Large-Time Behaviour of Curl-Free Compressible Navier-Stokes Equations

東北大学大学院 理学研究科 数学専攻
ダビ・オーヘー (Dáithí Ó hAodha)

1 Introduction

This presentation is concerned with the barotropic compressible Navier-Stokes system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^3, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu D(u) + \lambda \operatorname{div}(u)\mathrm{Id}) + \nabla p = 0, & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^3, \\ (u, \rho)|_{t=0} = (u_0, \rho_0), & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\rho : \mathbb{R}^3 \to [0, \infty)$, and $u : \mathbb{R}^3 \to \mathbb{R}^3$ are unknown functions, representing the density and velocity of a fluid respectively. $p : \mathbb{R}^3 \to \mathbb{R}$ is the pressure in the fluid, and the barotropic assumption gives us $p \coloneqq P(\rho)$, for some smooth function P. μ, λ are viscosity coefficients, taken such that

$$\mu \geq 0, \quad 2\mu + \lambda \geq 0.$$

We define the deformation tensor

$$D(u) \coloneqq \frac{1}{2}\Big(Du + Du^T\Big).$$

In what follows, we shall assume that the density approaches 1 at infinity; and so we are concerned with strong solutions which are small perturbations from a constant state $(\rho, u) = (1, 0)$. We shall also assume that μ, λ are constant, and set $a \coloneqq \rho - 1$. Our system (1.1) can thus be rewritten into the following linearised problem:

$$\begin{cases} \partial_t a + \operatorname{div}(u) = f & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^3, \\ \partial_t u - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div}(u) + P'(1)\nabla a = g & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^3, \\ (a, u)\Big|_{t=0} = (a_0, u_0) & \text{in } \mathbb{R}^3. \end{cases}$$

We apply the orthogonal projections $\mathcal{P}$ and $\mathcal{Q}$ over the divergence and curl-free fields, respectively (these will be defined later). Then, setting $\alpha \coloneqq P'(1)$ and $\nu \coloneqq \lambda + 2\mu$, we get the system

$$\begin{cases} \partial_t a + \operatorname{div}(\mathcal{Q}u) = f & \text{in } \mathbb{R}_{>0}\times\mathbb{R}^3, \\ \partial_t \mathcal{Q}u - \nu\Delta\mathcal{Q}u + \alpha\nabla a = \mathcal{Q}g & \text{in } \mathbb{R}_{>0}\times\mathbb{R}^3, \\ \partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = \mathcal{P}g & \text{in } \mathbb{R}_{>0}\times\mathbb{R}^3. \end{cases} \tag{1.2}$$

We note that $\mathcal{P}u$ satisfies a simple heat equation, independent of a and $\mathcal{Q}u$. We thus focus on a and $\mathcal{Q}u$ in the first two equations. We set

$$v := |D|^{-1}\operatorname{div}(u), \text{ where } \mathcal{F}\Big[|D|^s u\Big](\xi) := |\xi|^s\hat{u}(\xi).$$

We note that one can obtain v from $\mathcal{Q}u$ by a 0-order Fourier multiplier. Thus, bounding v is equivalent to bounding $\mathcal{Q}u$ (see [1], Lemma 2.2).

Since the following rescaling solves (1.2) with $\alpha = \nu = 1$, we shall set those constants as such:

$$a(t,x) = \tilde{a}\Big(\frac{\alpha}{\nu}t, \frac{\sqrt{\alpha}}{\nu}x\Big), \quad u(t,x) = \sqrt{\alpha}\,\tilde{u}\Big(\frac{\alpha}{\nu}t, \frac{\sqrt{\alpha}}{\nu}x\Big).$$

Thus we get that (a, v) solves the following system:

$$\begin{cases} \partial_t a + |D|v = f & \text{in } \mathbb{R}_{>0}\times\mathbb{R}^3, \\ \partial_t v - \Delta v - |D|a = h := |D|^{-1}\operatorname{div}(g) & \text{in } \mathbb{R}_{>0}\times\mathbb{R}^3. \end{cases}$$

In this presentation, we shall focus on the homogeneous case, where $f = g = 0$, and afterwards discuss the inhomogeneous case. We now introduce our main results, which feature some notation to be explained in the following section. The estimates from above are analogous to those found in [2, 3]. The estimate from below is original.

Theorem 1.1. Let $s \in \mathbb{R}$, $p \in [2,\infty]$, $q \in [1,\infty]$, and $t > 2$. Then

$$\left\|\mathcal{F}^{-1}\begin{bmatrix}\hat{a}(t)\\ \hat{v}(t)\end{bmatrix}\right\|_{\dot{B}^s_{p,q}} \le Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}(1-\frac{2}{p})}\left\|\mathcal{F}^{-1}\begin{bmatrix}\hat{a}_0\\ \hat{v}_0\end{bmatrix}\right\|^l_{\dot{B}^s_{1,q}} + Ce^{-t}\left\|\mathcal{F}^{-1}\begin{bmatrix}\hat{a}_0\\ \hat{v}_0\end{bmatrix}\right\|^h_{\dot{B}^{s+7/2}_{1,q}}.$$

For the high-frequency norm, we also have

$$\left\|\mathcal{F}^{-1}\begin{bmatrix}\hat{a}(t)\\ \hat{v}(t)\end{bmatrix}\right\|^h_{\dot{B}^s_{p,q}} \le Ce^{-t}\left\|\mathcal{F}^{-1}\begin{bmatrix}\hat{a}_0\\ \hat{v}_0\end{bmatrix}\right\|^h_{\dot{B}^{s+1/2}_{p,q}}.$$

Also, there exists initial data such that for all sufficiently large $t > 0$,

$$\left\|\mathcal{F}^{-1}\begin{bmatrix}\hat{a}(t)\\ \hat{v}(t)\end{bmatrix}\right\|_{\infty} \ge Ct^{-2}.$$

2 Preliminaries

2.1 L^p Spaces

Definition 1. (L^p Space)

Let (Ω, M, μ) be a measure space. The set $L^p(\Omega)$, with $1 \le p < \infty$, is defined as the set of all functions, $f : \Omega \to \mathbb{R}$, such that

$$\|f\|_p := \Big(\int_\Omega |f(x)|^p \,\mathrm{d}\mu\Big)^{1/p} < \infty.$$

For $p = \infty$, the set is defined as the set of all functions whose essential supremum is finite.

$$\|f\|_\infty := \inf\{C \ge 0 \mid |f(x)| \le C, \text{ for almost all } x \in \Omega\}.$$

Functions that agree almost everywhere (i.e. everywhere in Ω except on a subset with 0 measure) are considered a single element of $L^p(\Omega)$.

Notation. For $1 \le p \le \infty$, we denote by p' the conjugate exponent. That is,

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

with the convention that $\frac{1}{\infty} := 0$ in this context.

Proposition 2.1. (Hölder's Inequality)

Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, where $1 \le p \le \infty$. Then $fg \in L^1$, and

$$\int_\Omega |fg| \,\mathrm{d}\mu \le \|f\|_p \|g\|_{p'}.$$

Proposition 2.2. (Young's Convolution Inequality)

Let $f \in L^p$ and $g \in L^q$, where $1 \le p, q \le r \le \infty$, such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

Then

$$\|f * g\|r \le \|f\|_p \|g\|_q.$$

Here, $(f * g)$ denotes the convolution of f and g. That is,

$$(f * g)(x) := \int_{\mathbb{R}^2} f(x-y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}^2} f(y)g(x-y)\,\mathrm{d}y.$$

2.2 The Fourier Transform

For a function, f, we define the Fourier transform of f as follows:

$$\mathcal{F}[f](\xi) := \hat{f}(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x.$$

The inverse Fourier transform is then defined as

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi.$$

For the purpose of calculating inequalities, we will frequently omit the factor of $1/2\pi$.

Proposition 2.3. (Plancherel Theorem)
Let $f \in L^1 \cap L^\infty$. Then the L^2 norm of f is invariant under the Fourier transform. That is,

$$\|f\|_2 = \|\hat{f}\|_2.$$

Proposition 2.4. (Orthogonal Projections on the divergence and curl-free fields)
We define the projection mappings $\mathcal{P}, \mathcal{Q}$ using the Fourier transform and the Kronecker delta, defined as follows:

$$\delta_{i,j} := \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

The projection mapping $\mathcal{P}$ is a matrix with each component defined as follows for $i, j \in \{1, 2, 3\}$:

$$(\mathcal{P})_{i,j} := \delta_{i,j} + (-\Delta)^{-1}\partial_i\partial_j.$$

We then define $\mathcal{Q} := 1 - \mathcal{P}$. For $f \in (\dot{B}^s_{p,q}(\mathbb{R}^3))^3$, with $s \in \mathbb{R}$, and $p, q \in [1, \infty]$, we may write

$$\mathcal{P}f := (1 + (-\Delta)^{-1}\nabla\mathrm{div})f.$$

2.3 Besov Spaces

Definition 2. We use the Littlewood-Paley decomposition of unity to define homogeneous Besov spaces. Let $\{\hat{\phi}_k\}_{k\in\mathbb{Z}}$ be a set of non-negative measurable functions such that

1. $\sum_{k\in\mathbb{Z}} \hat{\phi}_k(\xi) = 1,$ for all $\xi \in \mathbb{R}^2\backslash\{0\}$,
2. $\hat{\phi}_k(\xi) = \hat{\phi}_0(2^{-k}\xi)$,
3. $\mathrm{supp}\ \hat{\phi}_k(\xi) \subseteq \{\xi \in \mathbb{R}^2 \mid 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$.

For $f \in \mathcal{S}'/\mathcal{P}$, we write

$$\dot{\Delta}_k f := \mathcal{F}^{-1}[\hat{\phi}_k \hat{f}],$$

The Besov norm is then defined as follows: for $1 \leq p, q \leq \infty$, and $s \in \mathbb{R}$, we define the

$$\|f\|_{\dot{B}^s_{p,q}} := \Big(\sum_{k\in\mathbb{Z}} 2^{sqk}\|\dot{\Delta}_k f\|_p^q\Big)^{1/q}.$$

The set $\dot{B}^s_{p,q}$ is defined as the set of functions, $f \in \mathcal{S}'/\mathcal{P}$, whose Besov norm is finite. Finally, we also regularly use the following notation for so-called high and low-frequency norms:

$$\|f\|^h_{\dot{B}^s_{p,q}} := \Big(\sum_{k\geq 3} 2^{sqk}\|\dot{\Delta}_k f\|_p^q\Big)^{1/q}, \quad \|f\|^l_{\dot{B}^s_{p,q}} := \Big(\sum_{k\leq 2} 2^{sqk}\|\dot{\Delta}_k f\|_p^q\Big)^{1/q}.$$

3 Proof of Theorem

We begin our analysis of (a, v) which solves the system

$$\begin{cases} \partial_t a + |D|v = 0 & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^3, \\ \partial_t v - \Delta v - |D|a = 0 & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^3. \end{cases}$$

Taking the Fourier transform over space, x, we can write the above system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{a}\\ \hat{v}\end{bmatrix} = M_{|\xi|}\begin{bmatrix}\hat{a}\\ \hat{v}\end{bmatrix}, \quad \text{with} \quad M_{|\xi|} := \begin{bmatrix}0 & -|\xi| \\ |\xi| & -|\xi|^2\end{bmatrix}. \tag{3.1}$$

Then, by Duhamel's principle, we write the following formula for the solution to (3.1):

$$\begin{bmatrix}\hat{a}(t)\\ \hat{v}(t)\end{bmatrix} = e^{tM_{|\xi|}}\begin{bmatrix}\hat{a}_0\\ \hat{v}_0\end{bmatrix}.$$

We obtain the following eigenvalues for $M_{|\xi|}$, which differ between high and low frequencies:

$$\lambda_\pm(\xi) := \begin{cases} -\frac{|\xi|^2}{2}\Big(1 \pm i\sqrt{\frac{4}{|\xi|^2} - 1}\Big), & \text{for } |\xi| < 2, \\ -\frac{|\xi|^2}{2}\Big(1 \pm \sqrt{1 - \frac{4}{|\xi|^2}}\Big), & \text{for } |\xi| > 2, \end{cases}$$

from which we obtain the matrices of eigenvectors

$$P = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22}\end{bmatrix} = \begin{cases} \frac{1}{\sqrt{2}}\begin{bmatrix} \Big(1 - i\sqrt{\frac{4}{|\xi|^2}-1}\Big)^{1/2} & \Big(1 + i\sqrt{\frac{4}{|\xi|^2}-1}\Big)^{1/2} \\ \Big(1 + i\sqrt{\frac{4}{|\xi|^2}-1}\Big)^{1/2} & -\Big(1 - i\sqrt{\frac{4}{|\xi|^2}-1}\Big)^{1/2}\end{bmatrix}, & \text{for } |\xi| < 2, \\ \\ \frac{1}{\sqrt{2}}\begin{bmatrix} \Big(1 - \sqrt{1-\frac{4}{|\xi|^2}}\Big)^{1/2} & \Big(1 + \sqrt{1-\frac{4}{|\xi|^2}}\Big)^{1/2} \\ \Big(1 + \sqrt{1-\frac{4}{|\xi|^2}}\Big)^{1/2} & -\Big(1 - \sqrt{1-\frac{4}{|\xi|^2}}\Big)^{1/2}\end{bmatrix}, & \text{for } |\xi| > 2. \end{cases}$$

Using these eigenvalues and eigenvectors, we can rewrite the fundamental solution:

$$e^{tM_{|\xi|}} = P\begin{bmatrix}e^{t\lambda_+} & 0 \\ 0 & e^{t\lambda_-}\end{bmatrix}P^{-1}.$$

Looking again at our solution, if we left-multiply both sides of (3.1) by P^{-1}, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{m}_+\\ \hat{m}_-\end{bmatrix}=\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{bmatrix}\begin{bmatrix}\hat{m}_+\\ \hat{m}_-\end{bmatrix}, \text{ where } \begin{bmatrix}\hat{m}_+\\ \hat{m}_-\end{bmatrix}:=P^{-1}\begin{bmatrix}\hat{a}\\ \hat{v}\end{bmatrix}.$$

We thus obtain

$$\begin{bmatrix}\hat{m}_+(t)\\ \hat{m}_-(t)\end{bmatrix}=\begin{bmatrix}e^{t\lambda_+} & 0\\ 0 & e^{t\lambda_-}\end{bmatrix}\begin{bmatrix}\hat{m}_+(0)\\ \hat{m}_-(0)\end{bmatrix}.$$

Before proceeding, we give some remarks on the relationship between a_0, v_0, and $\hat{m}_+(0), \hat{m}_-(0)$. We note the following relations for all $p \in [1,\infty]$:

$$\left\|\mathcal{F}^{-1}[\hat{\phi}_j v_{11} f]\right\|_p \sim 2^{-j/2}\left\|\mathcal{F}^{-1}[\hat{\phi}_j f]\right\|_p, \text{ for all } j\in\mathbb{Z},$$
$$\left\|\mathcal{F}^{-1}[\hat{\phi}_j v_{21} f]\right\|_p \sim 2^{-j/2}\left\|\mathcal{F}^{-1}[\hat{\phi}_j f]\right\|_p, \text{ for all } j<3,$$
$$\left\|\mathcal{F}^{-1}[\hat{\phi}_j v_{21} f]\right\|_p \sim \left\|\mathcal{F}^{-1}[\hat{\phi}_j f]\right\|_p, \text{ for all } j\geq 3.$$

The last inequality is due to the fact that

$$\left(1+\sqrt{1-(4/|\xi|^2)}\right)^{1/2}\sim 1$$

for high frequencies. This causes a small loss of regularity for our purposes, as we are forced to take advantage of the following chain of inequalities in order to obtain our estimates:

$$\|f\|_{\dot{B}^s_{p,q}} \leq C\|Pf\|_{\dot{B}^{s+1/2}_{p,q}} \leq C\|f\|_{\dot{B}^s_{p,q}\cap\dot{B}^{s+1/2}_{p,q}}.$$

Proposition 3.1. ($L^1 - L^\infty$ estimates for high and low frequencies)
For all sufficiently large $t>1$,

$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_+}\hat{\phi}_j\right]\right\|_\infty \leq Ce^{-t}, \text{ for all } j\geq 3$$
$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_-}\hat{\phi}_j\hat{m}_-(0)\right]\right\|_\infty \leq 2^{3j}Ce^{-t}\|\Delta_j m_-(0)\|_1, \text{ for all } j\geq 3$$
$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_\pm}\hat{\phi}_j\right]\right\|_\infty \leq Ct^{-2}, \text{ for all } j\leq 2.$$

Proof. We start with the high-frequency case $j\geq 3$. Note that in this case we have

$$\text{supp } \hat{\phi}_j \subseteq \{\xi\in\mathbb{R}^3 | 2<|\xi|\},$$

and thus we are dealing with the high-frequency definitions of our eigenvalues $\lambda_\pm$. That is,

$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_-}\hat{\phi}_j\hat{m}_-(0)\right]\right\|_\infty = \left\|\int_{|\xi|>2} e^{ix\cdot\xi}e^{-\frac{t}{2}|\xi|^2\left(1-\sqrt{1-\frac{4}{|\xi|^2}}\right)}\hat{\phi}(2^{-j}\xi)\hat{m}_-(0,\xi)\,\mathrm{d}\xi\right\|_{L^\infty_x}$$

$$= \left\| \mathcal{F}^{-1}\left[e^{-\frac{t}{2}|\xi|^2\left(1-\sqrt{1-\frac{4}{|\xi|^2}}\right)} \hat{\Phi}_j \hat{\phi}_j \hat{m}_-(0) \right] \right\|_{L_x^\infty}$$
$$= \left\| \mathcal{F}^{-1}\left[e^{-\frac{t}{2}|\xi|^2\left(1-\sqrt{1-\frac{4}{|\xi|^2}}\right)} \hat{\Phi}_j \right] * \mathcal{F}^{-1}\left[\hat{\phi}_j \hat{m}_-(0) \right] \right\|_{L_x^\infty},$$

where $\hat{\Phi}_j = \hat{\phi}_{j-1} + \hat{\phi}_j + \hat{\phi}_{j+1}$. We finally end by using Young's convolution inequality

$$\begin{aligned}
&\left\| \mathcal{F}^{-1}\left[e^{-\frac{t}{2}|\xi|^2\left(1-\sqrt{1-\frac{4}{|\xi|^2}}\right)} \hat{\Phi}_j \right] * \mathcal{F}^{-1}\left[\hat{\phi}_j \hat{m}_-(0) \right] \right\|_{L_x^\infty} \\
&\leq \left\| \mathcal{F}^{-1}\left[e^{-\frac{t}{2}|\xi|^2\left(1-\sqrt{1-\frac{4}{|\xi|^2}}\right)} \hat{\Phi}_j \right] \right\|_\infty \|\dot{\Delta}_j m_-(0)\|_1 \\
&\leq \int_{\xi\in\mathbb{R}^3} e^{-2t\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-1}} \hat{\Phi}_j(\xi)\,\mathrm{d}\xi \|\dot{\Delta}_j m_-(0)\|_1 \\
&\leq 2^{3j+6} C e^{-t} \|\dot{\Delta}_j m_-(0)\|_1,
\end{aligned}$$

and

$$\begin{aligned}
\left\| \mathcal{F}^{-1}\left[e^{t\lambda_+} \hat{\phi}_j \right] \right\|_\infty &= \sup_{x\in\mathbb{R}^3} \left| \int_{2^{j-1}<|\xi|<2^{j+1}} e^{ix\cdot\xi} e^{-\frac{t}{2}|\xi|^2\sqrt{1-\frac{4}{|\xi|^2}}} e^{-\frac{t}{2}|\xi|^2} \hat{\phi}(2^{-j}\xi)\,\mathrm{d}\xi \right| \\
&\leq C \int_{2^{j-1}<|\xi|<2^{j+1}} \left| e^{-\frac{t}{2}|\xi|^2} \hat{\phi}(2^{-j}\xi) \right| \mathrm{d}\xi \\
&\leq C e^{-t} \int_{\mathbb{R}^3} e^{-t|\xi|^2/4}\,\mathrm{d}\xi \\
&\leq C e^{-t}, \text{ for all } j \geq 3,\ t > 1. \qquad (3.2)
\end{aligned}$$

Next, for middling frequencies, $-1 < j \leq 2$, the result follows similarly to the above. Finally, we are left with the low frequencies $j \leq -1$. We start by extracting the heat-like decay by a change of variables.

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^3} \left| \int_{|\xi|\leq 1} e^{ix\cdot\xi} e^{\pm it|\xi|^2\sqrt{\frac{4}{|\xi|^2}-1}} e^{-t|\xi|^2} \hat{\phi}(2^{-j}\xi)\,\mathrm{d}\xi \right| \\
&= t^{-3/2} \sup_{x\in\mathbb{R}^3} \left| \int_{|\xi|\leq t^{1/2}} e^{i(t^{-1/2}x)\cdot\xi} e^{\pm it^{1/2}|\xi|^2\sqrt{\frac{4}{|\xi|^2}-t^{-1}}} e^{-|\xi|^2} \hat{\phi}(2^{-j}t^{-1/2}\xi)\,\mathrm{d}\xi \right|.
\end{aligned}$$

Thus it remains to extract a further decay of $t^{-1/2}$ from the L^∞ norm above. We fix $j \leq -1$, and consider two cases with respect to time t.

First, let $\epsilon > 0$, and let $t_0 \geq C$, for some constant $C > 2$, such that

$$2^j \geq t_0^{-1/2} + \epsilon.$$

In this case, we easily obtain

$$\sup_{x\in\mathbb{R}^3} \left| \int_{2^{j-1}t^{1/2}<|\xi|<2^{j+1}t^{1/2}} e^{i(t^{-1/2}x)\cdot\xi} e^{\pm it^{1/2}|\xi|^2\sqrt{\frac{4}{|\xi|^2}-t^{-1}}} e^{-|\xi|^2} \hat{\phi}(2^{-j}t^{-1/2}\xi)\,\mathrm{d}\xi \right|$$

$$\leq \int_{2^{j-1}t^{1/2}<|\xi|<2^{j+1}t^{1/2}} e^{-|\xi|^2}\, \mathrm{d}\xi$$
$$\leq \int_{|\xi|>(\epsilon t^{1/2})/2} e^{-|\xi|^2}\, \mathrm{d}\xi \leq t^{-1/2}.$$

Thus, for each fixed j, there exists a constant t_0 large enough such that for all $t \geq t_0$ our claim holds. This, however, is not sufficient, as this constant increases as $j \to -\infty$. We thus must consider the inverse case where, for $j \leq -1$ and $t \geq C$, for some constant $C > 2$, we have

$$2^{j+1} \leq t^{-1/2}.$$

Note that in this case, thanks to the presence of $\hat{\phi}_j$ in our integrand, we need only integrate over $|\xi| < 2^{j+1}t^{1/2} \leq 1$. The rest of this proof is focused on this one remaining case.

First note that, by radial symmetry, we can assume without loss of generality that $x = (x_1, 0, 0)$. Next, we decompose the frequency space into four parts.

$$\begin{aligned}\mathbb{R}^3 =& \{\xi \in \mathbb{R}^3 \ : \ |\xi_2|, |\xi_3| \leq t^{-1/4}\} \cup \{\xi \in \mathbb{R}^3 \ : \ |\xi_2| \geq t^{-1/4}, |\xi_3| \leq t^{-1/4}\} \\ & \cup \{\xi \in \mathbb{R}^3 \ : \ |\xi_2| \leq t^{-1/4}, |\xi_3| \geq t^{-1/4}\} \cup \{\xi \in \mathbb{R}^3 \ : \ |\xi_2|, |\xi_3| \geq t^{-1/4}\} \\ & := B_1 \cup B_2 \cup B_3 \cup B_4.\end{aligned}$$

We estimate our integral over each of these four subsets, and as the calculations involved are elementary, we skip the details for this report. □

Proposition 3.2. Let $j \in \mathbb{Z}$, $p \in [2, \infty]$, $s \in \mathbb{R}$, $q \in [1, \infty]$, and $t \geq C > 2$. Then

$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_+}\hat{m}_+(0)\right]\right\|_{\dot{B}^s_{p,q}} \leq Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}(1-\frac{2}{p})}\|m_+(0)\|^l_{\dot{B}^s_{1,q}} + Ce^{-t}\|m_+(0)\|^h_{\dot{B}^s_{1,q}},$$
$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_-}\hat{m}_-(0)\right]\right\|_{\dot{B}^s_{p,q}} \leq Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}(1-\frac{2}{p})}\|m_-(0)\|^l_{\dot{B}^s_{1,q}} + Ce^{-2t}\|m_-(0)\|^h_{\dot{B}^{s+3}_{1,q}}.$$

The above proposition follows from interpolation between the L^∞-norm, which we've bounded by Proposition 3.1, and the L^2-norm, which decays at the same rate as the heat kernel.

Proposition 3.3. ($L^p - L^p$ estimates for high frequencies)

$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_+}\hat{\phi}_j\hat{m}_+(0)\right]\right\|_p \leq Ce^{-t}\|\dot{\Delta}_j m_+(0)\|_p, \text{ for all } j \geq 3,$$
$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_-}\hat{\phi}_j\hat{m}_-(0)\right]\right\|_p \leq Ce^{-t}\|\dot{\Delta}_j m_-(0)\|_p, \text{ for all } j \geq 3.$$

Proof. Since the proofs are similar, we only show for $\hat{m}_-(0)$. First, as noted in the previous proposition, we have

$$e^{-\frac{t}{2}|\xi|^2\left(1-\sqrt{1-\frac{4}{|\xi|^2}}\right)} = e^{-2t\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-1}}$$

Next, note that

$$-t\big(1+\sqrt{1-4/|\xi|^2}\big)^{-1} = -\frac{t}{2} - \frac{2t}{|\xi|^2}\big(1+\sqrt{1-4/|\xi|^2}\big)^{-2}.$$

Thus, we can rewrite

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left[e^{t\lambda_-}\hat{\phi}_j\hat{m}_-(0)\right]\right\|_p &= e^{-t}\left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{|\xi|^2}\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-2}}\hat{\phi}_j\hat{m}_-(0)\right]\right\|_p \\ &\le e^{-t}\left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{|\xi|^2}\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-2}}\hat{\Phi}_j\right]\right\|_1 \|\dot{\Delta}_j m_-(0)\|_p. \qquad (3.3)\end{aligned}$$

We note that, for all $j \ge 4$,

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{|\xi|^2}\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-2}}\hat{\Phi}_j\right]\right\|_1 &= \left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{2^{2j}|\xi|^2}\left(1+\sqrt{1-\frac{4}{2^{2j}|\xi|^2}}\right)^{-2}}\hat{\Phi}_0\right]\right\|_1 \\ &\le C\left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{2^{2j}|\xi|^2}\left(1+\sqrt{1-\frac{4}{2^{2j}|\xi|^2}}\right)^{-2}}\hat{\Phi}_0\right]\right\|_{W^{2,2}} \\ &\le C.\end{aligned}$$

The above sobolev norm is divergent for $j = 3$, and so we must instead use the following inequality:

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{|\xi|^2}\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-2}}\hat{\Phi}_3\right]\right\|_1 \le \left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{|\xi|^2}\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-2}}\hat{\Phi}_3\right]\right\|_{\dot{B}^{3/2}_{2,\infty}} \\ &\le \left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{|\xi|^2}\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-2}}\hat{\Phi}_3\right]\right\|_{\dot{B}^{1}_{2,\infty}}\left\|\mathcal{F}^{-1}\left[e^{-\frac{4t}{|\xi|^2}\left(1+\sqrt{1-\frac{4}{|\xi|^2}}\right)^{-2}}\hat{\Phi}_3\right]\right\|_{\dot{B}^{2}_{2,\infty}} \le Ce^{-t}.\end{aligned}$$

The final step is calculated similarly to (3.2) using the presence of fixed $\hat{\Phi}_3$ inside the above norms. Thus, returning to (3.3), we have

$$\left\|\mathcal{F}^{-1}\left[e^{t\lambda_-}\hat{\phi}_j\hat{m}_-(0)\right]\right\|_p \le Ce^{-t}\|\dot{\Delta}_j m_-(0)\|_p.$$

□

We now discuss the optimality of the estimates in the previous section. In particular, we will prove that the low-frequency bound obtained is optimal.

Proposition 3.4. (Optimality of Low-Frequency Linear Estimate)

Let t be sufficiently large. There exist $j < 0$ and $C > 0$ independent of t, j such that

$$\|\mathcal{F}^{-1}[e^{t\lambda_\pm}\hat{\phi}_j]\|_\infty \ge Ct^{-2}.$$

Proof.

First, we observe that for all $t > 2$, there exists $j < 0$ such that

$$[1/4, 7/8] \subseteq [2^{j-1}t^{1/2}, 2^{j+1}t^{1/2}].$$

We fix $t > 2$ and take such $j < 0$. Next note that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^3} \Big| \int_{|\xi|\le 1} e^{ix\cdot\xi} e^{\pm it|\xi|^2\sqrt{\frac{4}{|\xi|^2}-1}} e^{-t|\xi|^2} \hat{\phi}(2^{-j}\xi)\,\mathrm{d}\xi \Big| \\
&\ge t^{-3/2} \Big| \int_{|\xi|\sim 2^j t^{1/2}} e^{\pm 2it^{1/2}\xi_1} e^{\pm it^{1/2}|\xi|^2\sqrt{\frac{4}{|\xi|^2}-t^{-1}}} e^{-|\xi|^2} \hat{\phi}(2^{-j}t^{-1/2}\xi)\,\mathrm{d}\xi \Big|,
\end{aligned}$$

where in the above, we chose the specific value $x = (\pm 2t, 0, 0)$.

Next, we divide the whole space into four sections as in the previous section.

$$\begin{aligned}
\mathbb{R}^3 =& \{\xi \in \mathbb{R}^3 \ : \ |\xi_2|, |\xi_3| \le t^{-1/4}\} \cup \{\xi \in \mathbb{R}^3 \ : \ |\xi_2| \ge t^{-1/4}, |\xi_3| \le t^{-1/4}\} \\
&\cup \{\xi \in \mathbb{R}^3 \ : \ |\xi_2| \le t^{-1/4}, |\xi_3| \ge t^{-1/4}\} \cup \{\xi \in \mathbb{R}^3 \ : \ |\xi_2|, |\xi_3| \ge t^{-1/4}\} \\
&\coloneqq B_1 \cup B_2 \cup B_3 \cup B_4.
\end{aligned}$$

Additionally, we divide B_1 further into two parts.

$$\begin{aligned}
B_1 &= B_{1,1} \cup B_{1,2}, \\
B_{1,1} &\coloneqq \{\xi \in B_1 \mid \xi_1 < -2^{j+1}t^{1/2} + 1/8\}, \\
B_{1,2} &\coloneqq B_1 \backslash B_{1,1}.
\end{aligned}$$

In our proof we obtain a bound the integral over $B_{1,1}$ from below by $t^{-1/2}$ times a harmless constant. We then may obtain bounds from above that decay at a faster rate for the integrals over every other set $B_{1,2}, B_2, B_3, B_4$ by calculations similar to those of Proposition 3.1. □

We lastly must find some suitable initial data (u_0, a_0) such that

$$\left\| \mathcal{F}^{-1} \begin{bmatrix} \hat{a}(t) \\ \hat{v}(t) \end{bmatrix} \right\|_\infty \ge Ct^{-2}.$$

We propose that for sufficiently small a_0, the following initial velocity satisfies our main theorem:

$$u_0 \coloneqq e^{-|x|^2}\vec{1}, \text{ with } \vec{1} \coloneqq [1, 1, 1]^T.$$

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