On Efimov K-theory

Department of Mathematics, Graduate School of Science, Nagoya University
He Li

Abstract

We give the definition of Efimov (or continuous) K-theory of dualizable presentable stable ∞-categories and introduce some basic properties of Efimov K-theory.

1 Introduction

There is a long history about the algebraic K-theory. Grothendieck firstly defined the K_0-group and Quillen later defined the higher algebraic K-theory. After the development of the theory of ∞-categories established by Joyal, Lurie and so on, Barwick defined the algebraic K-theory for Waldhansen ∞-categories in [2] and Blumberg-Gepner-Tabuada defined the algebraic K-theory for small stable ∞-categories in [1].
Recently, Efimov defined the continuous extension of this algebraic K-theory to the ∞-category of dualizable presentable stable ∞-categories, which we call it *Efimov* K-*theory*. See [6].

2 Introduction to ∞-categories

We will give a basic introduction to the theory of ∞-categories. For a much more thorough introduction, see [3].
Let $[n] = \{0 < 1 < \cdots < n\}$ and let Δ be the category of objects of the form $[n]$ and the maps are given by the nondecreasing maps.

Definition 2.1. A *simplicial set* X is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$.

Definition 2.2. A simplicial set X is called an ∞*-category*, if it has the extension property for all inner horn inclusion $\Lambda_i^n \to \Delta^n$, for any n and any $0 < i < n$.

Example 2.3. If all arrows in an ∞-category X are equivalences, then we refer to X as an ∞*-groupoid*, or *space*, or *anima*. The ∞-category of all (small) anima is denoted by An.

There is a special kind of ∞-categories, called stable ∞-categories, which is fundamental. We refer to [4] for a thorough introduction.

Definition 2.4. We say an ∞-category $\mathcal{C}$ is *pointed*, if $\mathcal{C}$ amdits zero objects. That is, the initial objects and final objects coincide. We say a pointed ∞-category $\mathcal{C}$ is *stable*, if $\mathcal{C}$ admits finite limits and finite colimits, and finite limits coincide with finite colimits.

Definition 2.5. For a pointed ∞-category $\mathcal{C}$, and $X \in \mathcal{C}$, we define $\Sigma(X) := \mathrm{fib}(0 \to X)$ and $\Omega(X) := \mathrm{cofib}(X \to 0)$. Hence we have two endofunctors $\Sigma : \mathcal{C} \to \mathcal{C}$ and $\Omega : \mathcal{C} \to \mathcal{C}$. One can show that if $\mathcal{C}$ is a stable ∞-category, then Σ and Ω are equivalences.

Example 2.6. The ∞-category Sp of all spectra is a stable ∞-category.

In order to give the definition of presentable ∞-categories, we need to fix a Grothendieck universe.

Definition 2.7. We say an ∞-category $\mathcal{C}$ is *κ-accessible* if $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}_0)$ for some small ∞-category $\mathcal{C}_0$. And we say $\mathcal{C}$ is *accessible* if $\mathcal{C}$ is κ-accessible for some cardinal κ.

Definition 2.8. An ∞-category $\mathcal{C}$ is *presentable* if $\mathcal{C}$ is accessible and admits small colimits.

Example 2.9. We say ∞-category $\mathcal{C}$ is *κ-compactly generated* if $\mathcal{C}$ is presentable and κ-accessible. In particular, if $\kappa = \omega$, we will simply refer to κ-compactly generated as compactly generated.

We always consider the presentable stable ∞-categories. We denote the ∞-category of all presentalbe stable ∞-categories by $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$, where the functors between two presentalbe stable ∞-categories are colimit-preserving functors.

Definition 2.10. We say a presentable stable ∞-category $\mathcal{C}$ is *dualizable*, if $\mathcal{C}$ is a retract of compactly generated stable ∞-category in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$. That is, there exists a compactly generated stable ∞-category $\mathcal{D}$, with colimit-preserving functors $U : \mathcal{C} \to \mathcal{D}$ and $V : \mathcal{D} \to \mathcal{C}$, s.t. $V \circ U \simeq \mathrm{id}_{\mathcal{C}}$. We denote the ∞-category of all dualizable presentable stable ∞-categories with compact functors by $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{st}}$. Here, A functor F is *compact*, if F admits a right adjoint G and G preserves filtered colimits.

3 Algebraic K-theory

In this section, we recall the algebraic K-theory of a small stable ∞-category. There are many equivalent definitions for algebraic K-theory. Here we adapt the definition given by the Q-construction. See [9] for example.

For an ∞-category $\mathcal{C}$, we have the mapping anima functor $\mathrm{Map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$. By Grothendieck construction (see [8] for example), the right fibration classifying $\mathrm{Map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ is given by the twisted arrow category $(s, t) : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{D} \times \mathcal{D}^{\mathrm{op}}$.

Definition 3.1. For an ∞-category $\mathcal{C}$ with finite limits, we define $\mathrm{Q}_n(\mathcal{C})$ to be the full subcategory of $\mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$ spanned by those diagrams which take every square of the form

$$\begin{array}{ccc} (i \le \ell) & \longrightarrow & (j \le \ell) \\ \downarrow & & \downarrow \\ (i \le k) & \longrightarrow & (j \le k) \end{array}$$

to a cartesian square in $\mathcal{C}$.

Hence, we get a simplicial category $\mathrm{Q}_\bullet(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{lex}}_\infty$. One can show that $\mathrm{Q}_\bullet(\mathcal{C})$ is functorial in $\mathcal{C}$, hence we get the functor

$$\mathrm{Q}_\bullet(-) : \mathrm{Cat}^{\mathrm{lex}}_\infty \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{lex}}_\infty); \mathcal{C} \mapsto \mathrm{Q}_\bullet(\mathcal{C}).$$

For the ∞-category $\mathrm{Q}_n(\mathcal{C})$, taking the maximal subgroupoid of $\mathrm{Q}_n(\mathcal{C})$, we get an anima $\mathrm{Q}_n(\mathcal{C})^{\simeq} \in \mathrm{An}$. Hence $\mathrm{Q}_\bullet(\mathcal{C})^{\simeq}$ is a simplicial anima. By taking the geometric realization of the simplicial anima $\mathrm{Q}_\bullet(\mathcal{C})^{\simeq}$, we get the anima $|\mathrm{Q}_\bullet(\mathcal{C})^{\simeq}|$. And we can make the following definition.

Definition 3.2. For a small stable ∞-category $\mathcal{C}$, we define

$$\mathrm{K}(\mathcal{C}) := \Omega|\mathrm{Q}_\bullet(\mathcal{C})^{\simeq}|.$$

Hence, we have the functor

$$\mathrm{K} : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{An}.$$

Next, we introduce the universal property of algebraic K-theory. We first introduce the concept of grouplike additive functors.

Definition 3.3. Let $\mathcal{C}$ be an ∞-category with finite products. We define $\mathrm{CMon}(\mathcal{C})$ to be the full subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$ spanned by all functors $X : \mathrm{Fin}_* \to \mathcal{C}$, s.t.

$$X(\langle n \rangle) \simeq \prod_n X(\langle 1 \rangle),$$

for any n. Objects in $\mathrm{CMon}(\mathcal{C})$ are called *commutative monoid objects* in $\mathcal{C}$.

In particular, we have the ∞-category $\mathrm{CMon}(\mathrm{An})$ of commutative monoid objects in An. Note that we have the functor $\pi_0 : \mathrm{An} \to \mathrm{Set}$, which induces a functor

$$\pi_0 : \mathrm{CMon}(\mathrm{An}) \to \mathrm{CMon}(\mathrm{Set}) = \mathrm{CMon}.$$

Definition 3.4. $X \in \mathrm{CMon}(\mathrm{An})$ is *grouplike*, if $\pi_0(X)$ is a group. We denote the full subcategory of $\mathrm{CMon}(\mathrm{An})$ spanned by grouplike commutative monoid objects in An by $\mathrm{CMon}(\mathrm{An})^{\mathrm{gp}}$. And we say a functor $F : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{An}$ is *grouplike*, if for any $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{ex}}$, $F(\mathcal{C})$ lies in $\mathrm{CMon}(\mathrm{An})^{\mathrm{gp}}$.

Definition 3.5. Consider the functor $F : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{An}$ with $F(0) \simeq *$. We say F is *additive* if F sends every split exact sequence $\mathrm{Cat}_\infty^{\mathrm{ex}}$ to a fiber sequence in An.

Theorem 3.6 (Blumberg-Gepner-Tabuada, [1])**.** $\mathrm{K} : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{An}$ *is the initial grouplike additive functor under* $(-)^{\simeq} : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{An}$.

4 Efimov K-theory

For a dualizable presentable stable ∞-category $\mathcal{C}$, the yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ factors through $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C}^\kappa)$, which admits a left adjoint $\mathrm{colim} : \mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$. We define the Calkin category of $\mathcal{C}$ to be

$$\mathrm{Calk}_\kappa(\mathcal{C}) := \ker(\mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}).$$

One can show that $\mathrm{Calk}_\kappa(\mathcal{C})$ is a compactly generated stable ∞-category. That is, $\mathrm{Calk}_\kappa(\mathcal{C}) = \mathrm{Ind}(\mathrm{Calk}_\kappa(\mathcal{C})^\omega)$. In particular, $\mathrm{Calk}_\kappa(\mathcal{C})^\omega$ is a small stable ∞-category. Hence, we can make the following definition.

Definition 4.1 (Efimov K-theory)**.** For a dualizable presentable stable ∞-category $\mathcal{C}$, we define the *Efimov* K-*theory* or *continuous* K-*theory* to be

$$\mathrm{K}_{\mathrm{cont}}(\mathcal{C}) := \Omega \mathrm{K}(\mathrm{Calk}_\kappa(\mathcal{C})^\omega).$$

In fact, for any additive invariant and localizing invariant $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$, we can defnie the continuous extension of E to $\mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{dual}}$.

Example 4.2. If $\mathcal{C}$ is a compactly generated stable ∞-category, then $\mathrm{K}_{\mathrm{cont}}(\mathcal{C}) \simeq \mathrm{K}(\mathcal{C}^\omega)$.

Similar to the main result in [1], we have the following result.

Theorem 4.3 ([7])**.** *The functor* $(\mathcal{U}_{\mathrm{add}})_{\mathrm{cont}} : \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{dual}} \to \mathrm{Mot}_{\mathrm{add}}$ *is the universal additive invariant.*

And we also have the following co-representability.

Theorem 4.4 (Co-representability, [7]). *Let $\mathcal{C}$ be a dualizable presentable stable ∞-category, then we have*

$$\mathrm{map}((\mathcal{U}_{\mathrm{add}})_{\mathrm{cont}}(\mathrm{Sp}), (\mathcal{U}_{\mathrm{add}})_{\mathrm{cont}}(\mathcal{C})) \simeq \mathrm{K}_{\mathrm{cont}}(\mathcal{C}).$$

And we have similar results from algebraic K-theory.

Proposition 4.5 ([7]). *The functors $\mathrm{K}_{\mathrm{cont}}$ admits a lax symmetric monoidal structure and $\mathrm{K}_{\mathrm{cont}}$ is initial in $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{add}}(\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{st}}, \mathrm{Sp})$.*

Theorem 4.6 ([7]). *Let $\{\mathcal{C}_i\}_{i\in I}$ be a family of dualizable presentable stable ∞-categories, then we have*

$$\mathrm{K}_{\mathrm{cont}}(\prod_{i\in I}\mathcal{C}_i) \simeq \prod_{i\in I}\mathrm{K}_{\mathrm{cont}}(\mathcal{C}_i).$$

Definition 4.7 (Efimov). Let $\{\mathcal{C}_n\}$ be an inverse sequence of dualizable presentable stable ∞-categories, s.t. the transition functors $f_{nm}, n \geq m$ are compact. Let g_{nm} the right adjoint functors of f_{nm}. We say that $\{\mathcal{C}_n\}$ is a *Mittag-Leffler inverse system*, if the following conditions hold:

(1) For any $n \geq 0$, the inverse sequence $(f_{kn}g_{kn})_{k\geq n}$is essentially constant in the category $\mathrm{Fun}(\mathcal{C}_n, \mathcal{C}_n)$.

(2) For any $n, m \geq 0$, the functor

$$\varprojlim_{k\geq n,m} f_{km}g_{kn} : \mathcal{C}_n \to \mathcal{C}_m$$

is compact and has a left adjoint.

The following result of Efimov justifies the name of *continuous* K-*theory*, which means that it commutes with the Mittag-Leffler inverse limits.

Theorem 4.8 (Efimov, [10] Theorem 7). *If $\{\mathcal{C}_i\}$ is a Mittag-Leffler sequence of dualizable presentable stable ∞-categories, then we have*

$$\mathrm{K}_{\mathrm{cont}}(\varprojlim_i{}^{\mathrm{dual}}\mathcal{C}_i) \simeq \varprojlim_i \mathrm{K}_{\mathrm{cont}}(\mathcal{C}_i).$$

In particular, Efimov computed the continuous K-theory of nuclear $A^{\wedge}_I$-modules for a noetherian ring A with an ideal I. Note that the concept of nuclear modules is introduced by Clausen-Scholze in condensed mathematices. See [5].

Theorem 4.9 (Efimov, [10]). *Let A be a Noetherian ring and $I \subset A$ is an ideal, then we have*

$$\mathrm{K}_{\mathrm{cont}}(\mathrm{Nuc}(A^{\wedge}_I)) \simeq \varprojlim_n \mathrm{K}(A/I^n).$$

The following result us given by Efimov is about Efimov K-theory of sheaves on locally compact Hausdorff spaces.

Theorem 4.10 (Efimov, [6] Theorem 15). *Let $\mathcal{C}$ be a dualizable presentable stable ∞-category, then we have*

$$\mathrm{K}_{\mathrm{cont}}(\mathrm{Shv}(\mathbb{R}^n, \mathcal{C})) \simeq \Omega^n\mathrm{K}_{\mathrm{cont}}(\mathcal{C}).$$

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