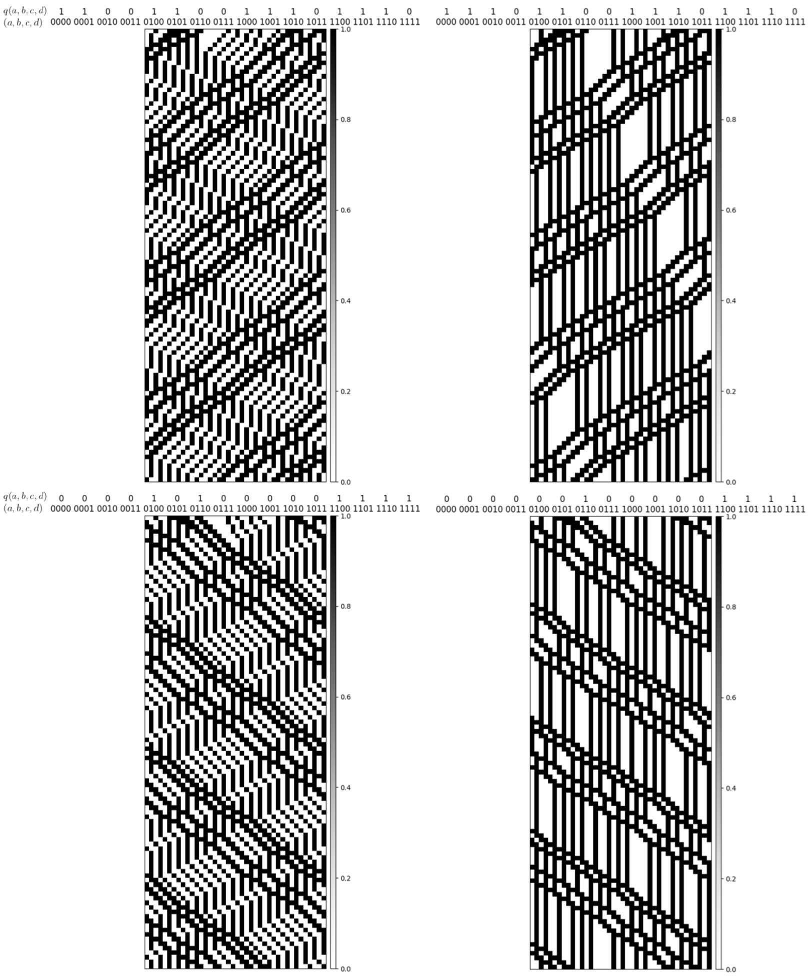

Figure 3: Four non-trivial systems exhibiting conserved quantity #011

It is evident that the system at the top and the one at the bottom share notable similarities.

4 Symmetry of the system with 5-neighbours

We observed that the particle systems discussed in Section 3 possess corresponding symmetric versions.

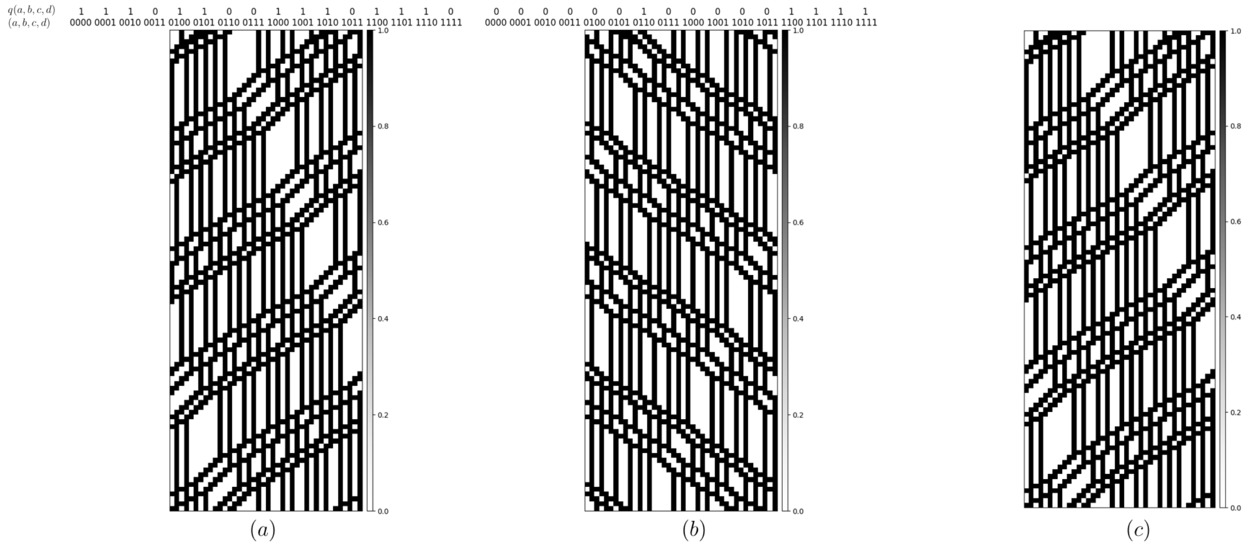

Figure 4: Comparison of particle systems in section 3: reversing the initial condition and mirroring the time evolution of (b) recreates (a) as (c).

By reversing the initial condition and evolving the system, we obtain a mirrored version of another 5-neighbours system.

This symmetry can be explained as follows: if we define the evolution equation of a system as $u_j^{n+1} = u_j^n + q\left(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n\right) - q\left(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n\right)$, then the corresponding symmetric system will have the evolution equation as

$$\begin{aligned} u_j^{n+1} &= u_j^n + \left[1 - q\left(u_{j+1}^n, u_j^n, u_{j-1}^n, u_{j-2}^n\right)\right] - \left[1 - q\left(u_{j+2}^n, u_{j+1}^n, u_j^n, u_{j-1}^n\right)\right] \\ &= u_j^n + \tilde{q}\left(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n\right) - \tilde{q}\left(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n\right), \end{aligned} \tag{4}$$

where $\tilde{q}\left(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n\right) = 1 - q\left(u_{j+1}^n, u_j^n, u_{j-1}^n, u_{j-2}^n\right)$. Hence, the symmetric system shares the same structure of the evolution equation as the original one.

We can represent the flux q in matrix form to facilitate the calculation of the rule for a symmetric system. If we define the matrix Q as

$$Q = \begin{pmatrix} q(1,1,1,1) & q(1,1,1,0) & q(1,1,0,1) & q(1,1,0,0) \\ q(0,1,1,1) & q(0,1,1,0) & q(0,1,0,1) & q(0,1,0,0) \\ q(1,0,1,1) & q(1,0,1,0) & q(1,0,0,1) & q(1,0,0,0) \\ q(0,0,1,1) & q(0,0,1,0) & q(0,0,0,1) & q(0,0,0,0) \end{pmatrix},$$

then the flux $\tilde{q}$ for the symmetric system will be the transpose of Q subtracted by 1:

$$\tilde{Q} = 1 - Q^t = \begin{pmatrix} 1-q(1,1,1,1) & 1-q(0,1,1,1) & 1-q(1,0,1,1) & 1-q(0,0,1,1) \\ 1-q(1,1,1,0) & 1-q(0,1,1,0) & 1-q(1,0,1,0) & 1-q(0,0,1,0) \\ 1-q(1,1,0,1) & 1-q(0,1,0,1) & 1-q(1,0,0,1) & 1-q(0,0,0,1) \\ 1-q(1,1,0,0) & 1-q(0,1,0,0) & 1-q(1,0,0,0) & 1-q(0,0,0,0) \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{q}(1,1,1,1) & \tilde{q}(1,1,1,0) & \tilde{q}(1,1,0,1) & \tilde{q}(1,1,0,0) \\ \tilde{q}(0,1,1,1) & \tilde{q}(0,1,1,0) & \tilde{q}(0,1,0,1) & \tilde{q}(0,1,0,0) \\ \tilde{q}(1,0,1,1) & \tilde{q}(1,0,1,0) & \tilde{q}(1,0,0,1) & \tilde{q}(1,0,0,0) \\ \tilde{q}(0,0,1,1) & \tilde{q}(0,0,1,0) & \tilde{q}(0,0,0,1) & \tilde{q}(0,0,0,0) \end{pmatrix}.$$

In other words, we have the new symmetric rule table for the flux $\tilde{q}$.

We will illustrate the process of constructing symmetric systems using a previous particle system as an example. If we consider system (a) in Figure 4, the matrix Q takes the form

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Hence Q^t and $\tilde{Q}$ are

$$Q^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$
$$\tilde{Q} = 1 - Q^t = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can represent the rule for the new system in table format:

(a,b,c,d)	1111	1110	1101	1100	1011	1010	1001	1000
$\tilde{q}(a,b,c,d)$	1	1	1	1	0	0	0	0
(a,b,c,d)	0111	0110	0101	0100	0011	0010	0001	0000
$\tilde{q}(a,b,c,d)$	0	1	0	0	0	0	0	0

Table 2: Rule set for $\tilde{q}$ in the symmetric system

Table 2 exactly represents the rule of system (b) in Figure 4, which is the symmetric version of system (a).

It is crucial to emphasize that this symmetry phenomenon is not limited to the four systems in Figure 3 but extends to the other 22 newly discovered systems. This is because the new evolution equation (4) is guaranteed for any system that shares the same evolution equation.

5 Conclusion

In conclusion, this paper successfully employs a general method to establish the existence of a second conserved quantity in a particle system with 5 neighbours, as proposed by Endo et al. [2]. Furthermore, we leverage this method to numerically generate additional 5-neighbours systems exhibiting the same second conserved quantity. The final section of our work delves into a discussion on the symmetry exhibited by systems governed by this specific evolution equation.

Moving forward, the methods outlined in Sections 3 and 4 can be applied to systematically generate particle systems with m-neighbours while ensuring desired conserved quantities. This involves combining the evolution equation as described:

$$u_j^{n+1} = u_j^n + q\underbrace{\left(u_{j-2}^n, u_{j-1}^n, \cdots, u_{j+m-4}^n, u_{j+m-3}^n\right)}_{m-1 \text{ terms}} - q\underbrace{\left(u_{j-1}^n, u_j^n, \cdots, u_{j+m-3}^n, u_{j+m-2}^n\right)}_{m-1 \text{ terms}}$$

Additionally, it is of significant interest to explore the relationship between the number of neighbours and the number of conserved quantities by generating more $m-$neighbours systems. Notably, we observe that a 5-neighbours system can possess 2 conserved quantities, while the ultradiscrete KdV system, characterized as an ∞-neighbours system, exhibits an infinite number of conserved quantities. This observation prompts further investigation into the intriguing correlation between the system's neighbour count and the extent of conserved quantities.

Finally, the meaning of the newly discovered identities presented in Endo et al.'s system, as listed in the propositions, remains unexplored. Investigating how these propositions connect with Endo et al.'s pattern analysis represents an intriguing avenue for further exploration.

References

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