Rational function semifields of tropical curves are finitely generated over the tropical semifield

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概 要

We prove that the rational function semifield of a tropical curve is finitely generated as a semifield over the tropical semifield $\boldsymbol{T} := (\boldsymbol{R} \cup \{-\infty\}, \max, +)$ by giving a specific finite generating set.

1 Introduction

This manuscript is on tropical geometry. Tropical geometry is an algebraic geometry over the *tropical semifield* $\boldsymbol{T} := (\boldsymbol{R} \cup \{-\infty\}, \max, +)$, which is developing from the bigining of this century. In tropical geometry, we study tropical varieties. Tropical varieties are polyhedral complexes obtained from algebraic varieties by a limit operation called *tropicalization*. Tropical curves, one-dimensional tropical varieties, correspond to algebraic curves. In this manuscript, an *(abstract) tropical curve* means a metric graph, i.e., the underlying metric space of the pair (G, l), where G is an unweighted, undirected, finite, connected, nonempty multigraph that may have loops (in what follows, we call such a multigraph a *graph* simply), $l : E(G) \to \boldsymbol{R}$ is a *length function*, and $E(G)$ stands for the set of edges of G. For a tropical curve $\varGamma$, a map $f : \varGamma \to \boldsymbol{R} \cup \{-\infty\}$ is a *rational function* on $\varGamma$ if f is a continuos piecewise affine function with integer slopes, or a constant function of $-\infty$. The set $\mathrm{Rat}(\varGamma)$ of all rational functions on $\varGamma$ becomes a semifield over $\boldsymbol{T}$ with the pointwise maximum operation as its addition $\oplus$ and the pointwise usual addition as its multiplication $\odot$. We call $\mathrm{Rat}(\varGamma)$ the *rational function semifield* of $\varGamma$. The following theorem is our main theorem:

Theorem 1. *Let $\varGamma$ be a tropical curve. Then, the rational function semifield $\mathrm{Rat}(\varGamma)$ is finitely generated as a semifield over $\boldsymbol{T}$.*

This is a tropical analogue of the fact that the function field of an algebraic curve over $\boldsymbol{C}$ is generated by two elements over $\boldsymbol{C}$. To prove this theorem, the following lemma is our key:

Lemma 2 ([1, Lemma 2])**.** *Let $\varGamma$ be a tropical curve. Then, $\mathrm{Rat}(\varGamma)$ is generated by all chip firing moves and all constant functions as a group with $\odot$ as its binary operation.*

Here, a *chip firing move* $\mathrm{CF}(\varGamma_1, l)$ is the rational function defined by the pair of a subgraph $\varGamma_1$ and a number $l \in \boldsymbol{R}_{>0}$ as follows: $\mathrm{CF}(\varGamma_1, l)(x) := -\min\{\mathrm{dist}(\varGamma_1, x), l\}$, where a *subgraph* is a closed subset with a finite number of connected components and $\mathrm{dist}(\varGamma_1, x)$ denotes the distance between Γ_1 and x. By this lemma, it is enough to find a finite set of rational functions which generates all chip firing moves as a semifield over $\boldsymbol{T}$.

Note that this manuscript is based on [2]. In [2], a tropical curve may have edges of length ∞. Even a tropical curve has an edge of length ∞, the assertion of Theorem 1 holds.

2 Proof of Theorem 1

Let Γ be the tropical curve obtained from the pair (G, l) of a graph G and a length function l. Then, the pair (G, l) is called a *model* for Γ. There are many possible models for Γ. We frequently identify a vertex (resp. an edge) of G with the corresponding point (resp. the corresponding closed subset) of Γ. The *canonical model* $(G_\circ, l_\circ)$ for Γ is defined as follows: if Γ is not homeomorphic to a circle, then let $V(G_\circ) := \{x \in \Gamma \mid \mathrm{val}(x) \neq 2\}$; otherwise, let $V(G_\circ)$ be consist of arbitrary one point of Γ, where $\mathrm{val}(x)$ denotes the minimum number of the connected components of $U \setminus \{x\}$ with any neighborhood U of x. Let $G_\circ$ be the graph which has $V(G_\circ)$ as its set of vertices and the set of all closures of connected components of $\Gamma \setminus V(G_\circ)$ as its set of edges, and $l_\circ$ the length function which maps each edge of $G_\circ$ to its length. Fix a direction on edges of $G_\circ$. Let each $e \in E(G_\circ)$ be identified with the interval $[0, l_\circ(e)]$ with this direction. For each edge $e \in E(G_\circ)$, let $x_e = \frac{l_\circ(e)}{4}$, $y_e = \frac{l_\circ(e)}{2}$, and $z_e = \frac{3l_\circ(e)}{4}$. We define rational functions

$$f_e := \mathrm{CF}\left(\{y_e\}, \frac{l_\circ(e)}{2}\right), g_e := \mathrm{CF}\left(\{x_e\}, \frac{l_\circ(e)}{4}\right), h_e := \mathrm{CF}\left(\{z_e\}, \frac{l_\circ(e)}{4}\right).$$

Let d be the diameter of Γ, i.e., $d = \sup\{\mathrm{dist}(x, y) \mid x, y \in \Gamma\} = \max\{\mathrm{dist}(x, y) \mid x, y \in \Gamma\}$. Let R be the semifield generated by f_e, g_e, h_e for any $e \in E(G_\circ)$ and $\mathrm{CF}(\{v\}, d)(= -\mathrm{dist}(v, \cdot))$ for any $v \in V(G_\circ)$ over $\boldsymbol{T}$. We will show that this semifield R coincides with $\mathrm{Rat}(\Gamma)$.

In Algorithm 1, $\overline{e_i \setminus S}$ denotes the closure of $e_i \setminus S$, and if S consists of only one point x, then we write l_x instead of $l_{\{x\}}$.

Remark 3. Let Γ be a tropical curve and S_1 a proper connceted subgraph of Γ. Let $l \leq l_{S_1}$ and $S_2 := \{x \in \Gamma \mid \mathrm{dist}(S_1, x) \leq l\}$. With $a := \min\{k \in \boldsymbol{Z}_{>0} \mid l/k \leq l_{S_2}\}$, $m := \min\{k \in \boldsymbol{Z}_{>0} \mid l_{S_2}/k \leq l/a\}$ and any $l' > 0$, by the definition of chip firing moves, we have

$$\mathrm{CF}\left(S_2, \frac{l}{a}\right) = \mathrm{CF}\left(S_1, \frac{l}{a}\right) \odot \bigodot_{k=1}^{a} \bigodot_{\substack{x' \in \Gamma: \\ \mathrm{dist}(S_1, x') = \frac{kl}{a}}} \left\{\mathrm{CF}\left(\{x'\}, \frac{l}{a}\right) \odot \frac{l}{a}\right\},$$

$$\begin{aligned}\mathrm{CF}\left(\left\{x \in \Gamma \mid \mathrm{dist}(S_2, x) \leq \frac{kl_{S_2}}{m}\right\}, \frac{l_{S_2}}{m}\right) = \; & \mathrm{CF}\left(\left\{x \in \Gamma \mid \mathrm{dist}(S_2, x) \leq \frac{(k-1)l_{S_2}}{m}\right\}, \frac{l_{S_2}}{m}\right) \\ & \odot \bigodot_{\substack{x' \in \Gamma: \\ \mathrm{dist}(S_2, x') = \frac{kl_{S_2}}{m}}} \left\{\mathrm{CF}\left(\{x'\}, \frac{l_{S_2}}{m}\right) \odot \frac{l_{S_2}}{m}\right\},\end{aligned}$$

$$\mathrm{CF}(S_2, l_{S_2}) = \left\{\mathrm{CF}\left(S_2, \frac{l}{a}\right) \oplus \left(-\frac{l_{S_2}}{m}\right)\right\} \odot \bigodot_{k=1}^{m-1} \mathrm{CF}\left(\left\{x \in \Gamma \mid \mathrm{dist}(S_2, x) \leq \frac{kl_{S_2}}{m}\right\}, \frac{l_{S_2}}{m}\right),$$

and

$$\mathrm{CF}(S_1, l + l') = \mathrm{CF}(S_1, l) \odot \mathrm{CF}(S_2, l').$$

Algorithm 1

Input: Γ : a tropical curve

$E(G_\circ) = \{e_1, \dots, e_n\}$: a labeling of edges of the canonical model for Γ

S : a proper connected subgraph of Γ

Output: l_S

$i \leftarrow 1$

while $i \leq n$ **do**

  if $e_i \cap S = \varnothing$ **then**

    $l_i \leftarrow$ (the diameter of Γ), $i \leftarrow i+1$

  else $\{S \supset e_i\}$

    $l_i \leftarrow$ (the diameter of Γ), $i \leftarrow i+1$

  else $\{S \supset \partial e_i\}$

    $l_i \leftarrow$ (the length of $\overline{e_i \setminus S}$)$/2, i \leftarrow i+1$

  else $\{S \subset e_i^\circ\}$

    $l_i \leftarrow \min\{\mathrm{dist}(S, x) \mid x$ is one of the endpoints of $e_i\}, i \leftarrow i+1$

  else

    $l_i \leftarrow$ (the length of $\overline{e_i \setminus S}$), $i \leftarrow i+1$

  end if

end while

$l_S \leftarrow \min\{l_1, \dots, l_n\}$

return l_S

Lemma 4. *Let e be an edge of $G_\circ$ and v, w the endpoint(s) of e (possibly $v = w$). Let x be in $e \setminus \{v, w\}$. Then,* $\mathrm{CF}(\{x\}, l_x) \in R$.

Proof. If x is the midpoint of e, then $\mathrm{CF}(\{x\}, l_x) = f_e \in R$. Suppose that x is not the midpoint of e. Assume that $0 < l_x \leq \frac{l_\circ(e)}{4}$ and $g_e(x) = -\frac{l_\circ(x)}{4}$. Then

$$\begin{aligned}\mathrm{CF}(\{x\}, l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)} \\ &\quad \odot \left(-\frac{l_\circ(e)}{4}\right) \odot g_e^{\odot(-1)} \oplus (-l_x) \in R.\end{aligned}$$

Similarly, if $0 < l_x \leq \frac{l_\circ(e)}{4}$ and $h_e(x) = -\frac{l_\circ(e)}{4}$, then $\mathrm{CF}(\{x\}, l_x) \in R$.
When $\frac{l_\circ(e)}{4} < l_x \leq \frac{l_\circ(e)}{3}$ and $g_e(x) = -\frac{l_\circ(e)}{4}$, we have

$$\begin{aligned}\mathrm{CF}(\{x\}, l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)} \\ &\quad \odot \left\{\left(-\frac{l_\circ(e)}{4}\right) \odot g_e^{\odot(-1)}\right\}^{\odot 2} \oplus (-l_x) \in R.\end{aligned}$$

Similarly, if $\frac{l_\circ(e)}{4} < l_x \leq \frac{l_\circ(e)}{3}$ and $h_e(x) = -\frac{l_\circ(e)}{4}$, then $\mathrm{CF}(\{x\}, l_x) \in R$.

When $\frac{l_\circ(e)}{3} < l_x < \frac{l_\circ(e)}{2}$ and $g_e(x) = -\frac{l_\circ(e)}{4}$, we have

$$\begin{aligned}\mathrm{CF}(\{x\}, l_\circ(e) - 2l_x) &= \left\{\left(\frac{l_\circ(e)}{2} - l_x\right) \odot f_e \oplus \left(l_x - \frac{l_\circ(e)}{2}\right) \odot f_e^{\odot(-1)}\right\}^{\odot(-1)} \\ &\odot \left\{\left(-\frac{l_\circ(e)}{4}\right) \odot g_e^{\odot(-1)}\right\}^{\odot 2} \oplus (2l_x - l_\circ(e)) \in R.\end{aligned}$$

Similarly, if $\frac{l_\circ(e)}{3} < l_x < \frac{l_\circ(e)}{2}$ and $h_e(x) = -\frac{l_\circ(e)}{4}$, then $\mathrm{CF}(\{x\}, l_\circ(e) - 2l_x) \in R$.

Let x be in the fifth case. Since

$$\begin{aligned}&\mathrm{CF}\left(\left\{x_1 \in \Gamma \mid \mathrm{dist}(x, x_1) \leq \frac{l_\circ(e)}{2} - l_x\right\}, \frac{l_\circ(e)}{2} - l_x\right) \\ = &\mathrm{CF}\left(\{x\}, \frac{l_\circ(e)}{2} - l_x\right) \odot \bigodot_{\substack{x_1 \in e: \\ \mathrm{dist}(x,x_1) = \frac{l_\circ(e)}{2} - l_x}} \left\{\mathrm{CF}\left(\{x_1\}, \frac{l_\circ(e)}{2} - l_x\right) \odot \left(\frac{l_\circ(e)}{2} - l_x\right)\right\} \in R,\end{aligned}$$

with inputs $l = \frac{l_\circ(e)}{2} - l_x$, $S_1 = \{x_1 \in \Gamma \mid \mathrm{dist}(x, x_1) \leq l_\circ(e)/2 - l_x\}$ in Remark 3, we have $\mathrm{CF}(\{x\}, l_x) \in R$.

When x is in the sixth case, by the same argument, we have $\mathrm{CF}(\{x\}, l_x) \in R$. □

Note that l_x coincides with $\min(\mathrm{dist}(x, v), \mathrm{dist}(x, w))$ in the setting of Lemma 4.

By Remark 3 and Lemma 4, we prove the following three lemmas.

Lemma 5. *For any $x \in \Gamma$ and any positive real number l, $\mathrm{CF}(\{x\}, l)$ is in R.*

Proof. For any $x \in \Gamma$ and $l > 0$, by the definition of chip firing moves, we have $\mathrm{CF}(\{x\}, l) = \mathrm{CF}(\{x\}, d) \oplus (-l)$. Hence it is sufficient to check that $\mathrm{CF}(\{x\}, d) \in R$. If $x \in V(G_\circ)$, then $\mathrm{CF}(\{x\}, d) \in R$.

Suppose that there exists an edge $e \in E(G_\circ)$ which does not have x as an endpoint. Considering Remark 3 with $l = l_x, S_1 = \{x\}$, by Lemma 4, we have

$$\mathrm{CF}(S_2, l_{S_2}) \in R$$

and

$$\mathrm{CF}(S_1, l + l_{S_2}) = \mathrm{CF}(S_1, l) \odot \mathrm{CF}(S_2, l_{S_2}) \in R.$$

Since S_2 contains a lot of whole edges of $G_\circ$ more than S_1 and the set of edges of $G_\circ$ is finite, by repeating inputs of $l = l_{S_2}$, $S_1 = S_2$ in Remark 3, we have $\mathrm{CF}(\{x\}, d) \in R$. □

Lemma 6. *For any proper connected subgraph Γ_1 and any positive real number l, $\mathrm{CF}(\Gamma_1, l)$ is in R.*

Proof. By Lemma 5, if Γ_1 consists of only one point, then we have the conclusion. Assume that Γ_1 does not consist of only one point.

Suppose that Γ_1 contains no whole edges of $G_\circ$ and that there exists an edge $e \in E(G_\circ)$ containing Γ_1. Let x_1 and x_2 be the endpoints of Γ_1. Let x be the midpoint of Γ_1. By Lemma 5, for any positive real number l, we have

$$\mathrm{CF}(\Gamma_1, l) = \left[\{\mathrm{CF}(\{x\}, l + \mathrm{dist}(x_1, x)) \odot \mathrm{dist}(x_1, x)\}^{\odot(-1)} \oplus 0\right]^{\odot(-1)} \in R.$$

Suppose $\varGamma_1$ contains p edges. Let $\partial\varGamma_1 \cup (V(G_\circ) \cap \varGamma_1) = \{x_1, \ldots, x_q\}$. We may assume that $x_1, \ldots, x_q$ are distinct. Let $\varGamma_{11}, \ldots, \varGamma_{1s}$ be connected components of $\varGamma_1 \setminus \{x_1, \ldots, x_q\}$. For a sufficiently small positive real number ε, let $\varGamma'_{1i}$ be the connected subgraph $\{x \in \varGamma_{1i} \,|\, \text{for any } j, \mathrm{dist}(x, x_j) \geq \varepsilon\}$ of $\varGamma$. Then, we have

$$\mathrm{CF}(\varGamma_1, \varepsilon) = \left\{ \bigoplus_{k=1}^{q} \mathrm{CF}(\{x_k\}, \varepsilon) \right\} \odot \bigodot_{k=1}^{s} \left(\varepsilon \odot \mathrm{CF}(\varGamma'_{1k}, \varepsilon) \right).$$

The last divisor is in the first case, and thus it is in R. By inputting $l = \varepsilon$, $S_1 = \varGamma_1$ and by repeating inputs $l = l_{S_2}$, $S_1 = S_2$ in Remark 3, we have $\mathrm{CF}(\varGamma_1, d) \in R$. From this, for any $l > 0$, we have $\mathrm{CF}(\varGamma_1, l) = \mathrm{CF}(\varGamma_1, d) \oplus (-l) \in R$. □

Lemma 7. *For any proper subgraph $\varGamma_1$ and any positive real number l, $\mathrm{CF}(\varGamma_1, l)$ is in R.*

Proof. Let $\varGamma_1$ be a proper subgraph of $\varGamma$. Let s be the number of connected components of $\varGamma_1$. If $s = 1$, then the conclusion follows Lemma 6. Assume $s \geq 2$. Let $\varGamma'_1, \ldots, \varGamma'_s$ be all the distinct connected components of $\varGamma_1$. For $l' > 0$, let $\varGamma'_k(l') := \{x \in \varGamma \mid \mathrm{dist}(\varGamma'_k, x) \leq l'\}$. If l' is sufficiently small, then the intersection of $\varGamma'_1(l'), \ldots, \varGamma'_s(l')$ is empty. Let l'_1 be the minimum value of l' such that this intersection is nonempty. By induction on s, $\mathrm{CF}\left(\bigcup_{k=1}^{s} \varGamma'_k(l'_1), d\right) \in R$. On the other hand,

$$\mathrm{CF}(\varGamma_1, l'_1) = \bigoplus_{k=1}^{s} \mathrm{CF}(\varGamma'_k, l'_1) \in R.$$

Hence

$$\mathrm{CF}(\varGamma_1, d) = \mathrm{CF}(\varGamma_1, l'_1) \odot \mathrm{CF}\left(\bigcup_{k=1}^{s} \varGamma'_k(l'_1), d \right) \in R.$$

In conclusion, for any $l > 0$, we have

$$\mathrm{CF}(\varGamma_1, l) = \mathrm{CF}(\varGamma_1, d) \oplus (-l) \in R.$$

□

From Lemmas 2, 5, 6, 7, we have Theorem 1.

参考文献

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