Applications of p-harmonic transplantation for functional inequalities involving a Finsler norm

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Abstract

We prove several inequalities such as Sobolev, Poincaré, logarithmic Sobolev, which involve a general norm (Finsler norm) with accurate information of extremals. These inequalities are valid for Finsler radially symmetric functions. We use a special case of p-harmonic transplantation between symmetric functions. This manuscript is based on our recent paper [7].

1 Introduction

The Sobolev inequality

$$S_{N,p}\left(\int_{\mathbb{R}^N}|u|^{p^*}dx\right)^{p/p^*}\le\int_{\mathbb{R}^N}|\nabla u|^p dx \tag{1}$$

which holds for every $u\in W^{1,p}(\mathbb{R}^N)$, where $N\ge 2$, $1\le p<N$, and $p^*=\frac{Np}{N-p}$, is one of the fundamental tools in analysis. The best constant $S_{N,p}$ is known as

$$\begin{cases} S_{N,p}=\pi^{p/2}N\left(\frac{N-p}{p-1}\right)^{p-1}\left(\frac{\Gamma(\frac{N}{p})\Gamma(1+\frac{N}{p'})}{\Gamma(N)\Gamma(1+\frac{N}{2})}\right)^{p/N}, \quad 1<p<N,\\ S_{N,1}=\pi^{1/2}\frac{N}{\left(\Gamma(1+\frac{N}{2})\right)^{1/N}}, \quad p=1, \end{cases} \tag{2}$$

where $p'=\frac{p}{p-1}$, see [1] and [12]. It is well-known that the best constant for $1<p<N$ is achieved by a family of functions of the form

$$U(x)=\left(a+b|x|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}}, \quad a,b>0 \tag{3}$$

and its translation $U(x-x_0)$ for $x_0\in\mathbb{R}^N$. If we replace $\mathbb{R}^N$ by a domain Ω different from $\mathbb{R}^N$, then still Sobolev's inequality with the same best constant holds true for functions in $W_0^{1,p}(\Omega)$, however, the attainability of the constant is lost.

Recently, Ioku [9] obtained a new Sobolev type inequality for radially symmetric functions on the ball B_R with radius $R>0$, which admits extremals for the best constant in the inequality.

Theorem A (Ioku [9])

Let $N\ge 2$, $1<p<N$ and $p^*=\frac{Np}{N-p}$. Then for any radially symmetric function $v\in W_0^{1,p}(B_R)$, the inequality

$$S_{N,p}\left(\int_{B_R}\frac{|v(y)|^{p^*}}{\left(1-(\frac{|y|}{R})^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}}dy\right)^{p/p^*}\le\int_{B_R}|\nabla v(y)|^p dy$$

holds true. Here $S_{N,p}$ on the left-hand side is the same constant in (2). *The equality occurs if and only if v is of the form*

$$v(y) = \left(a + b \left(|y|^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}} \right)^{\frac{p}{p-N}} \right)^{\frac{p-N}{p}}$$

for some $a, b > 0$.

Ioku obtained the above theorem by exploiting a transformation between radially symmetric functions on B_R and on the whole $\mathbb{R}^N$. Then the inequality in Theorem A is the "disguised" version of (1) under the transformation. His transformation is a special case of p-harmonic transplantation initiated by J. Hersh [8], see also [3]. This point of view is pursued in [11].

In this paper, we use the same transformation by Ioku, but between radially symmetric functions on the whole space (resp. on the ball) and *Finsler radially symmetric* functions on the ball (resp. on the whole space). The resulting inequalities include a general norm (Finsler norm) with the precise information on the extremizers of the best constant involved. These inequalities will be fundamental tools on the analysis of *anisotropy* in many fields of science, where the anisotropy is realized to introduce a general norm, instead of the usual (Euclidean) norm.

2 An integral formula for Finsler symmetric functions

Let $H : \mathbb{R}^N \to [0, +\infty)$, $N \geq 2$ be a function such that H is convex, $H(\xi) \geq 0$, $H(\xi) = 0$ if and only if $\xi = 0$, and satisfies

$$H(t\xi) = |t| H(\xi), \quad \text{for any } \xi \in \mathbb{R}^N, \text{ and for any } t \in \mathbb{R}. \tag{4}$$

By (4), H must be even: $H(-\xi) = H(\xi)$ for all $\xi \in \mathbb{R}^N$. We assume in this paper that $H \in C^1(\mathbb{R}^N \setminus \{0\})$. We call H a *Finsler norm* on $\mathbb{R}^N$. Since all norms on $\mathbb{R}^N$ are equivalent to each other, we see the existence of positive constants α and β such that

$$\alpha |\xi| \leq H(\xi) \leq \beta |\xi|, \quad \xi \in \mathbb{R}^N.$$

The dual norm of H is the function $H^0 : \mathbb{R}^N \to [0, +\infty)$ defined by

$$H^0(x) = \sup_{\xi \in \mathbb{R}^N \setminus \{0\}} \frac{\xi \cdot x}{H(\xi)}.$$

It is well-known that H^0 is also a norm on $\mathbb{R}^N$ and satisfies the inequality

$$\frac{1}{\beta} |x| \leq H^0(x) \leq \frac{1}{\alpha} |x|, \quad \text{for any } x \in \mathbb{R}^N.$$

The set

$$\mathcal{W} = \{ x \in \mathbb{R}^N \mid H^0(x) < 1 \}$$

is called the *Wulff ball*, or the H^0-unit ball, and we denote $\kappa_N = \mathcal{H}^N(\mathcal{W})$, where $\mathcal{H}^N$ denotes the N-dimensional Hausdorff measure on $\mathbb{R}^N$. We also denote

$$\mathcal{W}_r = \{ x \in \mathbb{R}^N \mid H^0(x) < r \}$$

for any $r > 0$.

Here we just recall some properties of H and H^0. These will be proven by using the homogeneity property of H and H^0.

Proposition 2.1. *Let H be a Finsler norm on $\mathbb{R}^N$. Then the following properties hold true:*

(1) $|\nabla_\xi H(\xi)| \le C$ *for any* $\xi \neq 0$.

(2) $\nabla_\xi H(\xi) \cdot \xi = H(\xi)$, $\nabla_x H(x) \cdot x = H(x)$ *for any* $\xi \neq 0$, $x \neq 0$.

(3) $(\nabla_\xi H)(t\xi) = \frac{t}{|t|} (\nabla_\xi H)(\xi)$ *for any* $\xi \neq 0$, $t \neq 0$.

(4) $H\left(\nabla_x H^0(x)\right) = 1$, $H^0(\nabla_\xi H(\xi)) = 1$.

(5) $H^0(x)(\nabla_\xi H)\left(\nabla_x H^0(x)\right) = x$.

For a domain $\Omega \subset \mathbb{R}^N$ and a Borel set $E \subset \mathbb{R}^N$, the *anisotropic H-perimeter* of a set E with respect to Ω is defined as

$$P_H(E;\Omega) = \sup\left\{\int_{E\cap\Omega} \mathrm{div}\sigma dx \mid \sigma \in C_0^\infty(\Omega;\mathbb{R}^N), H^0(\sigma(x)) \le 1\right\}.$$

If E is Lipschitz, then it holds $P_H(E;\Omega) = \int_{\Omega\cap\partial^* E} H(\nu) d\mathcal{H}^{N-1}$, where $\partial^* E$ denotes the reduced boundary of the set E and $\nu(x)$ is the measure theoretic outer unit normal of $\partial^* E$.

If H^0 is Lipschitz, then $\partial\mathcal{W}$ is also Lipschitz and $\partial^*\mathcal{W} = \partial\mathcal{W}$ holds. In this case, the outer unit normal of $\mathcal{W}$ is given by $\nu = \frac{\nabla H^0}{|\nabla H^0|}$. Thus

$$\begin{aligned} P_H(\mathcal{W};\mathbb{R}^N) &= \int_{\partial\mathcal{W}} H(\nu(x))\, d\mathcal{H}^{N-1}(x) \\ &= \int_{\partial\mathcal{W}} H(\frac{\nabla H^0}{|\nabla H^0|})\, d\mathcal{H}^{N-1}(x) \\ &= \int_{\partial\mathcal{W}} \frac{1}{|\nabla H^0|}\, d\mathcal{H}^{N-1}(x). \end{aligned}$$

Here we have used $H(\nabla H^0(x)) = 1$ by Proposition 2.1 and the positive 1-homogeneity of H. Similarly, we have

$$P_H(\mathcal{W}_r;\mathbb{R}^N) = \int_{\partial\mathcal{W}_r} \frac{1}{|\nabla H^0|}\, d\mathcal{H}^{N-1}(x) \tag{5}$$

for any $r > 0$. On the other hand, by the fact $H^0(x) \equiv 1$ on $\partial\mathcal{W}$, the formula $x\cdot\nabla H^0(x) = H^0(x)$ by Proposition 2.1, and the divergence theorem, we have

$$\begin{aligned} P_H(\mathcal{W};\mathbb{R}^N) &= \int_{\partial\mathcal{W}} \frac{1}{|\nabla H^0|}\, d\mathcal{H}^{N-1}(x) \\ &= \int_{\partial\mathcal{W}} \frac{H^0(x)}{|\nabla H^0(x)|}\, d\mathcal{H}^{N-1}(x) \\ &= \int_{\partial\mathcal{W}} \frac{x\cdot\nabla H^0(x)}{|\nabla H^0(x)|}\, d\mathcal{H}^{N-1}(x) \\ &= \int_{\partial\mathcal{W}} x\cdot\nu\, d\mathcal{H}^{N-1}(x) \\ &= \int_{\mathcal{W}} \mathrm{div} x dx = N\mathcal{H}^N(\mathcal{W}). \end{aligned}$$

Thus we obtain

$$P_H(\mathcal{W};\mathbb{R}^N) = N\mathcal{H}^N(\mathcal{W}) = N\kappa_N. \tag{6}$$

Similarly, we have

$$P_H(\mathcal{W}_r;\mathbb{R}^N) = N\kappa_N r^{N-1} \quad (r > 0). \tag{7}$$

In the following, we call a function g of the form $g(x) = h(H^0(x))$ for some $h : \mathbb{R}_+ \to \mathbb{R}$ as H^0-symmetric , or *Finsler symmetric* function.

The following is a key fact in this paper. Though the statement is widely known, we prove it here for the sake of completeness.

Proposition 2.2. (Polar formula) *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be such that $h \circ H^0 \in L^1_{loc}(\mathbb{R}^N)$. Then it holds that*

$$\begin{aligned} \int_{H^0(x)<t} h(H^0(x))dx &= P_H(\mathcal{W}; \mathbb{R}^N) \int_0^t h(s)s^{N-1}ds \\ &= N\kappa_N \int_0^t h(s)s^{N-1}ds. \end{aligned} \tag{8}$$

In particular, if $h \circ H^0 \in L^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} h(H^0(x))dx = N\kappa_N \int_0^\infty h(s)s^{N-1}ds$$

holds.

For the proof of Proposition 2.2, we use the coarea formula in the following form.

Theorem 2.3. (Coarea formula) *Let $f : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz and let $g \in L^1(\mathbb{R}^N)$. Then it holds that*

$$\int_{f(x)<t} g(x)dx = \int_0^t \int_{f(x)=s} \frac{g(x)}{|\nabla f(x)|} d\mathcal{H}^{N-1}(x)ds. \tag{9}$$

Proof of Proposition 2.2.

For $h : \mathbb{R}_+ \to \mathbb{R}$ as above, put $g(x) = h(H^0(x))$, $f(x) = H^0(x)$ in Theorem 2.3. Then (9), (5), (7), and (6) yield that

$$\begin{aligned} \int_{H^0(x)<t} h(H^0(x))dx &\overset{(9)}{=} \int_0^t \int_{H^0(x)=s} \frac{h(H^0(x))}{|\nabla H^0(x)|} d\mathcal{H}^{N-1}(x)ds \\ &= \int_0^t h(s) \left(\int_{\partial \mathcal{W}_s} \frac{1}{|\nabla H^0(x)|} d\mathcal{H}^{N-1}(x) \right) ds \\ &\overset{(5)}{=} \int_0^t h(s) P_H(\mathcal{W}_s; \mathbb{R}^N)ds \\ &\overset{(7)}{=} \int_0^t h(s) P_H(\mathcal{W}; \mathbb{R}^N)s^{N-1}ds \\ &= P_H(\mathcal{W}; \mathbb{R}^N) \int_0^t h(s)s^{N-1}ds \\ &\overset{(6)}{=} N\kappa_N \int_0^t h(s)s^{N-1}ds. \end{aligned}$$

This completes the proof of Proposition 2.2. □

3 A transformation between symmetric functions

Let $u = u(x)$ be a radially symmetric function, thus, there exists a function U defined on $[0, +\infty)$ such that $u(x) = U(|x|)$. Also let $v = v(y)$ be a Finsler radially symmetric function on $\mathcal{W}_R$ of the form $v(y) = V(H^0(y))$ for some $V = V(s)$, $s \in [0, R)$, where $R > 0$ be any number. Fix $1 < p < N$ and assume that u and v are related with each other by the transformation

$$\begin{cases} r = |x|, x \in \mathbb{R}^N, \\ s = H^0(y), y \in \mathcal{W}_R \subset \mathbb{R}^N, \\ r^{\frac{p-N}{p-1}} = s^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}, \\ u(x) = U(r) = V(s) = v(y). \end{cases} \tag{10}$$

Throughout the paper, ω_{N-1} denotes the surface measure of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Under the transformation (10), we have the following equivalence.

Proposition 3.1. *Let $1 < p < N$ and let u, v be as above. Let $F : \mathbb{R} \to \mathbb{R}$ be continuous. Then we have*

$$\int_{\mathbb{R}^N} |\nabla u|^p dx = \frac{\omega_{N-1}}{N\kappa_N} \int_{\mathcal{W}_R} H(\nabla v)^p dy,$$
$$\int_{\mathbb{R}^N} F(u(x))dx = \frac{\omega_{N-1}}{N\kappa_N} \int_{\mathcal{W}_R} \frac{F(v(y))}{\left(1 - \left(\frac{H^0(y)}{R}\right)^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}} dy.$$

Proof. By (10), we see that if r runs from 0 to ∞, then s runs from 0 to R, and vice versa. Also by differentiating the relation $r^{-\frac{N-p}{p-1}} = s^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}}$ with respect to s, we have

$$\left(\frac{p-N}{p-1}\right) r^{\frac{p-N}{p-1}-1} \left(\frac{dr}{ds}\right) = \left(\frac{p-N}{p-1}\right) s^{\frac{p-N}{p-1}-1},$$

which implies

$$\frac{dr}{ds} = (r(s))^{\frac{N-1}{p-1}} s^{\frac{1-N}{p-1}}, \quad \frac{ds}{dr} = (s(r))^{\frac{N-1}{p-1}} r^{\frac{1-N}{p-1}}.$$

Since $U'(r) = V'(s)(\frac{ds}{dr})$, we compute

$$\begin{aligned} \int_0^\infty |U'(r)|^p r^{N-1} dr &= \int_0^R |V'(s)|^p \left(\frac{ds}{dr}\right)^p r(s)^{N-1} \left(\frac{dr}{ds}\right) ds \\ &= \int_0^R |V'(s)|^p \left(\frac{ds}{dr}\right)^{p-1} r(s)^{N-1} ds \\ &= \int_0^R |V'(s)|^p \left(s^{\frac{N-1}{p-1}} r(s)^{\frac{1-N}{p-1}}\right)^{p-1} r(s)^{N-1} ds \\ &= \int_0^R |V'(s)|^p s^{N-1} ds. \end{aligned} \tag{11}$$

Now, for $v(y) = V(H^0(y))$, $y \in \mathcal{W}_R$, we compute

$$\begin{aligned} &\nabla v(y) = V'(H^0(y))\nabla H^0(y), \\ &H(\nabla v(y)) = H\left(V'(H^0(y))\nabla H^0(y)\right) = |V'(H^0(y))| H(\nabla H^0(y)) = |V'(H^0(y))|, \end{aligned} \tag{12}$$

here we used Proposition 2.1. Recalling Proposition 2.2 (8) with $h(s) = |V'(s)|^p$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u|^p dx &= \omega_{N-1} \int_0^\infty |U'(r)|^p r^{N-1} dr \\
&\overset{(11)}{=} \frac{\omega_{N-1}}{N\kappa_N} N\kappa_N \int_0^R |V'(s)|^p s^{N-1} ds \\
&\overset{(8)}{=} \frac{\omega_{N-1}}{N\kappa_N} \int_{\mathcal{W}_R} |V'(H^0(y))|^p dy \\
&\overset{(12)}{=} \frac{\omega_{N-1}}{N\kappa_N} \int_{\mathcal{W}_R} H(\nabla v)^p dy.
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
\int_0^\infty F(U(r)) r^{N-1} dr &= \int_0^R F(V(s)) r(s)^{N-1} \left(\frac{dr}{ds}\right) ds \\
&= \int_0^R F(V(s)) r(s)^{N-1} r(s)^{\frac{N-1}{p-1}} s^{\frac{1-N}{p-1}} ds \\
&= \int_0^R F(V(s)) r(s)^{(N-1)(1+\frac{1}{p-1})} s^{\frac{1-N}{p-1}+1-N} s^{N-1} ds \\
&= \int_0^R F(V(s)) \left(\left(s^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}\right)^{\frac{p-1}{p-N}}\right)^{(N-1)(\frac{p}{p-1})} s^{\frac{p(1-N)}{p-1}} s^{N-1} ds \\
&= \int_0^R \frac{F(V(s))}{s^{\frac{p(N-1)}{p-1}} \left(s^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}\right)^{\frac{p(N-1)}{N-p}}} s^{N-1} ds \\
&= \int_0^R \frac{F(V(s))}{\left(1 - \left(\frac{s}{R}\right)^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}} s^{N-1} ds. \qquad (13)
\end{aligned}$$

Thus again Proposition 2.2 yields that

$$\begin{aligned}
\int_{\mathbb{R}^N} F(u(x)) dx &= \omega_{N-1} \int_0^\infty F(U(r)) r^{N-1} dr \\
&\overset{(13)}{=} \frac{\omega_{N-1}}{N\kappa_N} N\kappa_N \int_0^R \frac{F(V(s))}{\left(1 - \left(\frac{s}{R}\right)^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}} s^{N-1} ds \\
&\overset{(8)}{=} \frac{\omega_{N-1}}{N\kappa_N} \int_{\mathcal{W}_R} \frac{F(v(y))}{\left(1 - \left(\frac{H^0(y)}{R}\right)^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}} dy.
\end{aligned}$$

This completes the proof. □

Next, let us replace the roles of u and v in Proposition 3.1. That is, let $v = v(y)$ be a radially symmetric function on $B_R = \{x \in \mathbb{R}^N \ : \ |x| < R\}$, here $R > 0$. Thus, there exists a function V defined on $[0, R)$ such that $v(y) = V(|y|)$. Also let $u = u(x)$ be a Finsler radially symmetric function on $\mathbb{R}^N$ of the form $u(x) = U(H^0(x))$, $U = U(r)$, $r \in [0, +\infty)$. Assume u and v are related by the transformation

$$\begin{cases} r = H^0(x), x \in \mathbb{R}^N, \\ s = |y|, y \in B_R \subset \mathbb{R}^N, \\ r^{\frac{p-N}{p-1}} = s^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}, \\ u(x) = U(r) = V(s) = v(y), \end{cases} \tag{14}$$

where $1 < p < N$. Then as before, under the transformation (14), we have the following equivalence.

Proposition 3.2. *Let $1 < p < N$ and let u, v be as above. Let $F : \mathbb{R} \to \mathbb{R}$ be continuous. Then we have*

$$\int_{B_R} |\nabla v|^p dy = \frac{\omega_{N-1}}{N\kappa_N} \int_{\mathbb{R}^N} H(\nabla u)^p dx,$$

$$\int_{B_R} F(v(y))dy = \frac{\omega_{N-1}}{N\kappa_N} \int_{\mathbb{R}^N} \frac{F(u(x))}{\left(1 + \left(\frac{H^0(x)}{R}\right)^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}} dx.$$

The proof is similar as above, and we omit it here.

4 Functional inequalities for symmetric functions

In this section, we will prove several functional inequalities which hold for functions in the appropriate Sobolev space with some symmetry. Though many similar inequalities can be derived by the same idea, we record here few of them.

Following inequalities are direct consequences of Proposition 3.1, Proposition 3.2, and the known inequalities on $\mathbb{R}^N$ or B_R, with the information of extremals.

4.1 The sharp L^p-Sobolev inequality

First we treat the sharp L^p-Sobolev inequality.

Theorem 4.1. *Let $N \geq 2$, $1 < p < N$ and $p^* = \frac{Np}{N-p}$. Then for any Finsler radially symmetric function $v \in W_0^{1,p}(\mathcal{W}_R)$, the inequality*

$$\tilde{S}_{N,p} \left(\int_{\mathcal{W}_R} \frac{|v(y)|^{p^*}}{\left(1 - (\frac{H^0(y)}{R})^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}} dy \right)^{p/p^*} \leq \int_{\mathcal{W}_R} H(\nabla v(y))^p dy$$

holds true. Here

$$\tilde{S}_{N,p} = S_{N,p} \left(\frac{\omega_{N-1}}{N\kappa_N} \right)^{p/p^*-1}$$

and $S_{N,p}$ is defined in (2)*. The equality occurs if and only if v is of the form*

$$v(y) = \left(a + b \left((H^0(y))^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}} \right)^{\frac{p}{p-N}} \right)^{\frac{p-N}{p}}$$

for some $a, b > 0$.

Proof. For any Finsler radially symmetric function $v \in W_0^{1,p}(\mathcal{W}_R)$, define $u \in W^{1,p}(\mathbb{R}^N)$ as $u(x) = U(r) = V(s) = v(y)$, where $r = |x|$, $x \in \mathbb{R}^N$ and $s = H^0(y)$, $y \in \mathcal{W}_R$. U and V are defining functions of u, v respectively. Then the L^p-Sobolev inequality (1) for u (with the information of extremals) and Proposition 10 yield the result. □

4.2 The Euclidean L^p-logarithmic Sobolev inequality

The Euclidean L^p-logarithmic Sobolev inequality states that

$$\int_{\mathbb{R}^N} |u|^p \log |u|^p dx \le \frac{N}{p} \log \left(\mathcal{L}_p \int_{\mathbb{R}^N} |\nabla u|^p dx \right) \tag{15}$$

holds for any function $u \in W^{1,p}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |f|^p dx = 1$. This form of inequality was first proved by Weissler [13] for $p = 2$, Ledoux [10] for $p = 1$, del Pino and Dolbeault [5] for $1 \le p < N$, and finally generalized by Gentil [6] for $1 \le p < \infty$. Actually, [6] extends the result in [5] not only for all $p \ge 1$, but also for any norm on $\mathbb{R}^N$ other than usual Euclidean norm. Here the sharp constant $\mathcal{L}_p$ is given by

$$\begin{cases} \mathcal{L}_1 = \frac{1}{N} \pi^{-1/2} \left(\Gamma(N/2+1) \right)^{1/N}, \quad p = 1, \\ \mathcal{L}_p = \frac{p}{N} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left(\frac{\Gamma(N/2+1)}{\Gamma(N/p'+1)} \right)^{p/N}, \quad p > 1, \end{cases} \tag{16}$$

where $p' = \frac{p}{p-1}$ for $p > 1$. For $p = 1$, Beckner [4] proved that the extremal functions for (15) are the characteristic functions of balls. For $1 < p < N$, it is proved in [5] that the extremal functions of (15) must be of the form

$$u(x) = U(|x|) = C(N,p) \exp \left(-\frac{1}{\sigma} |x|^{p'} \right),$$

where $\sigma > 0$ and

$$C(N,p) = \left(\pi^{N/2} \left(\frac{\sigma}{p} \right)^{N/p'} \frac{\Gamma(N/p'+1)}{\Gamma(N/2+1)} \right)^{-1/p}, \tag{17}$$

and its translation. Finally, the same characterization of the equality case also holds true even when $p \ge N$; this fact is proved in the very recent paper [2], which solves a conjecture in [6].

When $1 < p < N$, by the same argument as in Theorem 4.1, we have

Theorem 4.2. *Let $1 < p < N$ and $R > 0$. Then*

$$\left(\frac{\omega_{N-1}}{N\kappa_N} \right) \int_{\mathcal{W}_R} \frac{|v(y)|^p \log |v(y)|^p}{\left(1 - \left(\frac{H^0(y)}{R} \right)^{\frac{N-p}{p-1}} \right)^{\frac{p(N-1)}{N-p}}} dy \le \frac{N}{p} \log \left(\tilde{\mathcal{L}}_p \int_{\mathcal{W}_R} H(\nabla v)^p dy \right)$$

holds true for any Finsler radially symmetric function v satisfying

$$\left(\frac{\omega_{N-1}}{N\kappa_N} \right) \int_{\mathcal{W}_R} \frac{|v(y)|^p}{\left(1 - \left(\frac{H^0(y)}{R} \right)^{\frac{N-p}{p-1}} \right)^{\frac{p(N-1)}{N-p}}} dy = 1.$$

Here $\tilde{\mathcal{L}}_p = \left(\frac{\omega_{N-1}}{N\kappa_N} \right) \mathcal{L}_p$, where $\mathcal{L}_p$ is as in (16)*. The equality holds if and only if v is of the form*

$$v(y) = C(N,p) \exp \left(-\frac{1}{\sigma} \left((H^0(y))^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}} \right)^{\frac{-p}{N-p}} \right),$$

where $C(N,p)$ is defined in (17)*.*

4.3 The Poincaré inequality on balls

The L^p-Poincaré inequality on balls states that

$$\lambda_1(B_1) \int_{B_R} |v(y)|^p dy \leq R^p \int_{B_R} |\nabla v(y)|^p dy \tag{18}$$

holds for any function $v \in W_0^{1,p}(B_R)$, where $B_R \subset \mathbb{R}^N$ is a ball with radius $R > 0$. Here, $\lambda_1(B_1)$ is the first eigenvalue of $-\Delta_p$ (p-Laplacian) with the Dirichlet boundary condition on the unit ball $B_1 \subset \mathbb{R}^N$. To the authors' knowledge, the explicit expression is not known for $\lambda_1(B_1)$ unless $p = 2$. The equality in (18) holds if and only if v is a constant multiple of the first eigenfunction of $-\Delta_p$ on B_R, which we denote $\phi_R \in W_0^{1,p}(B_R)$. Known regularity and symmetry results assure that the first eigenfunction of $-\Delta_p$ is $C^{1,\alpha}$ for some $\alpha \in (0,1)$ and radially symmetric. Thus we can write $\phi_R(y) = \Phi_R(|y|)$, $y \in B_R$, for some C^1-function Φ_R on $[0,R)$ with $\Phi_R(R) = 0$.

By these facts and Proposition 3.2, we have the following.

Theorem 4.3. *Let $1 < p < N$ and $R > 0$ be arbitrarily given. Then*

$$\lambda_1(B_1) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{\left(1 + \left(\frac{H^0(x)}{R}\right)^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}} dx \leq R^p \int_{\mathbb{R}^N} H(\nabla u)^p dx$$

holds true for any Finsler radially symmetric function $u(x) = U(H^0(x)) \in W^{1,p}(\mathbb{R}^N)$ such that $u(\infty) = 0$. For fixed $R > 0$, the equality holds if and only if u is the constant multiple of

$$\Phi_R\left(\left((H^0(x))^{\frac{p-N}{p-1}} + R^{\frac{p-N}{p-1}}\right)^{\frac{p-1}{p-N}}\right)$$

where $\Phi_R \in C^1([0,R))$ is such that $\phi_R(y) = \Phi_R(|y|)$, $y \in B_R$, is the first eigenfunction of $-\Delta_p$ on B_R.

Proof. The proof follows for a given Finsler radially symmetric function $u \in W^{1,p}(\mathbb{R}^N)$, define a radially symmetric function $v(y) = V(|y|)$ as $v(y) = V(s) = U(r) = u(x)$ where $s = |y|$, $y \in B_R$ and $r = H^0(x)$, $x \in \mathbb{R}^N$. Then use Proposition 3.2. □

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