Noetherian perfectoid towers and their tilts

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Abstract

The theory of perfectoid spaces is recognized also as a powerful tool for studying commutative ring theory, but it heavily relies on delicate nature of non-Noetherian rings. To establish a general framework to apply the perfectoid theory in a Noetherian setting, we introduce a certain class of sequences of ring extensions that provide Noetherian approximation of perfectoid rings, and discuss their "tilts". As an application, we prove a mixed characteristic analogue of Polstra's finiteness theorem on divisor class groups of strongly F-regular rings. This talk is based on a joint work with Shinnosuke Ishiro and Kazuma Shimomoto.

1 Introduction

Nowadays the theory of perfectoid spaces established by Scholze ([10]) is recognized also as a powerful tool for studying commutative ring theory. In this theory, a certain ring theoretic operation called *tilting* plays a significant role: it makes a bridge between objects in characteristic 0 and positive characteristic. However, this operation behaves well only for huge algebraic/geometric objects; those are highly non-Noetherian.

Our primary aim is to modify the tilting operation so that it can be applied to a certain tower of Noetherian rings, and the resulting positive characteristic object is also Noetherian and its singularities reflect some features coming from the original one. To achieve the goal, we first introduce a nice class of towers of rings, called *perfectoid towers* (Definition 3.2).

We then consider a variant of the tilting operation with several desired properties for perfectoid towers (Definition 3.3). We call the resulting positive characteristic objects *small tilts*. In this talk, we discuss several fundamental properties of perfectoid towers and their small tilts. The most important feature is recorded in Main Theorem 1.

We apply these results to the divisor class group of local log-regular rings. The divisor class group of Noetherian normal domains is an important invariant but it is often hard to compute (indeed, a classical result of Claborn [3] asserts that an arbitrary abelian group can be realized as a divisor class group of some Dedekind domain). However, if one restricts to a certain class of Noetherian rings, the type of a divisor class group appearing is limited. For example, Polstra recently proved that for a strongly F-regular local ring, the torsion part of the divisor class group is finite ([9]). As an application of perfectoid towers, we establish a mixed characteristic analogue of Polstra's result (Main Theorem 2).

Throughout this report, we follow the following convention.

- We consistently fix a prime $p > 0$. If we need to refer to another prime different from p, we denote it by ℓ.
- All rings are assume to be commutative and contain a unity (unless otherwise stated; cf. Main Theorem 1). We mean by a *ring map* a unital ring homomorphism.
- We mean by a *pair* a pair (A, I) consisting of a ring A and and an ideal $I \subseteq A$.

2 Perfectoid rings and tilting

Let us recall the definition of perfectoid rings in the sense of Bhatt-Morrow-Scholze ([1, Definition 3.5]). For a ring R, we set the inverse limit

$$R^{\text{frep}} := \varprojlim\{\cdots \to R/pR \to R/pR \to \cdots \to R/pR\},$$

where each transition map is the Frobenius endomorphism on R/pR. It is called the *inverse perfection* of R. Moreover, we denote by $W(R)$ the ring of p-typical Witt vectors over R. If R is p-adically complete and separated, we denote by $\theta_R : W(R^{\text{frep}}) \to R$ the unique ring map such that the diagram:

$$\begin{array}{ccc} W(R^{\text{frep}}) & \xrightarrow{\theta_R} & R \\ \downarrow & & \downarrow \\ R^{\text{frep}} & \longrightarrow & R/pR \end{array}$$

(where the vertical maps are induced by reduction modulo p and the bottom map is the first projection) commutes.

Definition 2.1. ([1, Definition 3.5]) A ring S is *perfectoid* if the following conditions hold.

1. S is ϖ-adically complete and separated for some element $\varpi \in S$ such that ϖ^p divides p.
2. The Frobenius endomorphism on S/pS is surjective.
3. The kernel of $\theta_S : W(S^{\text{frep}}) \to S$ is principal.

For a perfectoid ring S, the inverse perfection S^{frep} is called the *tilt of S* and denoted by $S^{\flat}$.

3 Axioms for perfectoid towers

To state our axioms for perfectoid towers, we first recall (or give) some notations.

Notation 3.1. Let R be a ring, and let $I \subseteq R$ be an ideal. Let M be an R-module.

1. We say that an element $m \in M$ is *I-torsion* if for every $x \in I$, there exists some integer $n(x) > 0$ such that $x^{n(x)}m = 0$. We denote by $M_{I\text{-tor}}$ the R-submodule of M consisting of all I-torsion elements in M.
2. We denote by $\varphi_{I,M} : M_{I\text{-tor}} \to M/IM$ the composition of natural A-linear maps:

$$M_{I\text{-tor}} \hookrightarrow M \twoheadrightarrow M/IM.$$

Now let us give our axioms for perfectoid towers.

Definition 3.2 (Perfectoid towers)**.** Let R be a ring, and let $I_0 \subseteq R$ be an ideal. Then we call a direct system of rings $\{R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} R_2 \xrightarrow{t_2} \cdots\}$ a *perfectoid tower arising from* (R, I_0) if it satisfies the following axioms.

(a) $R_0 = R$ and $p \in I_0$.

(b) For every $i \geq 0$, the ring map $\overline{t_i} : R_i/I_0R_i \to R_{i+1}/I_0R_{i+1}$ induced by t_i is injective.

(c) For every $i \geq 0$, the Frobenius endomorphism on R_{i+1}/I_0R_{i+1} factors as

$$R_{i+1}/I_0R_{i+1} \to R_i/I_0R_i \xrightarrow{\overline{t_i}} R_{i+1}/I_0R_{i+1}. \tag{3.1}$$

(d) For every $i \geq 0$, the former map of (3.1) (which will be denoted by F_i below) is surjective.

(e) For every $i \geq 0$, R_i is I_0-adically Zariskian.

(f) I_0 is a principal ideal, and R_1 contains a principal ideal I_1 that satisfies the following axioms.

  (f-1) $I_1^p = I_0R_1$.

  (f-2) For every $i \geq 0$, $\mathrm{Ker}(F_i) = I_1(R_{i+1}/I_0R_{i+1})$.

(g) For every $i \geq 0$, $I_0(R_i)_{I_0\text{-tor}} = (0)$. Moreover, there exists a bijective map $(F_i)_{\mathrm{tor}} : (R_{i+1})_{I_0\text{-tor}} \to (R_i)_{I_0\text{-tor}}$ such that the diagram:

$$\begin{array}{ccc} (R_{i+1})_{I_0\text{-tor}} & \xrightarrow{\varphi_{I_0,R_{i+1}}} & R_{i+1}/I_0R_{i+1} \\ {\scriptstyle (F_i)_{\mathrm{tor}}}\downarrow & & \downarrow{\scriptstyle F_i} \\ (R_i)_{I_0\text{-tor}} & \xrightarrow[\varphi_{I_0,R_i}]{} & R_i/I_0R_i \end{array}$$

commutes.

We then introduce a tower theoretic variant of the tilting operation, as follows.

Definition 3.3 (Tilts of perfectoid towers)**.** Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I_0).

1. For an integer $j \geq 0$, we define the *j-th small tilt of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *associated to* (R, I_0) as the limit:

$$(R_j)^{s.\flat}_{I_0} := \varprojlim\{\cdots \to (R_{j+i+1}/I_0R_{j+i+1}) \xrightarrow{F_{j+i}} (R_{j+i}/I_0R_{j+i}) \to \cdots \xrightarrow{F_j} R_j/I_0R_j\}.$$

2. For any $j \geq 0$, we define an injective ring map $(t_j)^{s.\flat}_{I_0} : (R_j)^{s.\flat}_{I_0} \hookrightarrow (R_{j+1})^{s.\flat}_{I_0}$ by the rule:

$$(t_j)^{s.\flat}_{I_0}((a_i)_{i\geq 0}) := (\overline{t_{j+i}}(a_i))_{i\geq 0}.$$

   Moreover, we call the resulting tower $(\{(R_i)^{s.\flat}_{I_0}\}_{i\geq 0}, \{(t_i)^{s.\flat}_{I_0}\}_{i\geq 0})$ *the tilt of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *associated to* (R, I_0).

Example 3.4. Let $(R, \mathfrak{m}, k)$ be a d-dimensional regular local ring whose residue field k is perfect and let $x_1, \ldots, x_d$ be a regular sequence of parameters. Then, by Cohen's structure theorem, R is isomorphic to

$$W(k)[|x_1, \ldots, x_d|]/(p - f)$$

where $f = x_1$ or $f \in (p, x_1, \ldots, x_d)^2$. For every $i \geq 0$, set $R_i = W(k)[|x_1^{1/p^i}, \ldots, x_d^{1/p^i}|]/(p - f)$, and let $t_i : R_i \to R_{i+1}$ be the natural injection. Then $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R, (p))$. Moreover, the tilt $(\{(R_i)^{s.\flat}_{(p)}\}_{i\geq 0}, \{(t_i)^{s.\flat}_{(p)}\}_{i\geq 0})$ is isomorphic to the tower:

$$k[|x_1, \ldots, x_d|] \hookrightarrow k[|x_1^{1/p}, \ldots, x_d^{1/p}|] \hookrightarrow k[|x_1^{1/p^2}, \ldots, x_d^{1/p^2}|] \hookrightarrow \cdots.$$

This method of constructing perfectoid towers is naturally extended to *complete local log-regular rings* (see [8] for foundations of local log-regular rings). The resulting perfectoid towers are applied to obtain our main result (Main Theorem 2).

4 Main results

In this section, we fix a perfectoid tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ arising from some pair (R_0, I_0), and let $I_1 \subseteq R_1$ denote the ideal given in the axiom (f) in Definition 3.2. Moreover, we denote by $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ the tilt of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R_0, I_0). For $j = 0, 1$, let $I_j^{s.\flat} \subseteq R_j^{s.\flat}$ denote the inverse image of $I_j(R_j/I_0R_j)$ via the first projection map $R_j^{s.\flat} \to R_j/I_0R_j$. In particular, by the axiom (d) in Definition 3.2, the projection $R_0^{s.\flat} \to R_0/I_0R_0$ induces an isomorphism:

$$R_0^{s.\flat}/I_0^{s.\flat}R_0^{s.\flat} \xrightarrow{\cong} R_0/I_0R_0. \tag{4.1}$$

The following result is at the technical core of our theory of perfectoid towers.

Main Theorem 1. *$I_0^{s.\flat}$ and $I_1^{s.\flat}$ are principal ideals, and $(I_1^{s.\flat})^p = I_0^{s.\flat}R_1^{s.\flat}$. Moreover, we have isomorphisms of (possibly) non-unital rings $(R_i^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \cong (R_i)_{I_0\text{-tor}}$ $(i \geq 0)$ that are compatible with $\{t_i\}_{i\geq 0}$ and $\{t_i^{s.\flat}\}_{i\geq 0}$.*

This result provides numerous important information on perfectoid towers. For example, the following corollary is essential for the application to analysis of divisor class groups.

Corollary 4.1. *Keep the notation as in Main Theorem 1. Set $R_\infty := \varinjlim_{i\geq 0} R_i$ and $R_\infty^{s.\flat} := \varinjlim_{i\geq 0} R_i^{s.\flat}$. Then the following assertions hold.*

1. *$(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ is a perfectoid tower arising from $(R_0^{s.\flat}, I_0^{s.\flat})$.*
2. *The I_0-adic completion $\widehat{R_\infty}$ of R_∞ and the $I_0^{s.\flat}$-adic completion $\widehat{R_\infty^{s.\flat}}$ of $R_\infty^{s.\flat}$ are perfectoid rings. Moreover, $\widehat{R_\infty^{s.\flat}} \cong (\widehat{R_\infty})^\flat$.*

By combining Corollary 4.1 with Cesnavicius-Scholze's comparison theorem for etale cohomology groups via tilting of perfectoid rings ([2]), we obtain a noetherian variant of it.

Proposition 4.2. *For every $i \geq 0$, suppose that R_i is I_0-adically Henselian, and $t_i : R_i \to R_{i+1}$ is a module-finite extension of Noetherian normal domains whose generic extension is of p-power degree. Fix a Zariski-open subset $U \subseteq \operatorname{Spec}(R_0)$ such that $\operatorname{Spec}(R_0) \setminus V(I_0) \subseteq U$ and the corresponding open subset $U^{s.\flat} \subseteq \operatorname{Spec}(R_0^{s.\flat})$ via the isomorphism (4.1). Let ℓ be a prime different from p. Then, for any fixed $k, n \geq 0$ such that $|H^k(U_{\mathrm{et}}^{s.\flat}, \mathbb{Z}/\ell^n\mathbb{Z})| < \infty$, one has*

$$|H^k(U_{\mathrm{et}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^k(U_{\mathrm{et}}^{s.\flat}, \mathbb{Z}/\ell^n\mathbb{Z})|.$$

In particular, if $H^k(U_{\mathrm{et}}^{s.\flat}, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$, then $H^k(U_{\mathrm{et}}, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$.

By applying Proposition 4.2 and Fujiwara-Gabber's formal base change theorem ([4], [5]), we establish the following result. It can be regarded as a mixed characteristic analogue of Polstra's theorem (Theorem 4.3).

Main Theorem 2. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring of mixed characteristic with perfect residue field k of characteristic $p > 0$, and denote by $\mathrm{Cl}(R)$ the divisor class group with its torsion subgroup $\mathrm{Cl}(R)_{\mathrm{tor}}$. Assume that $\widehat{R^{\mathrm{sh}}}[\frac{1}{p}]$ is locally factorial, where $\widehat{R^{\mathrm{sh}}}$ is the completion of the strict Henselization R^{sh}. Then $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}[\frac{1}{p}]$ is a finite group. In other words, the ℓ-primary subgroup of $\mathrm{Cl}(R)_{\mathrm{tor}}$ is finite for all primes $\ell \neq p$ and vanishes for almost all primes $\ell \neq p$.*

The proof is carried out by reduction to the positive characteristic case, which is settled by Polstra's theorem below.

Theorem 4.3 (Polstra)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local $\mathbb{F}_p$-algebra, where we set $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Suppose that R is an F-finite strongly F-regular domain. Then the torsion subgroup of the divisor class group of R is finite.*

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