# A recurring pattern in natural numbers of a certain property

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**概 要**

The concept of $v$-palindromes is recently introduced by the speaker. Let $b \geq 2$ and $n \geq 1$ be integers such that $b \nmid n$ and that $n$ is not a base $b$ palindrome. Our main theorem is that, if $n(k)$, for integers $k \geq 1$, denotes the $k$-copy repeated concatenation of the base $b$ digits of $n$, then whether $n(k)$ is a base $b$ $v$-palindrome depends only on $k$ modulo some integer $\omega \geq 1$.

## 1 Introduction

Since the concept of $v$-palindromes is introduced by the speaker [4, 3], there is not really any history or background on the subject. The usage of the word "palindrome" in the naming does, however, came from the idea that the $v$-palindromes can be viewed as an analogy of the usual palindromes. We first illustrate using the example of 1998.

$$\begin{array}{ccccc} 1998 & = & 2\cdot 3^3\cdot 37 & \longmapsto & 2+(3+3)+37 \\ \uparrow & & & & \| \\ \text{base 10 reverse} & & & & 45 \\ \downarrow & & & & \| \\ 8991 & = & 3^5\cdot 37 & \longmapsto & (3+5)+37 \end{array} \tag{1}$$

In the above, we prime factorized both 1998 and its base 10 reverse 8991. Then we summed the numbers "appearing" in each factorization (the prime divisors and their corresponding exponents larger than 1); notice that because an exponent of 1 does not "appear", i.e., we usually do not write it when doing an actual prime factorization, we do not sum it. Then perhaps somewhat surprisingly, the sums agree. In the following we rigorously define the concept of $v$-palindromes after stating some preliminary definitions.

Recall the base $b$ representation of an integer $n \geq 1$ as follows.

**Definition 1.** Let $b \geq 2$, $L \geq 1$, and $0 \leq a_{L-1}, a_{L-2}, \ldots, a_0 < b$ be integers, then we denote

$$(a_{L-1}a_{L-2}\cdots a_0)_b := a_{L-1}b^{L-1} + a_{L-2}b^{L-2} + \cdots + a_0. \tag{2}$$

**Theorem 1** ([1], Theorem 4.7)**.** *Let $b \geq 2$ be an integer. Then for every integer $n \geq 1$, there exist unique integers $L \geq 1$ and $0 \leq a_{L-1}, a_{L-2}, \ldots, a_0 < b$ with $a_{L-1} \neq 0$ such that*

$$n = (a_{L-1}a_{L-2}\cdots a_0)_b. \tag{3}$$

We give the definition of a palindrome.

**Definition 2.** Let $b \geq 2$ be an integer. An integer $n \geq 1$ given in the form (3) is a *base $b$ palindrome* if $a_i = a_{L-1-i}$ for each $0 \leq i \leq L-1$.

Intuitively speaking, a palindrome is a number whose digits read the same forwards and backwards, i.e., there is a left-right symmetry, such as 121 and 2332.

**Definition 3.** Let $b \geq 2$ be an integer and let $n \geq 1$ be an integer given in the form (3). Then the *base $b$ reverse* of $n$ is $r_b(n) = (a_0 \cdots a_{L-2}a_{L-1})_b$.

Intuitively speaking, the reverse is the number formed by writing the digits of the original number in reverse order. For example, $r_{10}(18) = 81$ and $r_{10}(20) = r_{10}(200) = 2$. With Definition 3, the definition of a base $b$ palindrome can be written simply as $n = r_b(n)$. We now define a function $v$ to denote "summing the numbers appearing in the factorization" as follows.

**Definition 4.** Let $n \geq 1$ be an integer with prime factorization

$$n = p_1^{a_1} \cdots p_s^{a_s} q_1 \cdots q_t, \tag{4}$$

where the $s, t \geq 0$ and $a_1, \ldots, a_s \geq 2$ are integers and the $p_1, \ldots, p_s, q_1, \ldots, q_t$ are distinct primes. We define

$$v(n) = \sum_{i=1}^{s}(p_i + a_i) + \sum_{j=1}^{t} q_j. \tag{5}$$

Recall that a function $f \colon \mathbb{N} \to \mathbb{C}$ is called *additive* if $f(mn) = f(m) + f(n)$ whenever $m, n \geq 1$ are coprime. Then $v$ is an additive function. We can now define $v$-palindromes.

**Definition 5.** Let $b \geq 2$ be an integer. An integer $n \geq 1$ is a *base $b$ $v$-palindrome* if

(i) $b \nmid n$,

(ii) $n \neq r_b(n)$,

(iii) $v(n) = v(r_b(n))$.

Here the notation $x \mid y$ means that $x$ divides $y$ and $x \nmid y$ that $x$ does not divide $y$. We explain why the $v$-palindromes can be viewed as an analogy to the usual palindromes. The definition of a base $b$ palindrome is $n = r_b(n)$. Condition (iii) above is obtained by applying a $v$ to $n = r_b(n)$. Condition (i) is included just for the aesthetic purpose that $n$ and $r_b(n)$ have the same number of digits and condition (ii) so that condition (iii) is nontrivial. Thus there is this slight connection between palindromes and $v$-palindromes. However it is conceivable that they will have very different behavior.

This talk is based on the speaker's paper [3], in which the main theorem [3, Theorem 1] indicates a periodic phenomenon involving $v$-palindromes and repeated concatenations. While [3] deals only with base ten, its content is easily adapted to a general base, and the main theorem of this talk is the adaptation of [3, Theorem 1] to a general base (Theorem 2 below).

We end this section by giving the definition of repeated concatenations.

**Definition 6.** Let $b \geq 2$ be an integer and let $n \geq 1$ be an integer given in the form (3). Then for integers $k \geq 1$, the *$k$-copy repeated concatenation in base $b$* of $n$ is

$$\begin{aligned} n(k)_b &= (\underbrace{a_{L-1}a_{L-2}\cdots a_0\, a_{L-1}a_{L-2}\cdots a_0\ \cdots\cdots\ a_{L-1}a_{L-2}\cdots a_0}_{k \text{ copies of } a_{L-1}a_{L-2}\cdots a_0})_b \\ &= n(1 + b^L + \cdots + b^{(k-1)L}) = n \cdot \frac{1 - b^{Lk}}{1 - b^L}. \end{aligned} \tag{6}$$

## 2 The main theorem

With the relevant definitions in the Introduction, we can now state the adaptation of [3, Theorem 1] to a general base as follows.

**Theorem 2** ([3, Theorem 1], for a general base)**.** *Let $b \geq 2$ be an integer and let $n \geq 1$ be an integer with $b \nmid n$ and $n \neq r_b(n)$. Then there exists an integer $\omega \geq 1$ such that for every integer $k \geq 1$, $n(k)_b$ is a base $b$ $v$-palindrome if and only if $n(k+\omega)_b$ is. In other words, whether $n(k)_b$ is a base $b$ $v$-palindrome depends only on $k$ modulo $\omega$.*

We illustrate the above theorem with the example of $b = 10$ and $n = 48$. The sequence of repeated concatenations in base ten of 48 is

$$48, 4848, 484848, \ldots. \tag{7}$$

Replacing each term above by 1 if it is a base ten $v$-palindrome and 0 if not, (7) becomes

$$0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots. \tag{8}$$

Now Theorem 2 says that the above sequence of 1's and 0's is periodic. It might seem that the smallest period is 3. However in fact, $48(21)_{10}$ is not a base ten $v$-palindrome and the smallest period is 21.

The proof of [3, Theorem 1] is only for $b = 10$, but is easily adapted to a general base. We give the first half of the general proof in Section 2.2 and briefly describe how the rest of the proof works in Section 2.3. For the rest of this section, we fix a base $b \geq 2$ and shall write $r_b$ simply as $r$, $n(k)_b$ simply as $n(k)$, and call a base $b$ $v$-palindrome simply a $v$-palindrome. We also fix an integer $n \geq 1$ with $b \nmid n$, $n \neq r(n)$, and $L$ base $b$ digits. Some notation which will be used later are as follows.

- For a prime $p$ and integer $n \neq 0$, $\operatorname{ord}_p(n)$ denotes the greatest integer $a$ such that $p^a \mid n$.
- sgn is the *sign function* defined by $\operatorname{sgn}(x) = 1$ if $x > 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$.
- $\vee$ and $\wedge$ are the logical symbols for *or* and *and*.

### 2.1 Preparation

Certain numbers $\rho_k$ are used in the proof and we give their definition as follows.

**Definition 7.** For integers $k \geq 1$, we denote

$$\rho_k = (\overbrace{1\underbrace{0\cdots0}_{L-1}1\underbrace{0\cdots0}_{L-1}1\ \cdots\cdots\ 1\underbrace{0\cdots0}_{L-1}1}^{k})_b, \tag{9}$$

where there are $k$ digits of 1 with $L-1$ digits of 0 between any consecutive digits of 1, and the number is regarded in base $b$.

Then it is clear that for integers $k \geq 1$, $n(k) = n\rho_k$. Also, certain functions $\varphi_{p,\delta}$ are used in the proof and we give their definition as follows.

**Definition 8.** For primes $p$ and integers $\delta \geq 1$, we define a function $\varphi_{p,\delta}\colon \mathbb{N} \cup \{0\} \to \mathbb{N}$ by setting, for integers $\alpha \geq 0$,

$$\varphi_{p,\delta}(\alpha) = v(p^{\alpha+\delta}) - v(p^{\alpha}). \tag{10}$$

The functions $\varphi_{p,\delta}$ can be described explicitly. We have

$$\varphi_{2,1}(\alpha) = \begin{cases} 2, & \text{if } \alpha = 0, 1, \\ 1, & \text{if } \alpha \geq 2. \end{cases} \tag{11}$$

For primes $p$,

$$\varphi_{p,1}(\alpha) = \begin{cases} p, & \text{if } \alpha = 0, \\ 2, & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha \geq 2. \end{cases} \tag{12}$$

For primes $p$ and integers $\delta \geq 2$,

$$\varphi_{p,\delta}(\alpha) = \begin{cases} p+\delta, & \text{if } \alpha = 0, \\ 1+\delta, & \text{if } \alpha = 1, \\ \delta, & \text{if } \alpha \geq 2. \end{cases} \tag{13}$$

We also give the following notation for the ranges of the functions $\varphi_{p,\delta}$.

**Definition 9.** For primes $p$ and integers $\delta \geq 1$, we denote $R_{p,\delta} = \varphi_{p,\delta}(\mathbb{N} \cup \{0\})$.

From (11), (12), and (13), it is clear that $R_{p,\delta}$ is of size 2 or 3.

## 2.2 First part of the proof

Let the prime factorizations of $n$ and $r(n)$ be

$$n = p_1^{a_1} \cdots p_m^{a_m}, \tag{14}$$
$$r(n) = p_1^{b_1} \cdots p_m^{b_m}, \tag{15}$$

where $m \geq 1$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \geq 0$ are integers with $(a_i, b_i) \neq (0,0)$ for each $1 \leq i \leq m$, and $p_1 < \cdots < p_m$ are primes. For integers $k \geq 1$, let the prime factorization of $\rho_k$ be

$$\rho_k = p_1^{g_1} \cdots p_m^{g_m} c, \tag{16}$$

where $g_1, \ldots, g_m \geq 0$ and $c \geq 1$ are integers with $(c, p_1 \cdots p_m) = 1$. The $g_1, \ldots, g_m, c$ varies as $k$ increases, but we suppress this dependency on $k$ from our notation for simplicity. We have, for integers $k \geq 1$,

$$n(k) = n\rho_{k,L} = p_1^{a_1+g_1} \cdots p_m^{a_m+g_m} c, \tag{17}$$
$$r(n(k)) = r(n)\rho_{k,L} = p_1^{b_1+g_1} \cdots p_m^{b_m+g_m} c. \tag{18}$$

Since $v$ is an additive function, applying $v$ to the above equalities yield

$$v(n(k)) = \sum_{i=1}^{m} v(p^{a_i+g_i}) + v(c), \tag{19}$$
$$v(r(n(k))) = \sum_{i=1}^{m} v(p^{b_i+g_i}) + v(c). \tag{20}$$

Hence $n$ is a $v$-palindrome if and only if

$$\sum_{i=1}^{m} (v(p_i^{a_i+g_i}) - v(p_i^{b_i+g_i})) = 0. \tag{21}$$

If $a_i = b_i$, then $v(p_i^{a_i+g_i}) - v(p_i^{b_i+g_i}) = 0$. Thus let those $1 \leq i \leq m$ with $a_i \neq b_i$ be ordered as $i_1 < \cdots < i_t$ (notice that because $n \neq r(n)$, $t \geq 1$). Then (21) becomes

$$\sum_{j=1}^{t} (v(p_{i_j}^{a_{i_j}+g_{i_j}}) - v(p_{i_j}^{b_{i_j}+g_{i_j}})) = 0. \tag{22}$$

But this is a cumbersome notation, so hereafter we denote $p_{i_j}$ by $p_j$, $a_{i_j}$ by $a_j$, $b_{i_j}$ by $b_j$, and $g_{i_j}$ by $g_j$. This will not cause any confusion because we will not be referring to the other prime factors or their corresponding exponents. Then (22) becomes

$$\sum_{j=1}^{t} (v(p_j^{a_j+g_j}) - v(p_j^{b_j+g_j})) = 0. \tag{23}$$

For each $1 \leq j \leq t$, put $\delta_j = a_j - b_j$ and $\mu_j = \min(a_j, b_j)$. Then the above can be rewritten, using the functions $\varphi_{p,\delta}$ of Section 2.1, as

$$\sum_{j=1}^{t} \operatorname{sgn}(\delta_j) \varphi_{p_j,|\delta_j|}(\mu_j + g_j) = 0. \tag{24}$$

Now let

$$\mathcal{U} = \left\{ (u_1, \ldots, u_t) \in R_{p_1,|\delta_1|} \times \cdots \times R_{p_t,|\delta_t|} \colon \sum_{j=1}^{t} \operatorname{sgn}(\delta_j) u_j = 0 \right\}. \tag{25}$$

Then for integers $k \geq 1$, $n(k)$ is a $v$-palindrome if and only if

$$\bigvee_{(u_1,\ldots,u_t)\in\mathcal{U}} \bigwedge_{j=1}^{t} (\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j). \tag{26}$$

The above expression depends only on $k$ and we see that only the $g_j = \operatorname{ord}_{p_j}(\rho_{k,L})$ $(1 \leq j \leq t)$ change as $k$ increases.

### 2.3 The rest of the proof

In Section 2.2, we showed that for integers $k \geq 1$, $n(k)$ is a $v$-palindrome if and only if (26) holds. Now the truth value of (26) depends only on the truth values of the individual $(\varphi_{p_j,|\delta_j|}(\mu_j+g_j) = u_j)$'s and each of them is equivalent to a simpler condition involving divisibility properties of $k$. We need to first define certain numbers $h_{p^\alpha}$ as follows.

**Definition 10.** For prime powers $p^\alpha$ with $(p, b) = 1$. The order of $b^L$ regarded as an element of $(\mathbb{Z}/p^{\alpha+\mathrm{ord}_p(b^L-1)}\mathbb{Z})^\times$ is denoted by $h_{p^\alpha}$.

Then it follows that we have the sequence of divisibility relations

$$h_p \mid h_{p^2} \mid h_{p^3} \mid \cdots . \tag{27}$$

Each $(\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j)$ is equivalent to one of the conditions

$$\begin{cases} h_{p_j} \nmid k, \\ h_{p_j} \mid k \text{ and } h_{p_j^2} \nmid k, \\ h_{p_j^2} \mid k, \\ h_{p_j} \mid k, \\ h_{p_j^2} \mid k, \\ \text{impossible}, \\ \text{always true}. \end{cases} \tag{28}$$

When $(p_j, b) = 1$, only the last two conditions are possible. The way to determine which of the above conditions $(\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j)$ is equivalent to is a bit tedious and the way in the case of $b = 10$ is described in [3]. Recall that if $x \mid z$, then for $y \in \mathbb{Z}$, $x \mid y$ if and only if $x \mid (y + z)$. Using this property and in view of (27), we see that a possible choice for $\omega$ in Theorem 2 is

$$\omega = \mathrm{lcm}\{h_{p_j^2} : (p_j, b) = 1\}. \tag{29}$$

# 3 The smallest period and order

The periodic phenomenon described by Theorem 2 raises the following questions.

**Question 1.** How to find the smallest possible $\omega$?

**Question 2.** Does there exist an integer $k \geq 1$ such that $n(k)_b$ is a base $b$ $v$-palindrome?

Based on the above questions, we give the following notation.

**Definition 11.** Let $b \geq 2$ and $n \geq 1$ be integers with $b \nmid n$ and $n \neq r_b(n)$. The smallest possible $\omega$ in Theorem 2 is denoted by $\omega_0(n)_b$. If there exists a $k \geq 1$ such that $n(k)_b$ is a base $b$ $v$-palindrome, then the smallest one is called the *order* of $n$ with respect to $b$ and denoted by $c(n)_b$; otherwise we set $c(n)_b = \infty$.

表 1: Table of $\omega_0(n)_{10}$ and $c(n)_{10}$.

| $n$ | $\omega_0(n)_{10}$ | $c(n)_{10}$ |
|---|---|---|
| 13 | 6045 | 15 |
| 17 | 337960 | 280 |
| 18 | 1 | 1 |
| 19 | 15561 | 819 |
| 26 | 6045 | 15 |
| 37 | 32412 | 12 |
| 39 | 6045 | 15 |
| 48 | 21 | 3 |
| 49 | 22701 | 3243 |
| 56 | 273 | 3 |
| 79 | 4781712 | 624 |

Although (29) provides a possible $\omega$, it is often too large. A way for finding $\omega_0(n)_{10}$ and $c(n)_{10}$ is given in the speaker's preprint [2], which is easily adapted to a general base. Whether that is the most efficient way, we are still not sure. In essence, a function called the indicator function is constructed from which both $\omega_0(n)_{10}$ and $c(n)_{10}$ can be derived quite easily. We describe the indicator function, for a general base. We need to first define certain functions $I_a$ as follows.

**Definition 12.** For integers $a \geq 1$, define a function $I_a\colon \mathbb{Z} \to \{0,1\}$ by

$$I_a(k) = \begin{cases} 1, & \text{if } a \mid k, \\ 0, & \text{if } a \nmid k. \end{cases} \tag{30}$$

That is, $I_a$ is the indicator function of $a\mathbb{Z}$ in $\mathbb{Z}$.

**Definition 13.** Let $b \geq 2$ and $n \geq 1$ be integers with $b \nmid n$ and $n \neq r_b(n)$. We define the *indicator function* $I_n^b\colon \mathbb{Z} \to \{0,1\}$ by

$$I_n^b(k) = \begin{cases} 1, & \text{if } n(k)_b \text{ is a base } b \text{ } v\text{-palindrome}, \\ 0, & \text{if } n(k)_b \text{ is not a base } b \text{ } v\text{-palindrome}, \end{cases} \tag{31}$$

for $k \geq 1$, and then extending periodically (which is possible due to Theorem 2) to the whole $\mathbb{Z}$.

Then we have the following.

**Theorem 3** ([2, Theorm 5.8], for a general base)**.** *Let $b \geq 2$ and $n \geq 1$ be integers with $b \nmid n$ and $n \neq r_b(n)$. Then there exist unique integers $q \geq 0$, $1 \leq a_1 < \cdots < a_q$, and $\lambda_1, \ldots, \lambda_q \neq 0$ such that*

$$I_n^b = \lambda_1 I_{a_1} + \lambda_2 I_{a_2} + \cdots + \lambda_q I_{a_q}. \tag{32}$$

From (32), $\omega_0(n)_b$ and $c(n)_b$ can be derived as follows.

**Theorem 4** ([2, Corollary 6.5 and Corollary 7.2], for a general base)**.** *Let $b \geq 2$ and $n \geq 1$ be integers with $b \nmid n$ and $n \neq r_b(n)$. Suppose that $I_n^b$ is expressed in the form (32), then*

$$\omega_0(n)_b = \operatorname{lcm}\{a_1, \ldots, a_q\}, \quad c(n)_b = \inf\{a_1, \ldots, a_q\}. \tag{33}$$

A definite procedure for expressing the indicator function $I_n^{10}$ in the form (32) is given in [2], which is easily adapted to a general base.

## 参考文献


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