On some analytic properties of a function associated with the Selberg class satisfying certain special conditions

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Abstract

In 2001, M.Rekoś described the analytic behavior of a function connected with the Euler totient function for the upper half-plane $\mathbb{H}$. In this paper, for Im $z > 0$ we describe the analytic behavior of the generalized function $f(z, F)$, where the function F belongs to a subclass of the Selberg class which has a polynomial Euler product expressions and satisfies some special conditions.

1 Introduction

1.1 Previous research for the Euler totient function

For $n \in \mathbb{N}$, let $\varphi(n)$ be the number of positive integers not exceeding n which are relatively prime to n. The function $\varphi(n)$ is called the *Euler totient function* and appears in various fields e.g. elementary number theory, group theory. In analytic number theory, studying the arithmetic mean of $\varphi(n)$ is a classical problem. Let

$$E(x) = \sum_{n \le x} \varphi(n) - \frac{3}{\pi^2} x^2 \tag{1.1.1}$$

be the associated error term. The error term (1.1.1) has been studied for a long time. P. G. Dirichlet proved the estimate

$$E(x) \ll x^{1+\epsilon} \tag{1.1.2}$$

for every positive ϵ. Here, we use the notation $f(x) \ll g(x)$, if there is a constant $C > 0$ such that $|f(x)| \le Cg(x)$ for all x in the appropriate range. The estimate (1.1.2) was improved by A. Walfisz to

$$E(x) \ll x(\log x)^{\frac{2}{3}} (\log\log x)^{\frac{4}{3}} \tag{1.1.3}$$

(see [15]). The estimate (1.1.3) is the best known result. H.L.Montgomary proved the best Ω-result for (1.1.1)

$$E(x) = \Omega_{\pm}(x\sqrt{\log\log x}) \tag{1.1.4}$$

(see [10]). Here, we use the notation $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ i.e. the inequalities $f(x) > Cg(x)$ and $f(x) < -Cg(x)$ hold respectively for some arbitrarily large values of x and a suitable positive constant C. Also, we use the notation $f(x) = \Omega_{\pm}(g(x))$ i.e.both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold.J.Kaczorowski and K.Wiertelak also studied (1.1.1) by

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splitting into two summands. J.Kaczorowski and K.Wiertelak considered there the following Volterra integral equation of second type for (1.1.1) (see [6]) :

$$F(x) - \int_0^\infty K(x,t)F(t)dt = E(x) \quad (x \geq 1), \tag{1.1.5}$$

where $F(x)$ is the unknown function and the kernel $K(x,t)$ is defined as follows:

$$K(x,t) = \begin{cases} 1/t & (0 < t \leq x), \\ 0 & (1 \leq x < t). \end{cases} \tag{1.1.6}$$

The equation (1.1.5) can be solved explicitly. Let us put

$$f(x) = -\sum_{n=1}^\infty \frac{\mu(n)}{n} \left\{\frac{x}{n}\right\} \tag{1.1.7}$$

for every $x \geq 0$, where $\mu(n)$ denotes the *Möbius function* i.e.

$$\mu(n) = \begin{cases} 1 & (n = 1), \\ (-1)^r & (n = p_1 p_2 \cdots p_r, p_i \ (1 \leq i \leq r) : \text{prime}), \\ 0 & (\text{otherwise}), \end{cases}$$

and $\{x\} = x - [x]$ is the fractional part of a real number x.

Theorem 1.1.1 (Theorem 1.1 in [6])**.** *The general solution of* (1.1.6) *is*

$$F(x) = (f(x) + A)x, \tag{1.1.8}$$

where A is an arbitrary constant.

In [6], $F(x) = xf(x)$ is claimed to be the unique solution of the integral equation (1.1.5), but this uniqueness does not hold even assuming the initial value condition at $x = 0$. Probably, the term Ax is missing to give the general solution (see [2]). For $x \geq 0$ let us write

$$g(x) = \sum_{n=1}^\infty \mu(n) \left\{\frac{x}{n}\right\}^2. \tag{1.1.9}$$

Theorem 1.1.2 (Theorem 1.2 in [6])**.** *For $x \geq 1$ we have*

$$E(x) = xf(x) + \frac{1}{2}g(x) + \frac{1}{2}.$$

According to Theorem1.1.2, for $x \geq 1$ we can split $E(x)$ as follows :

$$E(x) = E^{\mathrm{AR}}(x) + E^{\mathrm{AN}}(x), \tag{1.1.10}$$

where

$$E^{\mathrm{AR}}(x) = xf(x), \quad \text{and} \quad E^{\mathrm{AN}}(x) = \frac{1}{2}g(x) + \frac{1}{2}$$

with $f(x)$ and $g(x)$ given by (1.1.7) and (1.1.9) respectively. We call $E^{\mathrm{AR}}(x)$ and $E^{\mathrm{AN}}(x)$ the *arithmetic* and the *analytic part* of $E(x)$ respectively. J.Kaczorowski and K.Wiertelak proved the Ω-estimates for $E^{\mathrm{AR}}(x)$ and $E^{\mathrm{AN}}(x)$ (see [6]).

1.2 The associated he Euler totient function

J.Kaczorowski defined the associated Euler totient function for a class of generalized L-functions including the Riemann zeta function, Dirichlet L-functions and obtained an asymptotic formula (see [8]) : By a polynomial Euler product expressions we mean a function $F(s)$ of a complex variable $s = \sigma + it$ which for $\sigma > 1$ is defined by a Euler product expressions of the form

$$F(s) = \prod_p F_p(s) = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}, \tag{1.2.1}$$

where p runs over primes and $|\alpha_j(p)| \le 1$ for all p and $1 \le j \le d$. We take the smallest $d \in \mathbb{Z}_{>0}$ such that there is at least one prime number p_0 satisfying

$$\prod_{j=1}^{d} \alpha_j(p_0) \neq 0,$$

where d is called the *Euler degree* of F. Note that the L-functions from number theory including the Riemann zeta function $\zeta(s)$ and the Dirichlet L-functions $L(s, \chi)$ and Dedekind zeta and Hecke L-functions of algebraic number fields, also the (normalized) L-functions of holomorphic modular forms and, conjecturally, many other L-functions are polynomial Euler products expressions. For F in (1.2.1) we define *the associated Euler totient function* as follows :

$$\varphi(n, F) = n \prod_{p|n} F_p(1)^{-1} \quad (n \in \mathbb{N}). \tag{1.2.2}$$

Let

$$\gamma(p) = p\left(1 - \frac{1}{F_p(1)}\right), \tag{1.2.3}$$

$$C(F) = \frac{1}{2} \prod_p \left(1 - \frac{\gamma(p)}{p^2}\right), \tag{1.2.4}$$

and

$$\alpha(n) = \mu(n) \prod_{p|n} \gamma(p). \tag{1.2.5}$$

By (1.2.1) and (1.2.2), we see that $\varphi(n)$ corresponds to the case where F is $\zeta(s)$.

Theorem 1.2.1 (Theorem 1.1 in [8]). *For a polynomial Euler product F of degree d and $x \ge 1$ we have*

$$\sum_{n \le x} \varphi(n, F) = C(F)x^2 + O(x(\log 2x)^d). \tag{1.2.6}$$

Remark 1.2.1. Let us observe that $\alpha(n) \ll n^\epsilon$ for every positive ϵ. Hence the series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} \tag{1.2.7}$$

absolutely converges for $\sigma > 1$ (see p33 in [8]). Also, $\alpha(n)$ is multiplicative by (1.2.5). Therefore,

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^2} = 2C(F). \tag{1.2.8}$$

We now provide the *Selberg class* $\mathcal{S}$ defined as follows (see [11]) : $f \in \mathcal{S}$ if

(i) (*ordinary Dirichlet series*) $f(s) = \sum_{n=1}^{\infty} a_f(n) n^{-s}$, converges absolutely for $\sigma > 1$;

(ii) (*analytic continuation*) there exists $m \in \mathbb{Z}_{\geq 0}$ such that $(s-1)^m f(s)$ is entire of finite order;

(iii) (*functional equation*) $f(s)$ satisfies a functional equation of the form $\Phi(s) = \omega \overline{\Phi(1-\overline{s})}$, where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) f(s) = \gamma(s) f(s), \tag{1.2.9}$$

say, with $r \geq 0, Q > 0, \lambda_j > 0$, Re $\mu_j \geq 0$ and $|\omega| = 1$;

(iv) (*Ramanujan conjecture*) $a_f(n) \ll n^{\epsilon}$ holds for any $\epsilon > 0$;

(v) (*Euler product*) $f(s) = \prod_p \exp\left(\sum_{\ell=0}^{\infty} \frac{b_f(p^\ell)}{p^{\ell s}}\right)$ for $\sigma > 1$, where $b_f(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_f(n) \ll n^{\vartheta}$ for some $\vartheta < \frac{1}{2}$.

The empty products are hereafter to be equal to 1.

1.3 M.Rekoś's results

M.Rekoś described the analytic property of some function connected with the Euler totient function (see [12]) : We describe basic analytic properties of the function $f(z)$ defined for Im $z > 0$ as follows :

$$f(z) = \lim_{n\to\infty} \sum_{\substack{\rho \\ 0<\operatorname{Im}\rho<T_n}} \frac{e^{\rho z}\zeta(\rho-1)}{\zeta'(\rho)}, \tag{1.3.1}$$

where $\{T_n\}$ denotes a real sequence yielding appropriate groupings of the zeros, and the summation is over non-trivial zeros of $\zeta(s)$ with positive imaginary part. For simplicity we assume here that all the zeros of $\zeta(s)$ are simple. M.Rekoś showed the holomorphy of $f(z)$ for $\operatorname{Im} z > 0$, meromorphic continuation to the whole z-plane with its principal part, and a functional relation containing its reflection property. The functional equation for $f(z)$ connects the values of the function f at the points z and $\bar{z}$. Define a smooth curve $\tau : [0,1] \ni t \mapsto \tau(t) \in \mathbb{C}$ such that $\tau(0) = -1/4$, $\tau(1) = 5/2$ and $0 < \operatorname{Im} \tau(t) < 1$ for $t \in (0,1)$, and define it by $\ell(-1/4, 5/2)$. The analytic property of $f(z)$ is described by the following theorems :

Theorem 1.3.1 (Theorem 1.in [12]). *The function $f(z)$ is holomorphic on $\mathbb{H}$ and for $z \in \mathbb{H}$ we have*

$$2\pi i f(z) = f_1(z) + f_2(z) - e^{5z/2} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{5/2}(z - \log n)}, \tag{1.3.2}$$

where the functions

$$f_1(z) = \int_{-1/4+i\infty}^{-1/4} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds, \tag{1.3.3}$$

$$f_2(z) = \int_{\ell(-1/4,5/2)} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds \tag{1.3.4}$$

are holomorphic on $\mathbb{H}$ and on the whole z-plane respectively, and the last term is a meromorphic function on the whole z-plane, whose poles are at $z = \log n$ of the second order with the residues $-\varphi(n)/2\pi i$ $(n = 1, 2, \ldots)$. Here $\varphi(n)$ denotes Euler's totient function.

Theorem 1.3.2 (Theorem 2.in [12]). *The function $f(z)$ can be continued analytically to a meromorphic function on the whole z-plane, which satisfies the functional equation*

$$f(z) + \overline{f(\bar{z})} = B(z) \tag{1.3.5}$$

and

$$B(z) = -\frac{6}{\pi^2}e^{2z} + \frac{1}{2\pi^2}\sum_{k,n=1}^{\infty}\frac{\mu(k)}{n^2k}\left[\frac{1}{(nke^z-1)^2} + \frac{2}{nke^z-1} + \frac{1}{(nke^z+1)^2} - \frac{2}{nke^z+1}\right], \tag{1.3.6}$$

where $B(z)$ is a meromorphic function on the whole z-plane with the poles of the second order at $z = -\log nk$, $n, k = 1, 2, \ldots$ and $\mu(k)$ is the Möbius function.

2 The extension of $f(z)$ to the subclass of $\mathcal{S}$

2.1 The extension of $f(z)$ to the subclass of $\mathcal{S}$

If a functions $F \in \mathcal{S}$ has a polynomial Euler product expressions (1.2.1), the subclass of $\mathcal{S}$ of the functions with polynomial Euler product expressions is denoted by $\mathcal{S}^{\text{poly}}$. Establishing the results which extend Theorem1.3.1 and 1.3.2 to the function $F \in \mathcal{S}^{\text{poly}}$ are the main aim of the present paper. Let $\{\rho\}$ denote the non-trivial zeros of F with positive imaginary parts, and assume that each ρ is simple. Moreover, let $\{T_n\}$ denote a real sequence yielding appropriate groupings of the zeros, where its precise definition is to be given by (2.2.1) below. For Im $z > 0$ and $F \in \mathcal{S}^{\text{poly}}$, we consider the function defined by

$$f(z, F) = \lim_{n\to\infty} \sum_{\substack{\rho \\ 0<\operatorname{Im}\rho<T_n}} \frac{e^{\rho z}\zeta(\rho-1)}{F'(\rho)}. \tag{2.1.1}$$

If there are trivial zeros of $F(s)$ on the imaginary axis in $\mathbb{H}$, they are to be incorporated in the summation. The reason why $\zeta(s)$ appears in the numerator on the right hand side is that the Barnes type integral (5.1.3) below for the Whittaker function can then be applied under the hypothesis $(r, \lambda_j) = (1, 1)$ for all j in (1.2.9) (see Lemma 5.1.1).

Fact 2.1.1. *The limit in* (2.1.1) *exists for all $z \in \mathbb{H}$.*

2.2 Proof of Fact 2.1.1

We prove Fact 2.1.1, for which the following Lemma is used.

Lemma 2.2.1 (Lemma 4.in [14]). *Let $F \in \mathcal{S}$ and let T be sufficiently large, and fix $H = D\log\log T$ with a large constant $D > 0$. We take any subinterval $[n, n+1]$ with n chosen such that $n \in \mathbb{Z}_{>0} \cap [T-H, T+H]$. Then, there are the lines $t = t_0$ such that*

$$|F(\sigma + it_0)|^{-1} = O(\exp(C(\log T)^2)), \tag{2.2.1}$$

uniformly in $\sigma \geq -2$, where C is a positive constant.

Let T be sufficiently large. We fix $H = D\log\log T$, where D is a large positive constant. We take any subinterval $[n, n+1]$, where n is a positive integer in $[T-H, T+H]$. Then, by Lemma 2.2.1 there are the lines $t = T_n$ such that

$$|F(\sigma + iT_n)|^{-1} = O(\exp(C_1(\log T)^2)) \tag{2.2.2}$$

uniformly for $\sigma \geq -2$, where C_1 is a positive constant. Since T_n is contained in the interval $[T-H, T+H]$, we can see that $T_n \sim T$ as n tends to infinity. Let $\alpha = \frac{1}{2}\min\{\mathrm{Im}\,\rho > 0\}$ and $\mathscr{L}$ denote the contour consisting of the line segments

$$[b, b+iT_n],\ [b+iT_n, a+iT_n],\ [a+iT_n, a],\ \left[a, \frac{a+b}{2}+i\alpha\right],\ \left[\frac{a+b}{2}+i\alpha, b\right],$$

where $\max\left\{-\frac{3}{2}, \frac{1}{2}\max\{\mathrm{Re}\,\rho < 0\}\right\} < a < 0, b > 5/2$.We assume that the real part of $s = a+it\ (t \in \mathbb{R})$ does not coincide the poles of $\Gamma(s+\mu)\Gamma(s-\mu)$, where $\Gamma(s)$ is the Γ-function. We consider the following contour integral round $\mathscr{L}$:

$$\int_{\mathscr{L}} \frac{\zeta(s-1)}{F(s)} e^{sz} ds. \tag{2.2.3}$$

Since we assume the order of ρ is simple, we have by residue theorem

$$\begin{aligned}\int_{\mathscr{L}} \frac{\zeta(s-1)}{F(s)} e^{sz} ds &= \int_{a+iT_n}^{a} \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_{L} \frac{\zeta(s-1)}{F(s)} e^{zs} ds \\ &+ \int_{b}^{b+iT_n} \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_{b+iT_n}^{a+iT_n} \frac{\zeta(s-1)}{F(s)} e^{zs} ds \\ &= 2\pi i \sum_{\substack{\rho \\ 0<\mathrm{Im}\,\rho<T_n}} \frac{e^{\rho z}\zeta(\rho-1)}{F'(\rho)},\end{aligned} \tag{2.2.4}$$

where the path L above consists of joining the two line segments $\left[a, \frac{a+b}{2}+i\alpha\right]$ and $\left[\frac{a+b}{2}+i\alpha, b\right]$. We now estimate the integral along the line segment $[b+iT_n, a+iT_n]$. For $a \leq \sigma \leq b$, we have by (2.2.2),

$$|F(\sigma+iT_n)|^{-1} = O(\exp(C(\log T)^2)),$$

which, with the vertical estimate for $\zeta(s-1)$, shows with $z = x+iy$ that

$$\left|\int_{a+iT_n}^{b+iT_n} \frac{\zeta(s-1)}{F(s)} e^{zs} ds\right| \ll (b-a)T_n^c \exp\{C(\log T)^2 - yT_n + |x|(|a|+|b|)\} \tag{2.2.5}$$

for $T, T_n \geq 1$ with a constant $c = c(a,b) > 0$, where the last bound tends to zero as $n \to \infty$. By Theorem 3.1.1 below, the convergence of the other integrals in (2.2.4) are ensured (see (3.1.5)-(3.1.7)). The limit in (2.1.1) therefore exists. □

3 Main Results

3.1 Main Results

Letting $n \to \infty$, we have

$$\int_{a+i\infty}^{a} \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_{L} \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_{b}^{b+i\infty} \frac{\zeta(s-1)}{F(s)} e^{zs} ds = 2\pi i f(z, F), \tag{3.1.1}$$

with $f(z, F)$ in (2.1.1). To evaluate the integral along the vertical line with $s = b+it\ (t \geq 0)$, we prepare the Dirichlet series expansion of $\zeta(s-1)/F(s)$ for $\sigma > 2$.

Definition 3.1.1 (p.34 in [8])**.** For $\sigma > 1$ and $F \in \mathcal{S}^{\mathrm{poly}}$, we define the Dirichlet coefficients μ_F as follows :

$$\frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^s} = \prod_{p}\prod_{j=1}^{d}\left(1 - \frac{\alpha_j(p)}{p^s}\right). \tag{3.1.2}$$

Remark 3.1.1 (p.34 in [8]). By (3.1.2), $|\mu_F(n)| \le \tau_d(n)$, where $\tau_d(n)$ is the divisor function of order d, so that $\zeta^d(s) = \sum_{n=1}^{\infty} \tau_d(n)/n^s$ for $\sigma > 1$. In particular $\tau_1(n) = 1$ for all n.

Using (3.1.2) in $\sigma > 2$, we obtain

$$\frac{\zeta(s-1)}{F(s)} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \tag{3.1.3}$$

where

$$g(n) = \sum_{d|n} \mu_F(d)\frac{n}{d}. \tag{3.1.4}$$

Theorem 3.1.1 (Theorem4.1 in [3]). *Let* $\max\left\{-\frac{3}{2}, \frac{1}{2}\max\{Re\,\rho < 0\}\right\} < a < 0, b > 5/2$. *Then, the function* (2.1.1) *is holomorphic on* $\mathbb{H}$, *and for* $z \in \mathbb{H}$ *the formula*

$$2\pi i f(z,F) = f_1(z,F) + f_2(z,F) - e^{bz}\sum_{n=1}^{\infty} \frac{g(n)}{n^b(z - \log n)}, \tag{3.1.5}$$

is valid, where the functions

$$f_1(z,F) = \int_{a+i\infty}^{a} \frac{\zeta(s-1)}{F(s)} e^{sz} ds, \tag{3.1.6}$$

$$f_2(z,F) = \int_L \frac{\zeta(s-1)}{F(s)} e^{sz} ds \tag{3.1.7}$$

are holomorphic on $\mathbb{H}$ *and on the whole* z*-plane, and the last term on the right is a meromorphic function on the whole* z*-plane with the poles at* $z = \log n$ $(n = 1, 2, \ldots)$.

We need not use the condition of a which does not coincide the poles of $\Gamma(s+\mu)\Gamma(s-\mu)$ in proving of Theorem 3.1.1.

Theorem 3.1.2 (Theorem4.2 in [3]). *For any* $F \in \mathcal{S}^{poly}$ *with* $(r, \lambda_j) = (1,1)$ *for all* j *in* (1.2.9) *and* $0 \le \mu < 1$, *the function* (2.1.1) *has a meromorphic continuation to* $y > -\pi$.

The L-functions associated with holomorphic cusp forms and Dedekind zeta functions of the imaginary quadratic fields are the examples of F considered in Theorem 3.1.2. Letting

$$\mathbb{H}^- = \{z \in \mathbb{C} : \mathrm{Im}\ z < 0\}, \tag{3.1.8}$$

we next study the function, for $z \in \mathbb{H}^-$,

$$f^-(z,F) = \lim_{n\to\infty} \sum_{\substack{\rho \\ -T_n < \mathrm{Im}\,\rho < 0}} \frac{e^{\rho z}\zeta(\rho - 1)}{F'(\rho)}. \tag{3.1.9}$$

If there are trivial zeros of $F(s)$ on the imaginary axis in $\mathbb{H}^-$, they are incorporated in the summation. The existence of the limit on the right hand side of (3.1.9) is proved similarly to Fact 2.1.1.

Corollary 3.1.3 (Corollary4.3 in [3]). *For any* $F \in \mathcal{S}^{poly}$ *satisfying the same conditions as in Theorem 3.1.2, the function* (3.1.9) *has a meromorphic continuation to* $y < \pi$.

Theorem 3.1.4 (Theorem4.4 in [3]). *For any $F \in \mathcal{S}^{poly}$ satisfying the same conditions as in Theorem 3.1.2, the function* (2.1.1) *can be continued analytically to the whole z- plane. In addition to the condition as in Theorem 3.1.2, we assume that the Dirichlet series coefficients $a_F(n)$ of $F \in \mathcal{S}^{poly}$ is real-valued for all n. Then, the function* (2.1.1) *satisfies the functional equation*

$$f(z,F) + \overline{f(\overline{z},F)} = B(z,F), \tag{3.1.10}$$

where

$$B(z,F) = \frac{1}{2\pi i}(f_1(z,F) + f_1^-(z,F)) - \frac{e^{2z}}{F(2)} \tag{3.1.11}$$

for all $z \in \mathbb{C}$. Also, the function $f_1^-(z,F)$ is holomorphic on $\mathbb{H}^-$.

4 Motivation

Now, we mention the motivation for the main theorems in section 3. The author considered the Volterra integral equation for a remainder term in an asymptotic formula of an arithmetic function which satisfies the special conditions and the associated Euler totient function similar to [6].The remainder term can be split into the arithmetic part which is a solution of the Volterra integral equation and the analytic part which is not the solution of it. It seems that J.Kaczorowski and K.Wiertelak used the functional equation (1.3.5) in Theorem1.3.2 in the proof of the Ω-estimate for $E^{\mathrm{AN}}(x)$ in (1.1.10). It is not mentioned in [6] how use the functional equation (1.3.5) and it follows by repeating all steps in the proof of Theorem 1.1 in [7]. The author considers how to prove the Ω-estimate on an analytic part for a remainder term in the asymptotic formula of the associated Euler totient function using the functional equation (3.1.10) similar to [6] which will be necessary mainly. The author could prove the functional equation (3.1.10) which will be necessary, but it does not hold for all function $F \in \mathcal{S}^{\mathrm{poly}}$. Therefore, the author presents the results as Theorem 3.1.1, 3.1.2, 3.1.4 and Corollary 3.1.3 for the function F which satisfies the conditions already explained. More details are written in [4].

5 Some auxiliary results on the Whittaker function

5.1 The definition for the Whittaker function and the integral expression

We introduce the Whittaker function $W_{\kappa,\mu}(z)$ (via the confluent hypergeometric function $\Psi(\alpha,\gamma;z)$ below), which is necessary to prove our main theorems, and then prepare some auxiliary results, i.e.its integral expression and asymptotic expansions.

Definition 5.1.1 (The confluent hypergeometric function of the second kind ([1])). Let $\Psi(\alpha,\gamma;z)$ be the *confluent hypergeometric function of the second kind* defined by

$$\Psi(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)(e^{2\pi i\alpha}-1)} \int_{\infty}^{(0+)} e^{-zw} w^{\alpha-1}(1+w)^{\gamma-\alpha-1} dw \tag{5.1.1}$$

for any $(\alpha,\gamma) \in \mathbb{C}^2$ and for $|\arg z| < \pi$. Here the path of integration is a contour in the w-plane which consists of the upper real axis from $e^{0\pi i}\infty$ to $e^{0\pi i}\delta$ with a small $\delta > 0$, the circle with the center $w = 0$ and the radius δ through which $\arg w$ varies from 0 to 2π, and the lower real axis from $e^{2\pi i}\delta$ to $e^{2\pi i}\infty$.

Definition 5.1.2 (The Whittaker function ([9])). The *Whittaker function* $W_{\kappa,\mu}(z)$, which has large applicability, e.g. in number theory and physics, is defined by

$$W_{\kappa,\mu}(z) = z^{\mu+1/2} e^{-z/2} \Psi\left(\frac{1}{2} - \kappa + \mu, 2\mu+1; z\right) \quad (|\arg z| < \pi), \tag{5.1.2}$$

where (to avoid many-valuedness) the domain of z is to be restricted in the z-plane cut along the negative real axis with $|\arg z| < \pi$.

Lemma 5.1.1 (Barnes type integral for the Whittaker function ([13], [16]))**.** *The Barnes integral for $W_{\kappa,\mu}(z)$ asserts*

$$W_{\kappa,\mu}(z) = \frac{e^{-z/2}z^{\kappa}}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{\Gamma(s)\Gamma\left(-s-\kappa-\mu+\frac{1}{2}\right)\Gamma\left(-s-\kappa+\mu+\frac{1}{2}\right)}{\Gamma\left(-\kappa-\mu+\frac{1}{2}\right)\Gamma\left(-\kappa+\mu+\frac{1}{2}\right)} z^{s}ds, \tag{5.1.3}$$

for $|\arg z| < \frac{3}{2}\pi$, and $\kappa \pm \mu + \frac{1}{2} \neq 0, 1, 2, \ldots$; the contour has loops if necessary so that the poles of $\Gamma(s)$ from those of $\Gamma\left(-s-\kappa-\mu+\frac{1}{2}\right)\Gamma\left(-s-\kappa+\mu+\frac{1}{2}\right)$ are on opposite sides of it.

In (5.1.3), it holds for all finite values of c provided that the contour of integration can always be deformed so as to separate the poles $\Gamma(s)$ and those of the other Γ-factors. By Stirling's formula for $\Gamma(s)$ (cf. [5]), the integral in (5.1.3) represents a function of z which is holomorphic at all points in the domain $|\arg z| \leq \frac{3}{2}\pi - \alpha$ with any small $\alpha > 0$. The asymptotic expansions for $\Psi(\alpha;\gamma;z)$ as $z \to 0$ readily asserts the following proposition.

5.2 The asymptotic expansion

Proposition 5.2.1 (The asymptotic expansions for $\Psi(a;b;z)$ as $z \to 0$ ([13]))**.** *We have the asymptotic expansions, as $z \to 0$ through $|\arg z| < \pi$,*

$$\Psi(a;b;z) = \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b} + O(|z|^{\mathrm{Re}b-2}) \quad (\mathrm{Re}\, b \geq 2, b \neq 2), \tag{5.2.1}$$

$$= \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b} + O(|\log z|) \quad (b = 2), \tag{5.2.2}$$

$$= \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b} + O(1) \quad (1 < \mathrm{Re}\, b < 2), \tag{5.2.3}$$

$$= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b} + O(|z|) \quad (\mathrm{Re}\, b = 1, b \neq 1), \tag{5.2.4}$$

$$= -\frac{1}{\Gamma(a)}\left\{\log z + \frac{\Gamma'}{\Gamma}(a) + 2C_0\right\} + O(|z\log z|) \quad (b = 1), \tag{5.2.5}$$

where C_0 is Euler's constant.

By the definition (5.1.2) and Proposition 5.2.1, we have the following asymptotic expansions as $z \to 0$ for $W_{\kappa,\mu}(z)$.

Proposition 5.2.2 (The asymptotic expansions in $z \to 0$ for $W_{\kappa,\mu}(z)$)**.**

$$W_{\kappa,\mu}(z) = \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}z^{1/2-\mu} + O(z^{3/2-\mathrm{Re}\mu}) \quad \left(\mathrm{Re}\,\mu \geq \frac{1}{2}, \mu \neq \frac{1}{2}\right), \tag{5.2.6}$$

$$= \frac{1}{\Gamma(1-\kappa)} + O(|z\log z|) \quad \left(\mu = \frac{1}{2}\right), \tag{5.2.7}$$

$$= \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}z^{1/2-\mu} + O(|z|^{\mathrm{Re}\mu+1/2}) \quad \left(0 < \mathrm{Re}\,\mu < \frac{1}{2}\right), \tag{5.2.8}$$

$$= \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}z^{\mu+1/2} + \frac{\Gamma(2\mu)}{\Gamma\left(\mu+\frac{1}{2}-\kappa\right)}z^{-\mu+1/2} + O(|z|^{\mathrm{Re}\mu+3/2}) \quad (\mathrm{Re}\,\mu = 0, \mu \neq 0), \tag{5.2.9}$$

$$= -\frac{z^{1/2}}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left(\log z + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-\kappa\right) + 2C_0\right) + O(|z|^{3/2}|\log z|) \quad (\mu = 0). \tag{5.2.10}$$

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