Semi-Modules and Crystal Bases via Affine Deligne-Lusztig Varieties

Ryosuke Shimada (島田了輔)
東京大学大学院数理科学研究科

Abstract

There are two combinatorial ways of parameterizing the J_b-orbits of the irreducible components of affine Deligne-Lusztig varieties for GL_n and superbasic b. One way is to use the extended semi-modules introduced by Viehmann. The other way is to use the crystal bases introduced by Kashiwara and Lusztig. In this paper, we give an explicit correspondence between them using the crystal structure.

1 Introduction

Let F be a non-archimedean local field with finite field $\mathbb{F}_q$ of prime characteristic p, and let L be the completion of the maximal unramified extension of F. Let σ denote the Frobenius automorphism of L/F. Further, we write $\mathcal{O}$, $\mathfrak{p}$ for the valuation ring and the maximal ideal of L. Finally, we denote by ϖ a uniformizer of F (and L) and by v_L the valuation of L such that $v_L(\varpi) = 1$.

Let G be a split connected reductive group over $\mathbb{F}_q$ and let T be a split maximal torus of it. Let B be a Borel subgroup of G containing T. For a cocharacter $\mu \in X_*(T)$, let ϖ^μ be the image of $\varpi \in \mathbb{G}_m(F)$ under the homomorphism $\mu\colon \mathbb{G}_m \to T$.

Set $K = G(\mathcal{O})$. We fix a dominant cocharacter $\mu \in X_*(T)_+$ and $b \in G(L)$. Then the affine Deligne-Lusztig variety $X_\mu(b)$ is the locally closed reduced $\overline{\mathbb{F}}_q$-subscheme of the affine Grassmannian $\mathcal{G}r$ defined as

$$X_\mu(b)(\overline{\mathbb{F}}_q) = \{xK \in G(L)/K \mid x^{-1}b\sigma(x) \in K\varpi^\mu K\} \subset \mathcal{G}r(\overline{\mathbb{F}}_q).$$

Left multiplication by $g^{-1} \in G(L)$ induces an isomorphism between $X_\mu(b)$ and $X_\mu(g^{-1}b\sigma(g))$. Thus the isomorphism class of the affine Deligne-Lusztig variety only depends on the σ-conjugacy class of b.

The affine Deligne-Lusztig variety $X_\mu(b)$ carries a natural action (by left multiplication) by the group

$$J_b = \{g \in G(L) \mid g^{-1} b\sigma(g) = b\}.$$

For $\mu_\bullet = (\mu_1, \dots, \mu_d) \in X_*(T)_+^d$ and $b_\bullet = (1, \dots, 1, b) \in G^d(L)$ with $b \in G(L)$, we can similarly define $X_{\mu_\bullet}(b_\bullet) \subset \mathcal{G}r^d$ and $J_{b_\bullet}$ using $\sigma_\bullet$ given by

$$(g_1, \dots, g_d) \mapsto (g_2, \dots, g_d, \sigma(g_1)).$$

The geometric properties of affine Deligne-Lusztig varieties have been studied by many people. One of the most interesting results among such studies is an explicit description of the set $J_b \backslash \operatorname{Irr}^{\mathrm{top}} X_\mu(b)$ of J_b-orbits of $\operatorname{Irr}^{\mathrm{top}} X_\mu(b)$, where $\operatorname{Irr}^{\mathrm{top}} X_\mu(b)$ denotes the set of top-dimensional irreducible components of $X_\mu(b)$.

Remark 1.1. In the equal characteristic case, $X_\mu(b)$ is equi-dimensional, see [2]. In the mixed characteristic case, the equi-dimensionality is not fully established, see [1, Theorem 3.4].

Let $\widehat{G}$ be the Langlands dual of G defined over $\overline{\mathbb{Q}}_l$ with $l \neq p$. Denote V_μ the irreducible $\widehat{G}$-module of highest weight μ. The crystal basis $\mathbb{B}_\mu$ was first constructed by Lusztig and Kashiwara (cf. [4]). In $X_*(T)$, there is a distinguished element λ_b determined by b. It is the "best integral approximation" of the Newton vector of b, but we omit the precise definition. For this, see [1, §2.1] (in fact, [1, Example 2.3] is enough for our purpose). In [5], Nie proved that there exists a natural bijection

$$J_b \backslash \operatorname{Irr}^{\mathrm{top}} X_\mu(b) \cong \mathbb{B}_\mu(\lambda_b).$$

In particular, $|J_b \backslash \operatorname{Irr}^{\mathrm{top}} X_\mu(b)| = \dim V_\mu(\lambda_b)$. The proof is reduced to the case where $G = \mathrm{GL}_n$ and b is superbasic. So this case is particularly important. The existence of this bijection is first conjectured by Miaofen Chen and Xinwen Zhu. The last equality is also proved by Rong Zhou and Yihang Zhu. See [8, §1.2] for the history. In the case where $G = \mathrm{GL}_n$ and b is superbasic, Viehmann [6] defined a stratification of $X_\mu(b)$ using extended semi-modules. For $\mu \in X_*(T)_+$ and superbasic $b \in \mathrm{GL}_n(L)$, let $\mathbb{A}_{\mu,b}^{\mathrm{top}}$ be the set of equivalence classes of top extended semi-modules, that is, the semi-modules whose corresponding strata are top-dimensional. Then $J_b \backslash \operatorname{Irr}^{\mathrm{top}} X_\mu(b)$ is also parametrized by $\mathbb{A}_{\mu,b}^{\mathrm{top}}$.

In [5, Remark 0.10], Nie pointed out that it would be interesting to give an explicit correspondence between $\mathbb{A}_{\mu,b}^{\mathrm{top}}$ and $\mathbb{B}_\mu(\lambda_b)$. The purpose of this paper is to study this question (for the split case). More precisely, we will propose a way of constructing (the unique lifts of) all the top extended semi-modules from crystal elements, which was unclear before this work.

From now and until the end of this paper, we set $G = \mathrm{GL}_n$. Let T be the torus of diagonal matrices, and we choose the subgroup of upper triangular matrices B as Borel subgroup. Let us define the Iwahori subgroup $I \subset K$ as the inverse image of the *lower* triangular matrices under the projection $K \to G(\overline{\mathbb{F}}_q)$, $\varpi \mapsto 0$.

We assume b to be superbasic, i.e., its Newton vector $\nu_b \in X_*(T)_{\mathbb{Q}} \cong \mathbb{Q}^n$ is of the form $\nu_b = (\frac{m}{n}, \ldots, \frac{m}{n})$ with $(m, n) = 1$. Moreover, we choose b to be η^m, where $\eta = \begin{pmatrix} 0 & \varpi \\ 1_{n-1} & 0 \end{pmatrix}$. We often regard η (and hence b) as an element of the Iwahori-Weyl group $\widetilde{W}$. Then the action of η is given by $v \mapsto s_1 s_2 \cdots s_{n-1} v + (1, 0, \ldots, 0)$. Note that $\eta^n = \varpi^{(1,\ldots,1)}$. For superbasic b, the condition that $X_\mu(b)$ (resp. $X_{\mu_\bullet}(b_\bullet)$) is non-empty is equivalent to $v_L(\det(\varpi^\mu)) = v_L(\det(b))$ (resp. $v_L(\det(\varpi^{\mu_1 + \cdots + \mu_d})) = v_L(\det(b))$) (cf. [3, Theorem 3.1]). In this paper, we assume this.

Since $X_\mu(b) = X_{\mu+c}(\varpi^c b)$ for any central cocharacter c, we may assume that $\mu(1) \geq \cdots \geq \mu(n-1) \geq \mu(n) = 0$, where $\mu(i)$ denotes the i-th entry of μ.

For $\mu_\bullet \in X_*(T)_+^d$ and $b_\bullet = (1, \ldots, 1, b) \in G^d(L)$ with b superbasic, we define

$$\mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet} \coloneqq \{\lambda_\bullet \in X_*(T)^d \mid \dim X^{\lambda_\bullet}_{\mu_\bullet}(b_\bullet) = \dim X_{\mu_\bullet}(b_\bullet)\}.$$

Here $X^{\lambda_\bullet}_{\mu_\bullet}(b_\bullet)$ denotes $X_{\mu_\bullet}(b_\bullet) \cap I t^{\lambda_\bullet} K/K$. For $\lambda_\bullet, \lambda'_\bullet \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$, we write $\lambda_\bullet \sim \lambda'_\bullet$ if $\lambda_\bullet = \eta^k \lambda'_\bullet = (\eta^k \lambda'_1, \ldots, \eta^k \lambda'_d)$ for some $k \in \mathbb{Z}$. Let $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$ denote the set of equivalence classes with respect to $\sim$, and let $[\lambda_\bullet] \in \mathbb{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$ denote the equivalence class represented by $\lambda_\bullet \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$. Then $J_{b_\bullet} \backslash \mathrm{Irr}^{\mathrm{top}} X_{\mu_\bullet}(b_\bullet)$ is also parametrized by $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$.

For $\mu \in X_*(T)_+$, let $\mu_\bullet \in X_*(T)_+^d$ be certain minuscule dominant cocharacters with $\mu = \mu_1 + \cdots + \mu_n$. Note that $\{\mu_1, \ldots, \mu_n\}$ is uniquely determined by μ. Let $\mathrm{pr}\colon \mathcal{G}r^d \to \mathcal{G}r$ be the projection to the first factor. This induces $\mathrm{pr}\colon \mathbb{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet} \to \sqcup_{\mu' \leq \mu} \mathbb{A}^{\mathrm{top}}_{\mu', b}$. Then our main result is the following:

Theorem A. For $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$, using the crystal structure of $\mathbb{B}_\mu$, we can construct $\lambda^1_\bullet(\mathbf{b}), \lambda^2_\bullet(\mathbf{b}), \ldots, \lambda^n_\bullet(\mathbf{b}) \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$ such that $\lambda^i_\bullet(\mathbf{b}) = \eta^{i-1} \lambda^1_\bullet(\mathbf{b})$ and $[\lambda^1_\bullet(\mathbf{b})]$ is the unique equivalence class in $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$ whose image $\mathrm{pr}([\lambda^1_\bullet(\mathbf{b})])$ belongs to $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$ and maps to $\mathbf{b}$ under the bijection $J_b \backslash \mathrm{Irr}^{\mathrm{top}} X_\mu(b) \cong \mathbb{B}_\mu(\lambda_b)$ by Nie.

A crystal is a finite set with a weight map wt and Kashiwara operators $\tilde{e}_\alpha$ and $\tilde{f}_\alpha$ satisfying certain conditions. The merit of constructing $[\lambda^1_\bullet(\mathbf{b})]$ instead of constructing $\mathrm{pr}([\lambda^1_\bullet(\mathbf{b})])$ directly is that the J_b-orbit in $X_\mu(b)$ corresponding $[\lambda^1_\bullet(\mathbf{b})]$ is much more explicit. It is just $J_b \overline{X^{\lambda^1_\bullet(\mathbf{b})}_{\mu_\bullet}(b_\bullet)}$.

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