On the support of the generalized translation operator

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Abstract

The (k, a)-generalized Fourier analysis is a far-reaching generalization of classical Fourier analysis developed by S. Ben Saïd, T. Kobayashi and B. Ørsted, where the parameter k comes from Dunkl theory, and the parameter a comes from the "interpolation" of the two $sl(2, \mathbb{R})$ actions on the Weil representation of the metaplectic group and the minimal unitary representation of the conformal group. We will investigate the support of the generalized translation in (k, a)-generalized Fourier analysis of the functions supported in the balls centered at the origin for $a = 1$ and 2 respectively, as well as the support of the measure associated to the spherical mean of the generalized translation operator.

1 Introduction

Dunkl theory is a far-reaching generalization of Fourier analysis and special function theory about root system R with a rich structure parallel to ordinary Fourier analysis and (k, a)-generalized Fourier analysis is a further far-reaching generalization of Dunkl theory. The study of Dunkl theory originates from a generalization of spherical harmonics, in which the finite reflection groups G play the role of orthogonal group $O(N)$ in the classical theory of spherical harmonics. The Lebesgue measure dx, which is invariant under $O(N)$, is substituted by the Dunkl weight measure $dm_k(x) = h_k(x)dx$ which is invariant under the finite reflection group G and parameterized by a multiplicity function k, where

$$h_k(x) = \prod_{\alpha\in R} |\langle \alpha,\ x\rangle|^{k(\alpha)}.$$

The Dunkl operator T_i (see [4]) was constructed in such a way that the intersection of space of the homogeneous polynomials P_m of degree m with the kernel of the corresponding Laplacian $\triangle_k = \sum_{j=1}^{N} T_j^2$ is orthogonal to that of lower degree with respect to the Dunkl weight measure dm_k. And the restrictions of the spaces $\mathcal{H}_k^m\left(\mathbb{R}^N\right) := P_m \cap ker\triangle_k,\ m = 0, 1, \cdots$ to the unit sphere $\mathbb{S}^{N-1}$ are called spherical h-harmonics. For the normalized root system R such that $\langle \alpha, \alpha\rangle = 2$ for all vectors $\alpha \in R$, the Dunkl Laplacian $\triangle_k$ has the following explicit expression $\triangle_k = D_k - E_k$, with

$$D_k = \triangle f\left(x\right) + 2 \sum_{\alpha\in R^+} k\left(\alpha\right) \frac{\langle \nabla f, \alpha\rangle}{\langle \alpha, x\rangle},$$

where ∇ is the Euclidean gradient and R^+ is any fixed positive subsystem of R, and

$$E_k = 2 \sum_{\alpha\in R^+} k\left(\alpha\right) \frac{f\left(x\right) - f\left(\sigma_\alpha\left(x\right)\right)}{\langle \alpha, x\rangle^2}.$$

D_k is the G-invariant part of the Dunkl Laplacian. The Dunkl operators commute pairwise and they are in substitute of the ordinary partial derivatives in classical analysis. The joint eigenfunctions of Dunkl operators (or the eigenfunctions of the Dunkl Laplacian $\triangle_k$) take the place of the exponential functions in classical Fourier transform. The Dunkl transform was then defined correspondingly (see [5]) and has many similar properties with Fourier transform. The discovery of Dunkl operators also gave an explicit expression of the radial part of the Laplacian operator on a flat symmetric space unintentionally. Moreover, Dunkl theory has extensive application in algebra, probability theory and mathematical physics.

More recently, S. Ben Saïd, T. Kobayashi and B. Ørsted [2] gave a further far-reaching generalization of Dunkl theory by introducing a parameter $a>0$ arisen from the "interpolation" of two $sl(2,\mathbb{R})$ actions. The generalization was motivated by the definition of the classical Fourier transform on $L^2\left(\mathbb{R}^N\right)$ given by Howe [8], where the Fourier transform was defined using the harmonic oscillator $\mathbf{H} =: (\triangle - \|x\|^2)/2$ as

$$F := e^{i\pi N/4} \exp\left(\frac{\pi \mathrm{i}}{2}\mathbf{H}\right).$$

In [2], the authors deformed the Dunkl harmonic oscillator $\triangle_k - \|x\|^2$ via the parameter a such that the (k,a)-generalized harmonic oscillator $\triangle_{k,a} := \|x\|^{2-a}\triangle_k - \|x\|^a$ is symmetric on the Hilbert space $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$, where $\vartheta_{k,a}(x) = \|x\|^{a-2}h_k(x)$. In the case of $k\equiv 0$, such a-deformed harmonic oscillator is also a deformation of the operator $\|x\|\triangle - \|x\|$ studied by Kobayashi and Mano [9, 10]. The generalized Fourier transform was then defined via the (k,a)-generalized harmonic oscillator as

$$F_{k,a} = e^{i\pi\left(\frac{2\langle k\rangle + N + a - 2}{2a}\right)} \exp\left(\frac{\pi i}{2a}\triangle_{k,a}\right).$$

It reduces to the classical Fourier transform when $k\equiv 0$ and $a=2$, to the Kobayashi-Mano Hankel transform [9, 10] when $k\equiv 0$ and $a=1$, and to the Dunkl transform [5] when $k\geq 0$ and $a=2$. By Schwartz kernel theorem, the (k,a)-generalized Fourier transform has the following integral representation (see [2, (5.8)])

$$F_{k,a}f(\xi) = c_{k,a}\int_{\mathbb{R}^N} f(y)\,B_{k,a}(\xi,y)\,\vartheta_{k,a}(y)\,dy, \quad \xi\in\mathbb{R}^N,$$

where $c_{k,a} = \left(\int_{\mathbb{R}^N}\exp\left(-\|x\|^a\right)\vartheta_{k,a}(x)\,dx\right)^{-1}$ and $B_{k,a}(x,y)$ is a symmetric kernel. The integral kernel $B_{k,a}(x,y)$ of the (k,a)-generalized Fourier transform takes the place of the exponential function $e^{-i\langle x,y\rangle}$ in classical Fourier transform. It is the eigenfunction of the operator $\|x\|^{2-a}\triangle_k$ for any fixed y (see [2, Theorem 5.7]), i.e.,

$$\|x\|^{2-a}\Delta_k^x B_{k,a}(x,y) = -\|\xi\|^a B_{k,a}(x,y).$$

So, we can consider the operator $\|x\|^{2-a}\Delta_k$ as the a-deformed Dunkl Laplacian in (k,a)-generalized Fourier analysis.

Assume $2\langle k\rangle + N + a - 3 \geq 0$. For $a=\frac{2}{n}$, $n\in\mathbb{N}$, one can define the (k,a)-generalized translation on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ as

$$F_{k,a}(\tau_y f)(\xi) := B_{k,a}(y,\xi)\,F_{k,a}(f)(\xi), \quad \xi\in\mathbb{R}^N.$$

The above definition makes sense because for $a=\frac{2}{n}$, $n\in\mathbb{N}$, $F_{k,a}$ is an isometry on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx)$ from the inversion formula [2, Theorem 5.3], and in this case its integral kernel $B_{k,a}(x,y)$ is uniformly bounded by 1 (see [3]). In this case the (k,a)-generalized translation can also be written via an integral as

$$\tau_y f(x) = c_{k,a}\int_{\mathbb{R}^N} B_{k,a}(x,\xi)\,B_{k,a}(y,\xi)\,F_{k,a}(f)(\xi)\,\vartheta_{k,a}(\xi)\,d\xi$$

for $f \in \mathcal{L}_k^1\left(\mathbb{R}^N\right)$, where $\mathcal{L}_k^1\left(\mathbb{R}^N\right) := \left\{f \in L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right) : \; F_{k,a}(f) \in L^1\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)\right\}$. This formula holds true on Schwartz space $\mathcal{S}\left(\mathbb{R}^N\right)$ since $\mathcal{S}\left(\mathbb{R}^N\right)$ is a subspace of $\mathcal{L}_k^1\left(\mathbb{R}^N\right)$.

For the two particular cases when $a = 1$ and $a = 2$ (the Dunkl case) assuming that $2\langle k\rangle + N + a - 3 \geq 0$ of (k, a)-generalized Fourier analysis, the analytic structure is much richer because we have the formula of the generalized translation operator for radial functions. The radial formula for $a = 2$ was found by Rösler [11] and for $a = 1$ that was found by S. Ben Saïd [1]. The case when $a = 2$ (Dunkl theory) was intensively studied in the last twenty years and the study for case when $a = 1$ is still at its infancy. The (k, a)-generalized translation τ_x corresponds to the classical translation operator $f \mapsto f(x - \cdot)$ for $a = 1$, and corresponds to $f \mapsto f(x + \cdot)$ for $a = 2$. This is because for $a = 1$, the inversion formula of the generalized Fourier analysis is $F_{k,1}^{-1}(f) = F_{k,1}(f)$, and for $a = 2$, the inversion formula is $\left(F_{k,2}^{-1} f\right)(x) = (F_{k,2} f)(-x)$ (see [2, Theorem 5.3]). We will focus on the two particular cases for $a = 1$ and $a = 2$ in this report, and investigate the support of the generalized translation for the two cases respectively.

2 The support of the generalized translation of f, $suppf = B(0, r)$

2.1 The case of $a = 2$ (the Dunkl case)

The following is the formula of the Dunkl translation (the case of $a = 2$) for radial functions

$$\tau_x f(-y) = \int_{\mathbb{R}^N} (\widetilde{f} \circ A)(x,\ y,\ \eta)\, d\mu_x(\eta),\ x,\ y \in \mathbb{R}^N, \tag{1}$$

where $f(x) = \widetilde{f}(\|x\|)$ and

$$A(x,\ y,\ \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle y,\ \eta\rangle} = \sqrt{\|x\|^2 - \|\eta\|^2 + \|y - \eta\|^2},$$

It was first proved by Rösler [11] for Schwartz functions, and was then extended to all continuous radial functions in [6].

Denote $B(x, r)$ to be the closed ball $\{y \in \mathbb{R}^N : \|x - y\| \leq r\}$. We characterize the support of the the Dunkl translation of nonnegative radial functions on $L^2(m_k)$ in the following theorem.

Theorem 2.1. *([12]) If the multiplicity function $k > 0$ and let f be a nonnegative radial function on $L^2(m_k)$, $suppf = B(0, r)$, then for any $x \in \mathbb{R}^N$,*

$$supp\tau_x f(-\cdot) = \bigcup_{g \in G} B(gx,\ r).$$

Proof. Define the distance between the two orbits $G.x$ and $G.y$ for $a = 2$ as $d_G(x, y) := \min_{g \in G} ||g \cdot y - x||$. For the proof of $supp\tau_x f \subseteq \bigcup_{g \in G} B(gx, r)$, we only need to notice that

$$A(x,\ y,\ \eta) \geq d_G(x,\ y),\ \eta \in co(G.x).$$

For the converse part $\bigcup_{g \in G} B(gx, r) \subseteq supp\tau_x f$, we prove the theorem for continuous nonnegative radial functions from the radial formula (2) first, and then prove for all nonnegative radial function on $L^2\left(\mathbb{R}^N, \vartheta_{k,1}(x)\,dx\right)$. □

2.2 The case of $a = 1$

The following is the formula of the generalized translation for the case of $a = 1$ for radial functions For any radial function $f \in \mathcal{S}\left(\mathbb{R}^N\right)$, i.e., τ_y can be expressed as follows (see [1])

$$\begin{aligned}\tau_y f(x) =& \frac{\Gamma\left(\frac{N-1}{2} + \langle k\rangle\right)}{\sqrt{\pi}\Gamma\left(\frac{N-2}{2} + \langle k\rangle\right)} \times \\ & V_k\left(\int_{-1}^{1} f_0\left(\|x\| + \|y\| - \sqrt{2\left(\|x\|\,\|y\| + \langle\cdot, y\rangle\right)}u\right)\left(1-u^2\right)^{\frac{N}{2}+\langle k\rangle-2} du\right)(x), \qquad (2)\end{aligned}$$

where $f(x) = f_0\left(\|x\|\right)$ and $\langle k\rangle + \frac{N-2}{2} > 0$. This radial formula was prove for Schwartz dunctions in [1] and was then extended to all continuous radial functions in [13].

For $x, y \in \mathbb{R}^N$, define a function d from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}$ as

$$\begin{aligned} d\left(x,y\right) :&= \sqrt{\|x\| + \|y\| - \sqrt{2\left(\|x\|\,\|y\| + \langle x, y\rangle\right)}} \\ &= \sqrt{\|x\| + \|y\| - 2\sqrt{\|x\|\,\|y\|}\cos\frac{\theta}{2}} \geq \left|\sqrt{\|x\|} - \sqrt{\|y\|}\right|, \end{aligned}$$

where $\theta = \arccos\frac{\langle x,y\rangle}{\|x\|\|y\|}$, $0 \leq \theta \leq \pi$. It was shown in [13] that the function $d\left(x,y\right)$ is a metric and $\left(\mathbb{R}^N, d\right)$ is a complete metric space. Denote $B(x,r)$ to be the ball $B\left(x,r\right) := \left\{y \in \mathbb{R}^N : d\left(x,y\right) \leq r\right\}$. We can then give a characterization of support of the $(k,1)$-generalized translation of a function supported in $B(0,r) = \left\{y \in \mathbb{R}^N : \sqrt{\|y\|} \leq r\right\}$ via the metric $d(x,y)$.

Theorem 2.2. *([13]) Let $f = f_0\left(\|\cdot\|\right)$ be a nonnegative radial function on $L^2\left(\mathbb{R}^N, \vartheta_{k,1}\left(x\right)dx\right)$, $suppf = B(0,r)$, then*

$$supp\tau_x f = \bigcup_{g\in G} B(gx, r).$$

Proof. Define the distance between the two orbits $G.x$ and $G.y$ for $a = 1$ as $d_G\left(x,y\right) := \min\limits_{g\in G} d\left(gx, y\right)$. For the proof of $supp\tau_x f \subseteq \bigcup_{g\in G} B(gx,r)$, we only need to notice that

$$\sqrt{\|x\| + \|y\| - \sqrt{2\left(\|x\|\,\|y\| + \langle\eta, y\rangle\right)}u} \geq d_G\left(x,y\right),\ \eta \in co(G.x),\ u \in [-1,1].$$

For the converse part $\bigcup_{g\in G} B(gx,r) \subseteq supp\tau_x f$, we prove the theorem for continuous nonnegative radial functions from the radial formula (2) first, and then prove for all nonnegative radial function on $L^2\left(\mathbb{R}^N, \vartheta_{k,1}\left(x\right)dx\right)$. □

3 The support of the measure associated to spherical mean

We conjecture the product formula of the integral kernels of the (k,a)-generalized Fourier transform

$$B_{k,a}(x,z)B_{k,a}(y,z) = \int_{\mathbb{R}^N} B_{k,a}(\xi, z)d\nu_{x,y}^{k;a}(\xi),\ z \in \mathbb{C}^N,$$

where the measures $d\nu_{x,y}^{k,a}$ are signed Borel measures on $\mathbb{R}^N$. This implies the following integral representation of the (k,a)-generalized translation

$$\tau_x f(y) = \int_{\mathbb{R}^N} f(\xi)d\nu_{x,y}^{k;a}(\xi).$$

In the following we denote $\tau_x f(y) =: f(x * y)$ for convenience because it has the property $\tau_x f(y) = \tau_y f(x)$. Then we can define

$$B_{k,a}(x * y, z) := B_{k,a}(x, z) B_{k,a}(y, z)$$

from the above.

The generalized spherical mean operator $f \mapsto M_f$ on $C^\infty(\mathbb{R}^N)$ is defined as

$$M_f(x,t) := \frac{1}{d_{k,a}} \int_{\mathbb{S}^{N-1}} f(x * ty) \vartheta_{k,a}(y) d\sigma(y), \quad (x \in \mathbb{R}^N,\ t \geq 0),$$

where $d\sigma$ is the spherical measure and $d_{k,a} = \int_{\mathbb{S}^{N-1}} \vartheta_{k,a}(x)\, d\sigma(x)$.

Proposition 3.1. *([7, Proposition 5.7]) If $x, y \in \mathbb{R}^d$, $x = \rho x'$, $y = vy'$, then*

$$\frac{1}{d_{k,a}} \int_{\mathbb{S}^{d-1}} B_{k,a}(x, vy') p(y') \vartheta_{k,a}(x)\ d\sigma(y') = \frac{e^{-\frac{i\pi m}{a}} \Gamma(\lambda_a + 1)}{a^{2m/a}\, \Gamma(\lambda_{k,a,m} + 1)} v^m j_{\lambda_{k,a,m}} \left(\frac{2}{a} (\rho v)^{a/2} \right) p(x),$$

where p is a polynomial of degree m and $j_{\lambda_{k,a,m}}(z)$ is the Bessel function.

For f to be $B_{k,a}(\cdot, z)$ with $z \in \mathbb{R}^N$, from Proposition 3.1,

$$M_f(x,t) = B_{k,a}(x,z) j_{\lambda_a} \left(\frac{2}{a}\ |tz|^{a/2} \right). \tag{3}$$

And for $f \in \mathcal{S}(\mathbb{R}^N)$,

$$\begin{aligned} M_f(x,t) &= \frac{c_{k,a}}{d_{k,a}} \int_{\mathbb{S}^{N-1}} \left(\int_{\mathbb{R}^N} B_{k,a}(x,\xi) B_{k,a}(ty,\xi) F_{k,a}(f)(\xi) \vartheta_{k,a}(\xi) d\xi \right) \vartheta_{k,a}(y) d\sigma(y) \\ &= \frac{c_{k,a}}{d_{k,a}} \int_{\mathbb{R}^N} B_{k,a}(x,\xi) F_{k,a}(f)(\xi) \vartheta_{k,a}(\xi) d\xi \int_{\mathbb{S}^{N-1}} B_{k,a}(ty,\xi) \vartheta_{k,a}(y) d\sigma(y) \\ &= c_{k,a} \int_{\mathbb{R}^N} B_{k,a}(x,\xi) j_{\lambda_a} \left(\frac{2}{a}\ |t\xi|^{a/2} \right) F_{k,a}(f)(\xi) \vartheta_{k,a}(\xi) d\xi. \end{aligned} \tag{4}$$

For $\alpha \geq -1/2$, denote by A^t_α the singular Sturm-Liouville operator

$$A^t_\alpha := \partial_t^2 + \frac{2\alpha + 1}{t} \partial_t,\ t > 0.$$

For fixed $z \in \mathbb{C}$, the Bessel functions $j_\alpha(tz)$ are eigenfunctions of the Sturm-Liouville operator (see [11]). By substituting t by $\sqrt{\frac{2}{a}} t^{\frac{a}{2}}$, we get the a-deformed Sturm-Liouville operator

$$\begin{aligned} A^t_{a,\alpha} &= \frac{2}{a} \left(\frac{1}{t^{a-2}} \partial_t^2 + \left(1 - \frac{a}{2}\right) \frac{1}{t^{a-1}} \partial_t \right) + \frac{2\alpha + 1}{t^{a-1}} \partial_t \\ &= \frac{2}{a} \left(\frac{1}{t^{a-2}} \partial_t^2 + \frac{a\alpha + 1}{t^{a-1}} \partial_t \right). \end{aligned}$$

And for fixed $z \in \mathbb{C}$, the Bessel functions $j_\alpha \left(\frac{2}{a}\ (tz)^{a/2} \right)$ are eigenfunctions of the a-deformed Sturm-Liouville operator, i.e.,

$$A^t_{a,\alpha} j_\alpha \left(\frac{2}{a}\ (tz)^{a/2} \right) = -\frac{2}{a} z^a j_\alpha \left(\frac{2}{a}\ (tz)^{a/2} \right).$$

Therefore, $u = M_f(x,t)$ is the solution of the equation for the Darboux-type differential-reflection operator $2\|x\|^{2-a}\Delta_k^x - aA^t_{a,\lambda_a}$,

$$(2\|x\|^{2-a}\Delta_k^x - aA^t_{a,\lambda_a})u = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+; \tag{5}$$
$$u(x,0) = f(x),\ u_t(x,0) = 0 \quad \text{for all } x \in \mathbb{R}^N.$$

as $B_{k,a}(x,\cdot)$ is eigenfunction of the operator $\|x\|^{2-a}\triangle_k^x$.

For fixed $x \in \mathbb{R}^N$ and $t \geq 0$, consider the linear functional

$$\Phi_{x,t} : f \mapsto M_f(x,t).$$

It is represented by a compactly supported measure $\sigma_{x,t}^{k,a} \in M^1(\mathbb{R}^N)$, where $M^1(\mathbb{R}^N)$ stand for the spaces of regular bounded complex Borel measures on $\mathbb{R}^N$,

$$M_f(x,t) = \int_{\mathbb{R}^N} f\, d\sigma_{x,t}^{k,a} \quad \text{for all } f \in C^\infty(\mathbb{R}^N). \tag{6}$$

From (3) and (4), (6) is equivalent to the following product formula

$$B_{k,a}(x,z) j_{\lambda_a}\left(\frac{2}{a} t\,|z|^{a/2}\right) = \int_{\mathbb{R}^N} B_{k,a}(\xi,z)\, d\sigma_{x,t}^{k,a}(\xi).$$

We will investigate the support of the measure for $\sigma_{x,t}^{k,a}$ the case of $a = 2$ and $a = 1$ respectively.

3.1 The case of $a = 2$ (the Dunkl case)

We consider the domain of dependence of the wave equation associated to the G-invariant part D_k of the Dunkl Laplacian $\triangle_k$. Denote

$$C(x_0,t_0) = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}_+ : |x - x_0| \leq t_0 - t\}$$

to be the wave cone. The following theorem shows that $C(x_0,t_0)$ is the domain of dependence of the wave equation. That is to say, the values in $C(x_0,t_0)$ of the solution u depend only on the initial values for $t = 0$ of u and u_t in $|x - x_0| \leq t_0$, regardless of the perturbation of the values outside of $|x - x_0| \leq t_0$.

Theorem 3.2. *(see [11, Theorem 4.4]) Suppose that u is a C^2-solution of the wave equation $(D_k - \partial_t^2)u = 0$, defined in the cone $C(x_0,t_0)$ and satisfying*

$$u_t(x,0) = u(x,0) = 0 \quad \textit{for all } x \in \mathbb{R}^N \textit{ with } |x - x_0| \leq t_0.$$

Then u vanishes in $C(x_0,t_0)$.

Proof. Energy method. □

We can then characterize the support of the measure associated to the spherical mean for $a = 2$.

Theorem 3.3. *(see [11, Theorem 4.1] For $a = 2$, the support of the measure $\sigma_{x,t}^k$ associated to $M_f(x,t)$ satisfies*

$$\mathit{supp}\,\sigma_{x,t}^k \subseteq K(x,t) := \bigcup_{g\in G} \{\xi \in \mathbb{R}^N : |\xi - gx| \leq t\}.$$

Sketch of the Proof. We involve the Riemann-Liouville transform with parameter $\alpha > -1/2$ on $\mathbb{R}_+$. It is given by

$$\mathcal{R}_\alpha f(t) \;=\; \frac{2\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+1/2)}\int_0^1 f(st)(1-s^2)^{\alpha-1/2}ds \tag{7}$$

for $f \in C^\infty(\mathbb{R}_+)$, see [?]. The operator $\mathcal{R}_\alpha$ satisfies the intertwining property

$$A_\alpha \mathcal{R}_\alpha \;=\; \mathcal{R}_\alpha \frac{d^2}{dt^2}. \tag{8}$$

Put $u_f(x,t) := (\mathcal{R}_\lambda^t)^{-1} M_f(x,t)$, which is still G-invariant with respect to x. Then according to (5) and the above intertwining property, $u = u_f$ belongs to $C^\infty(\mathbb{R}^N \times \mathbb{R}_+)$ and solves the initial value problem

$$\begin{aligned}(D_k - \partial_t^2)u \;&=\; 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+;\\ u(x,0) = f(x),\; u_t(x,0) &= 0 \quad \text{for all } x \in \mathbb{R}^N.\end{aligned} \tag{9}$$

Now suppose in addition that $\operatorname{supp} f \cap K(x,t) = \emptyset$. Then Theorem 3.4 implies that $u_f(x,s) = 0$ for all $0 \le s \le t$. From the explicit form (7) of the Riemann-Liouville transform $\mathcal{R}_\lambda$ we further deduce that

$$\int_{\mathbb{R}^N} f\, d\widetilde{\sigma}_{x,t}^k \;=\; M_f(x,t) \;=\; \mathcal{R}_\lambda^t u_f(x,t) \;=\; 0$$

□

3.2 The case of $a = 1$ (Ongoing work)

Domain of dependence. Consider a class of special second-order linear partial differential equations of the form

$$u_{tt} + Lu = 0 \;\; (x \in \mathbb{R}^N,\; t > 0), \tag{10}$$

where L has the special form

$$Lu = -\sum_{i,j=1}^N a^{ij}(x) D_{ij} u,$$

with smooth symmetric coefficients $(a^{ij}(x))$ satisfying uniform ellipticity condition on $\mathbb{R}^N$. In this case, we say the operator $\partial_{tt} + L$ is uniformly hyperbolic.

Let $q(x)$ be a continuous function on $\mathbb{R}^N$, positive and smooth in $\mathbb{R}^N \backslash \{x_0\}$ and $q(x_0) = 0$, and assume that

$$\sum_{i,j=1}^N a^{ij}(x) D_i q(x) D_j q(x) \le 1,\; x \in \mathbb{R}^N \backslash \{x_0\}.$$

Consider the curved backward cone

$$C(x_0,t_0) = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}_+ \;: q(x) \le t_0 - t\}.$$

Then it is well known in PDE that $C(x_0,t_0)$ is the domain of dependence of the hyperbolic equation (10). Now let $L = \|x\| \triangle$ and $q(x) = \sqrt{\|x\| + \|x_0\| - \sqrt{2\left(\|x\|\,\|x_0\| + \langle x, x_0\rangle\right)}}$. Then it can be verified that

$$\sum_{i=1}^N \|x\|\, D_i q(x) D_i q(x) = 1,\; x \in \mathbb{R}^N \backslash \{x_0\}.$$

And so

$$C(x_0,t_0) = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}_+ \, : d(x,x_0) \le t_0 - t\},$$

where $d(x,x_0) = \sqrt{\|x\| + \|x_0\| - \sqrt{2\left(\|x\|\,\|x_0\| + \langle x, x_0\rangle\right)}}$, is the domain of dependence of the hyperbolic equation (10) for $L = \|x\| \triangle$.

Now we consider the domain of dependence hyperbolic equation (10) for the deformed Dunkl Laplacian $\|x\| \triangle_k$ in $(k,1)$-generalized Fourier analysis.

Theorem 3.4. *Let u be a smooth solution to the wave equation $u_{tt} + \|x\| \triangle_k u = 0$ $(x \in \mathbb{R}^N,\ t > 0)$. If $u_t(x,0) = u(x,0) = 0$ for all $x \in \mathbb{R}^N$ with $d(x,x_0) \le t_0$. Then u vanishes in $C(x_0,t_0)$.*

Proof. The proof is still ongoing. It will be given via a modification of the energy method for the hyperbolic equation (10). □

We can then characterize the support of the measure associated to the spherical mean for $a = 1$.

Theorem 3.5. *For $a = 1$, the support of the measure $\sigma^k_{x,t}$ associated to $M_f(x,t)$ satisfies*

$$supp\, \sigma^k_{x,t} \subseteq\ K(x, \sqrt{2t}),$$

where

$$K(x,t) := \bigcup_{g\in G} \{\xi \in \mathbb{R}^N : d(\xi, gx) \le t\}$$

and $d\,(x,y) = \sqrt{\|x\| + \|y\| - \sqrt{2\left(\|x\|\,\|y\| + \langle x, y\rangle\right)}}$.

Sketch of the Proof. The proof is ongoing. We substitude t by $\sqrt{2t}$ ($\sqrt{\frac{2}{a}}t^{\frac{a}{2}}$ for $a = 1$) in (8)

$$A_{1,\alpha}\mathcal{R}_\alpha \ = \ \mathcal{R}_\alpha \left(2t\frac{d^2}{dt^2} + \frac{d}{dt}\right).$$

Put $u_f(x,t) := (\mathcal{R}^t_\lambda)^{-1} M_f(x,t)$. It is G-invariant with respect to x. Then according to (5) and the above intertwining property, $u = u_f$ belongs to $C^\infty(\mathbb{R}^N \times \mathbb{R}_+)$ and solves the initial value problem

$$\begin{aligned} &(\|x\| \triangle_k - \left(2t\partial_t^2 + \partial_t\,\right))u \ = \ 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+; \\ &u(x,0) = f(x), \ \left(2tu_{tt} + u_t\,\right)(x,0) = 0 \quad \text{for all } x \in \mathbb{R}^N. \end{aligned} \tag{11}$$

Now suppose in addition that $\operatorname{supp} f \cap K(x,\sqrt{2t}) = \emptyset$. Then Theorem 3.4 implies that $u_f(x,s) = 0$ for all $0 \le \sqrt{2s} \le \sqrt{2t}$. From the explicit form (7) of the Riemann-Liouville transform $\mathcal{R}_\lambda$ we further deduce that

$$\int_{\mathbb{R}^N} f\, d\tilde{\sigma}^k_{x,t} \ = \ M_f(x,t) \ = \ \mathcal{R}^t_\lambda u_f(x,t) \ = \ 0$$

□

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