An Optimal Decay Estimate of Solutions to the Surface Quasi-Geostrophic Equation

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1 Introduction

In this presentation, we are concerned with the Surface Quasi-Geostrophic Equation.

$$\begin{cases} \partial_t \theta + (-\Delta)^{\alpha/2}\theta + (u \cdot \nabla)\theta = 0, & \text{in } (0,\infty) \times \mathbb{R}^2, \\ u = (-R_2\theta, R_1\theta), & \text{in } (0,\infty) \times \mathbb{R}^2, \\ \theta|_{t=0} = \theta_0, & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\theta : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is an unknown function, representing the potential temperature of a fluid parcel at a point (t, x) in spacetime; and u represents the velocity of a fluid parcel. $R_j = \partial_j(-\Delta)^{-1/2} = \mathcal{F}^{-1}[\, i\xi_j/|\xi|\,]*$ is the j-th Riesz transform; and $\alpha \in [1, 2]$. Discussions of the physical meaning and derivation of the equation can be found in references [4, 6, 7, 9, 16, 19], and we briefly introduce the derivation in Section 2.

In this presentation, we prove L^p-norm estimates for solutions to the equation with sufficiently regular initial data θ_0. We then show that these estimates are optimal given certain conditions on the shape of the initial data.

Let us recall several existing results related to the regularity of the solutions. In the subcritical case, $\alpha \in (1, 2]$, unique global existence and regularity can be shown, for initial data

$$\theta_0 \in L^1 \cap L^p,\ p \in \Big(\frac{2}{\alpha - 1}, \infty\Big],$$

by classical methods using the Banach fixed point theorem, similarly to [13, 21]. Global unique existence and regularity for the subcritical case are also proven on the torus in [17].

For the critical case, $\alpha = 1$, global well-posedness is an important problem due to criticality, and was proven independently in [8] and [2] (see also [18]).

Finally, for the supercritical case, $\alpha \in [0, 1)$, the paper [12] proves global regularity for all $\alpha \in [\alpha_0, 1)$, where α_0 grows with respect to the size of the initial data. Global regularity for large data in the supercritical case is an open problem. In this report, we discuss a sharp decay estimate of the nonlinear part of the solution in the subcritical and critical cases.

We begin our study of (1.1) by defining the following function:

$$P_{\alpha/2}(t,x) = \mathcal{F}^{-1}\left[e^{-t|\xi|^\alpha}\right](x), \text{ for } t>0, \ x\in\mathbb{R}^2, \tag{1.2}$$

which is the fundamental solution to the fractional heat equation, the linear part of (1.1).

Definition. (Mild Solution) A function, θ, is a mild solution of (1.1) if

$$\theta(t) = P_{\alpha/2}(t) * \theta_0 - \int_0^t P_{\alpha/2}(t-s) * (u(s)\cdot\nabla)\theta(s)\,\mathrm{d}s, \text{ for all } t>0, \tag{1.3}$$

$$\lim_{t\to 0^+}\theta(t) = \theta_0, \text{ in } L^p, \text{ for all } p\in[1,\infty), \tag{1.4}$$

$$\theta\in C([0,\infty);L^p(\mathbb{R}^2)), \text{ for all } p\in[1,\infty), \tag{1.5}$$

$$\theta\in C((0,\infty);W^{2,p}(\mathbb{R}^2))\cap C^1((0,\infty);L^p(\mathbb{R}^2)), \text{ for all } p\in[1,\infty]. \tag{1.6}$$

We also define M, the "mass" of the solution, θ, as $M := \int_{\mathbb{R}^2}\theta_0(x)\,\mathrm{d}x$, and denote the linear part of θ by $U(t) := P_{\alpha/2}(t)*\theta_0$. We briefly discuss the existence of global solutions.

Proposition 1.1. Let $\alpha\in[1,2]$ and $\theta_0\in W^{1,1}\cap W^{1,\infty}$. Then there exists a unique global mild solution $\theta\in C([0,\infty);W^{1,p})\cap C((0,\infty);W^{1,1}\cap W^{1,\infty})$ of (1.1), for all $1\le p<\infty$.

2 Preliminaries

We list some important definitions and results used for the proof in Section 3.

2.1 Preliminary Results

2.1.1 L^p Spaces

Definition. (L^p Space)

Let (Ω, M, μ) be a measure space. The set $L^p(\Omega)$, with $1\le p<\infty$, is defined as the set of all functions, $f:\Omega\to\mathbb{R}$, such that

$$\|f\|_p := \left(\int_\Omega |f(x)|^p\,\mathrm{d}\mu\right)^{1/p} < \infty.$$

For $p=\infty$, the set is defined as the set of all functions whose essential supremum is finite. That is,

$$\|f\|_\infty := \inf\{C\ge 0 \mid |f(x)|\le C, \text{ for almost all } x\in\Omega\}.$$

Functions that agree almost everywhere are considered a single element of $L^p(\Omega)$.

Notation. For $1\le p\le\infty$, we denote by p' the conjugate exponent. That is,

$$\frac{1}{p}+\frac{1}{p'}=1,$$

with the convention that $\frac{1}{\infty}:=0$ in this context.

Proposition 2.1. (Hölder's Inequality)

Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, where $1 \leq p \leq \infty$. Then $fg \in L^1$, and

$$\int_\Omega |fg| \,\mathrm{d}\mu \leq \|f\|_p \|g\|_{p'}.$$

Proposition 2.2. (Young's Convolution Inequality)

Let $f \in L^p$ and $g \in L^q$, where $1 \leq p, q \leq r \leq \infty$, such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

Then

$$\|f * g\|r \leq \|f\|_p \|g\|_q.$$

Here, $(f * g)$ denotes the convolution of f and g. That is,

$$(f * g)(x) := \int_{\mathbb{R}^2} f(x-y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}^2} f(y)g(x-y)\,\mathrm{d}y.$$

Proposition 2.3. (Gagliardo-Nirenberg Inequality for Fractional Laplacian [15])

Define

$$\dot{H}^s_p := \{f \in \mathcal{S}'/\mathcal{P} \mid \|f\|_{\dot{H}^s_p(\mathbb{R}^n)} := \|(-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ is the set of all polynomials over $\mathbb{R}^n$.

Let $0 \leq \sigma < s < \infty,\ 1 < q, r \leq \infty,\ 1 < p \leq \infty$ such that

$$\frac{n}{p} - \sigma = \theta\frac{n}{q} + (1-\theta)(\frac{n}{r} - s),$$

where $\theta \neq 0$ if $s - \sigma \geq n/r$. Then, for all $f \in L^q(\mathbb{R}^n) \cap \dot{H}^s_r(\mathbb{R}^n)$, we have $f \in \dot{H}^\sigma_p$. Moreover, we obtain the inequality

$$\|f\|_{\dot{H}^\sigma_p} \leq C\|f\|^\theta_q \|f\|^{1-\theta}_{\dot{H}^s_r} \text{ for all } f \in L^q(\mathbb{R}^n) \cap \dot{H}^s_r(\mathbb{R}^n). \tag{2.1}$$

2.1.2 The Fourier Transform

For a function, f, we define the Fourier transform of f as follows:

$$\mathcal{F}[f](\xi) := \hat{f}(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x.$$

The inverse Fourier transform is then defined as

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi.$$

For the purpose of calculating inequalities, we will frequently omit the factor of $1/2\pi$.

Proposition 2.4. (Plancherel Theorem)

Let $f \in L^1 \cap L^\infty$. Then the L^2 norm of f is invariant under the Fourier transform. That is,

$$\|f\|_2 = \|\hat{f}\|_2.$$

Proposition 2.5. (Hausdorff-Young Inequality)
Let $f \in L^1 \cap L^\infty$. Let $1 \le p \le 2$ with $1/p + 1/p' = 1$. Then

$$\|\hat{f}\|_{p'} \le \|f\|_p.$$

2.1.3 Besov Spaces

Definition. We use the Littlewood-Paley decomposition of unity to define homogeneous Besov spaces. Let $\{\phi_k\}_{k\in\mathbb{Z}}$ be a set of non-negative measurable functions such that

1. $\displaystyle\sum_{k\in\mathbb{Z}} \hat{\phi}_k(\xi) = 1, \text{ for all } \xi \in \mathbb{R}^2 \backslash \{0\},$
2. $\hat{\phi}_k(\xi) = \hat{\phi}_0(2^{-k}\xi),$
3. $\text{supp } \hat{\phi}_k(\xi) \subseteq \{\xi \in \mathbb{R}^2 \mid 2^{k-1} \le |\xi| \le 2^{k+1}\}.$

The Besov norm is then defined as follows. For $f \in \mathcal{S}'/\mathcal{P}$, $1 \le p, q \le \infty$, and $s \in \mathbb{R}$,

$$\|f\|_{\dot{B}^s_{p,q}} := \left\| \{2^{sk}\|\phi * f\|_p\}_{k\in\mathbb{Z}} \right\|_{l^q}.$$

Finally, the set $\dot{B}^s_{p,q}$ is defined as the set of functions, $f \in \mathcal{S}'/\mathcal{P}$, whose Besov norm is finite.

We refer to [10] for the proofs to the following propositions.

Proposition 2.6. Let $1 \le p, q \le \infty$, and $s \in \mathbb{R}$. Then for $f \in \dot{B}^{s+1}_{p,q}$,

$$\|\nabla f\|_{\dot{B}^s_{p,q}} \le C\|f\|_{\dot{B}^{s+1}_{p,q}}.$$

Proposition 2.7. Let $1 \le p \le \infty$. Then for $f \in \dot{B}^{2(1-\frac{1}{p})}_{1,1}$,

$$\|f\|_{\dot{B}^0_{p,1}} \le C\|f\|_{\dot{B}^{2(1-\frac{1}{p})}_{1,1}}.$$

3 Main Result

Theorem 3.1. Let $\alpha \in [1, 2]$, $p \in [1, \infty]$ and

$$b_{\alpha,p}(t) = \begin{cases} t^{\frac{2}{\alpha}(1-\frac{1}{p})+\frac{3}{\alpha}-1} & \text{if } \alpha \in (1, 2], \\ t^{2(1-\frac{1}{p})+2}\ln t & \text{if } \alpha = 1. \end{cases} \tag{3.1}$$

Let $\theta_0 \in W^{1,1} \cap W^{1,\infty}$, and let θ be a mild solution. Also assume $|x|^2\theta_0 \in L^1(\mathbb{R}^2)$. Then we have the following convergence:

$$b_{\alpha,p}(t)\Big\|\theta(t) - MP_{\alpha/2}(t) + \nabla P_{\alpha/2}(t)\cdot\int_{\mathbb{R}^2} y\theta_0(y)\,\mathrm{d}y - \sum_{i,j=1}^{2}\partial_{x_i}\partial_{x_j}P_{\alpha/2}(t)\int_{\mathbb{R}^2} y_iy_j\theta_0\,\mathrm{d}y \qquad (3.2)$$

$$+\int_0^t P_{\alpha/2}(t-s)*\nabla\cdot\Big((RU(s))U(s)\Big)\,\mathrm{d}s\Big\|_p \to 0,\ \text{as } t\to\infty.$$

Furthermore, there exists θ_0 such that the nonlinear component is optimally bounded by $b_{\alpha,p}(t)$ in the $p=2$ case. That is

$$\Big\|\int_0^t P_{\alpha/2}(t-s)*\nabla\cdot\Big((RU(s))U(s)\Big)\,\mathrm{d}s\Big\|_2 \simeq \frac{1}{b_{\alpha,2}(t)} \qquad (3.3)$$

for all sufficiently large t.

In this report, we will only be providing a full proof of (3.3). A full treatment of (3.2) can be found in [3]. The above result can be considered as an expansion of similar results, such as in [13, 21]. Roughly speaking, we have approximated the linear term in θ by a Taylor expansion, and constructed a linear approximation of the nonlinear term of θ.

Remark 3.2. If the initial data, θ_0, is radially symmetric, the nonlinear approximation term in (3.3) becomes 0 (see e.g. [14], page 46).

Proposition 3.3. Let $\theta_0 \in W^{1,1} \cap W^{1,\infty}$, and $1 \le p \le \infty$.

(i) For all $\alpha \in [1,2]$, there exists $C > 0$ such that for all $t > 0$

$$\|\theta(t)\|_p \le C(t+1)^{-\frac{2}{\alpha}(1-\frac{1}{p})}. \qquad (3.4)$$

(ii) For $\alpha = 1$, $\beta > 0$, there exists $C_\beta > 0$ such that for all $t \ge 1$

$$\||\nabla|^\beta\theta(t)\|_p \le C_\beta t^{-2(1-\frac{1}{p})-\beta}. \qquad (3.5)$$

For the proof of (i), see [1]. For the proof of (ii), see Proposition 4.3 in [20].

3.1 Estimating the Nonlinear Term from Above

Before we begin the proof, we note one more lemma, which we will apply in this report for brevity. However, it is in fact possible to complete the entire proof without using the following lemma. To see how, please refer to [3].

Lemma 3.4. ([11]) Let $0 < p, p_1, p_2 < \infty$, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $m = m(\xi-\eta,\eta) \in C^2(\mathbb{R}^2\backslash\{0\})$, such that $|\partial^\alpha_{\xi-\eta,\eta} m(\xi-\eta,\eta)| \le C(|\xi-\eta|+|\eta|)^{-|\alpha|}$, for all multi-indices α, with $|\alpha| \le 2$. Then

$$\Big\|\mathcal{F}^{-1}\Big[\int_{\mathbb{R}^2} m(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,\mathrm{d}\eta\Big]\Big\|_p \le C\|f\|_{p_1}\|g\|_{p_2}.$$

Lemma 3.5. Let $1 \le p \le \infty$, and $\alpha \in [1,2]$. Suppose that θ is a mild solution satisfying the decay property in Proposition 3.3. Then there exists $C > 0$ such that for all $t \ge 1$,

$$\left\| \int_0^t P_{\alpha/2}(t-s) * (u(s)\cdot\nabla)\theta(s)\,\mathrm{d}s \right\|_p \le C b_{\alpha,p}(t), \tag{3.6}$$

where $b_{\alpha,p}$ is defined by (3.1) (and the same inequality then obviously also holds for (3.3)).

Proof.

Step 1. ($p \ge 2$ case) We start on the second half of the time-interval. By Proposition 3.3, and by the boundedness of the Riesz transform, we obtain for all $\alpha \in [1,2]$

$$\left\| \int_{t/2}^t P_{\alpha/2}(t-s) * \nabla\cdot(u(s)\theta(s))\,\mathrm{d}s \right\|_p \le Ct^{-\frac{2}{\alpha}(1-\frac{1}{p})-\frac{3}{\alpha}+1}.$$

For the first half of the time-interval, we will take the Fourier transform inside the norm, and manipulate the resulting multipliers from the derivative and Riesz transform.

$$\begin{aligned}&\left\| \int_0^{t/2} P_{\alpha/2}(t-s) * \nabla\cdot(u(s)\theta(s))\,\mathrm{d}s \right\|_p \\ &= \left\| \int_0^{t/2} \sum_{j=1}^{2} \frac{1}{2}\mathcal{F}^{-1}\left[e^{-(t-s)|\xi|^\alpha} \int_{\mathbb{R}^2} \xi_j\eta_{3-j} \frac{|\xi-\eta|^2-|\eta|^2}{|\eta||\xi-\eta|(|\xi-\eta|+|\eta|)} \hat{\theta}(s,\xi-\eta)\hat{\theta}(s,\eta)\,\mathrm{d}\eta \right] \mathrm{d}s \right\|_p.\end{aligned}$$

We start with the $p=2$ case. The large multiplier inside the η integral is easily estimated from above by $2/|\xi-\eta|$. Next, we take the Littlewood-Paley decomposition of both θ functions:

$$\hat{\theta}(s,\xi-\eta) = \sum_{k\in\mathbb{Z}} \hat{\theta}_k(s,\xi-\eta), \qquad \hat{\theta}(s,\eta) = \sum_{l\in\mathbb{Z}} \hat{\theta}_l(s,\eta).$$

We also split the L^2 norm by Hölder's inequality.

$$\begin{aligned}&\left\| \int_0^{t/2} P_{\alpha/2}(t-s) * \nabla\cdot\Big(u(s)\theta(s)\Big)\,\mathrm{d}s \right\|_2 \\ &\le \sum_{k,l\in\mathbb{Z}} \int_0^{t/2} \| |\xi|^2 e^{-(t-s)|\xi|^\alpha} \|_2 \left\| \int_{\mathbb{R}^2} \frac{1}{|\xi-\eta|} \Big| \hat{\theta}_k(s,\xi-\eta)\hat{\theta}_l(s,\eta) \Big| \,\mathrm{d}\eta \right\|_\infty \mathrm{d}s \\ &\le Ct^{-\frac{2}{\alpha}(1-\frac{1}{2})-\frac{2}{\alpha}} \int_0^{t/2} \sum_{k,l\in\mathbb{Z}} \frac{1}{2^k} \|\hat{\theta}_k(s)\|_{\frac{4}{3}} \|\hat{\theta}_l(s)\|_4 \,\mathrm{d}s \\ &\le Ct^{-\frac{2}{\alpha}(1-\frac{1}{2})-\frac{2}{\alpha}} \int_0^{t/2} \sum_{k,l\in\mathbb{Z}} \|\theta_k(s)\|_{\frac{4}{3}} \|\theta_l(s)\|_{\frac{4}{3}} \,\mathrm{d}s.\end{aligned}$$

The above sum can be written as the product of Besov norms of the solution, θ. (3.6) for the $p=2$ case is then achieved by applying the following Besov norm estimate. ([3])

$$\|\theta(t)\|_{\dot{B}^0_{\frac{4}{3},1}} \le Ct^{-1/2\alpha}, \text{ for all } t \ge 1, \text{ and all } \alpha \in [1,2].$$

The result is then easily extended to all $p > 2$ by Young's convolution inequality.

Step 2. ($1 \leq p < 2$ case) Our goal is the following claim. Let $\alpha \in [1, 2]$ and $1 \leq p < 2$. Then

$$\left\| \mathcal{F}^{-1}\Big[\int_0^{t/2} e^{-(t-s)|\xi|^\alpha} \int_{\mathbb{R}^2} \xi_1^2 \frac{\eta_2(\xi_1 - 2\eta_1)}{|\eta||\xi-\eta|(|\xi-\eta|+|\eta|)} \hat{\theta}(s, \xi-\eta)\hat{\theta}(s,\eta)\, \mathrm{d}\eta\, \mathrm{d}s \Big] \right\|_p \leq C b_{\alpha,p}(t). \quad (3.7)$$

We split the Fourier multiplier into two parts:

$$\frac{\eta_2(\xi_1 - 2\eta_1)}{|\eta||\xi-\eta|(|\xi-\eta|+|\eta|)} = \frac{\eta_2(\xi_1-\eta_1) + \eta_1\eta_2}{|\eta||\xi-\eta|(|\xi-\eta|+|\eta|)} =: M_1(\xi-\eta,\eta) + M_2(\xi-\eta,\eta).$$

The desired result is then achieved by applying Lemma 3.4 to each of these multipliers. □

3.2 Optimal Decay of Non-Linear Term

We lastly discuss the optimality of our estimate for the L^p norm of the non-linear term in θ. The decay rate for our estimate from above is optimal if we can bound the non-linear estimate from below by the same power of t. That is, we need

$$\left\| \int_0^t P_{\alpha/2}(t-s) * \nabla \cdot \Big((RU(s))U(s) \Big)\, \mathrm{d}s \right\|_2 \geq \begin{cases} Ct^{1-\frac{4}{\alpha}}, & \text{for all } \alpha \in (1,2], \\ Ct^{-3}\ln t, & \text{for } \alpha = 1. \end{cases} \quad (3.8)$$

Since we are taking the L^2-norm, taking the Fourier Transform inside the norm does not change its value.

$$\begin{aligned} &\left\| \int_0^t P_{\alpha/2}(t-s) * \nabla \cdot ((RU(s))U(s))\, \mathrm{d}s \right\|_2 \\ &= \left\| \int_0^t \sum_{j=1}^2 \xi_j e^{-(t-s)|\xi|^\alpha} \frac{(-1)^j}{2} \int_{\mathbb{R}^2} \left(\frac{\eta_{3-j}}{|\eta|} + \frac{\xi_{3-j} - \eta_{3-j}}{|\xi-\eta|} \right) e^{-s|\xi-\eta|^\alpha} e^{-s|\eta|^\alpha} \right. \\ &\quad \left. \hat{\theta}_0(\xi-\eta)\hat{\theta}_0(\eta)\, \mathrm{d}\eta\, \mathrm{d}s \right\|_2 . \end{aligned}$$

We rewrite the divergence operator and Riesz transform as two separate Fourier multipliers.

$$\begin{aligned} &\sum_{j=1}^2 \xi_j \frac{(-1)^j}{2} \left(\frac{\eta_{3-j}}{|\eta|} + \frac{\xi_{3-j} - \eta_{3-j}}{|\xi-\eta|} \right) \\ =& \frac{2\xi_1\xi_2}{|\xi-\eta|} \left(\frac{\eta_1^2 - \eta_2^2}{|\eta|(|\eta|+|\xi-\eta|)} \right) + \frac{2\eta_1\eta_2}{|\xi-\eta|} \left(\frac{\xi_2^2 - \xi_1^2}{|\eta|(|\eta|+|\xi-\eta|)} \right) \\ &+ \sum_{j=1}^2 \frac{|\xi|^2 \xi_j \eta_{3-j}}{|\eta||\xi-\eta|(|\eta|+|\xi-\eta|)} \\ =:& m_1(\xi-\eta,\eta) + m_2(\xi-\eta,\eta) \end{aligned}$$

The key difference between these two multipliers is that the numerator of m_1 features a second-order derivative, whereas that of m_2 has a third-order derivative. We show that, for some initial data θ_0, the first part with m_1 has the optimal decay and the remainder with m_2 is smaller.

Lemma 3.6. Let $\delta, \epsilon > 0$. Let $\theta_0 \in W^{1,1} \cap W^{1,\infty}$ as before, but with the following additional conditions:

- $\hat{\theta}_0 \geq 0, \text{ on } \mathbb{R}^2$,
- $\text{supp } \hat{\theta}_0 \subseteq \{\xi \in \mathbb{R}^2 \mid |\xi_2| < \delta|\xi_1|\}$,
- $\hat{\theta}_0(\xi) \geq C, \text{ for some } C > 0, \text{ for all } \xi \in \text{ supp } \hat{\theta} \cap \{\xi \in \mathbb{R}^2 \mid |\xi| \leq 1\}$.

Then, for sufficiently small δ and ϵ, we have

$$\left\| \int_0^t e^{-(t-s)|\xi|^\alpha} \int_{\mathbb{R}^2} m_1(\xi - \eta, \eta)\, e^{-s|\xi-\eta|^\alpha} e^{-s|\eta|^\alpha} \,\mathrm{d}\eta \,\mathrm{d}s \right\|_{L^2(|\xi| \leq \epsilon t^{-1/\alpha})} \geq \begin{cases} Ct^{1-\frac{4}{\alpha}}\epsilon^3, \text{ for } \alpha \in (1,2], \\ Ct^{-3}\epsilon^3 \ln(t), \text{ for } \alpha = 1, \end{cases} \tag{3.9}$$

$$\left\| \int_0^t e^{-(t-s)|\xi|^\alpha} \int_{\mathbb{R}^2} m_2(\xi - \eta, \eta)\, e^{-s|\xi-\eta|^\alpha} e^{-s|\eta|^\alpha} \,\mathrm{d}\eta \,\mathrm{d}s \right\|_{L^2(|\xi| \leq \epsilon t^{-1/\alpha})} \leq C\epsilon^{7/2} t^{1-\frac{4}{\alpha}}, \text{ for all } \alpha \in [1,2]. \tag{3.10}$$

Proof. We consider (3.10) and (3.9) separately. Beginning with (3.10), we consider just the $j = 1$ part, as the estimates of both terms are identical. Since $\hat{\theta}_0$ is bounded, we need only consider the integral

$$\int_{\mathbb{R}^2} \frac{\eta_2}{|\eta||\xi - \eta|(|\eta| + |\xi - \eta|)} e^{-s|\xi-\eta|^\alpha} e^{-s|\eta|^\alpha} \,\mathrm{d}\eta.$$

Splitting the above into two parts, one where $|\eta|$ is close to $|\xi|$ and one where they are far apart, we can bound the above integral by

$$C + C|\xi|^{-1//2} s^{-1/2\alpha}$$

Therefore, we obtain

$$\begin{aligned}
&\left\| \int_0^t |\xi|^3 e^{-(t-s)|\xi|^\alpha} \int_{\mathbb{R}^2} \frac{\eta_2}{|\eta||\xi - \eta|(|\eta| + |\xi - \eta|)} e^{-s|\xi-\eta|^\alpha} e^{-s|\eta|^\alpha} \,\mathrm{d}\eta \,\mathrm{d}s \right\|_{L^2(|\xi| \leq \epsilon t^{-1/\alpha})} \\
&\leq \left\| \int_0^t |\xi|^3 e^{-(t-s)|\xi|^\alpha} (C + C|\xi|^{-1/2} s^{-1/2\alpha}) \,\mathrm{d}s \right\|_{L^2(|\xi| \leq \epsilon t^{-1/\alpha})} \\
&\leq \left\| |\xi|^3 \int_0^t C \,\mathrm{d}s \right\|_{L^2(|\xi| \leq \epsilon t^{-1/\alpha})} + \left\| |\xi|^{5/2} \int_0^t C s^{-1/2\alpha} \,\mathrm{d}s \right\|_{L^2(|\xi| \leq \epsilon t^{-1/\alpha})} \\
&\leq C(\epsilon^4 + \epsilon^{7/2}) t^{1-\frac{4}{\alpha}}.
\end{aligned}$$

We note that all three of these terms are smaller than $\epsilon^{7/2} t^{1-\frac{4}{\alpha}}$ for $\epsilon < 1$ and $t > 1$, and thus we have (3.10).

We next will show (3.9), for sufficiently small ϵ, and for sufficiently large t. We begin by labelling the two terms in our integral.

$$\begin{aligned}
&\Bigg\| \int_0^t e^{-(t-s)|\xi|^\alpha} \int_{\mathbb{R}^2} \left(\frac{2\xi_1\xi_2}{|\xi-\eta|}\left(\frac{\eta_1^2-\eta_2^2}{|\eta|(|\eta|+|\xi-\eta|)} \right) + \frac{2\eta_1\eta_2}{|\xi-\eta|}\left(\frac{\xi_2^2-\xi_1^2}{|\eta|(|\eta|+|\xi-\eta|)} \right) \right) \\
&\hat{U}(s,\xi-\eta)\hat{U}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \Bigg\|_{L^2(|\xi|\le\epsilon t^{-1/\alpha})} \\
&=: \|(L)+(R)\|_{L^2(|\xi|\le\epsilon t^{-1/\alpha})}.
\end{aligned}$$

Our plan is to make (L) the larger term. This is accomplished by the conditions on θ_0 that we have imposed. Taking the norm of (L) on its own,

$$\begin{aligned}
&\left\| \int_0^t e^{-(t-s)|\xi|^\alpha} \int_{\mathbb{R}^2} \frac{2\xi_1\xi_2}{|\xi-\eta|}\left(\frac{\eta_1^2-\eta_2^2}{|\eta|(|\eta|+|\xi-\eta|)} \right) e^{-s|\xi-\eta|^\alpha}\hat{\theta}_0(\xi-\eta)e^{-s|\eta|^\alpha}\hat{\theta}_0(\eta)\,\mathrm{d}\eta\,\mathrm{d}s \right\|_{L^2(|\xi|\le\epsilon t^{-1/\alpha})} \\
&\ge C\left\| \xi_1\xi_2 \int_0^t e^{-(t-s)|\xi|^\alpha} \int_{\mathbb{R}^2} \frac{\eta_1^2 e^{-s|\xi-\eta|^\alpha}\hat{\theta}_0(\xi-\eta)e^{-s|\eta|^\alpha}\hat{\theta}_0(\eta)}{|\eta||\xi-\eta|(|\eta|+|\xi-\eta|)}\,\mathrm{d}\eta\,\mathrm{d}s \right\|_{L^2(|\xi|\le\epsilon t^{-1/\alpha})} \\
&\ge C\left\| \xi_1\xi_2 \int_1^t e^{-\epsilon} \int_{2|\xi|<|\eta|<1} \frac{\eta_1^2 e^{-cs|\eta|^\alpha}}{|\eta|^3}\,\mathrm{d}\eta\,\mathrm{d}s \right\|_{L^2(|\xi|\le\epsilon t^{-1/\alpha})} \\
&\ge \left\| \xi_1\xi_2 \int_1^t e^{-\epsilon} \int_{2|\xi|}^1 e^{-cs\rho^\alpha}\,\mathrm{d}\rho\,\mathrm{d}s \right\|_{L^2(|\xi|\le\epsilon t^{-1/\alpha})} \\
&= \left\| \xi_1\xi_2 \int_1^t e^{-\epsilon} \int_{2|\xi|s^{1/\alpha}}^{s^{1/\alpha}} e^{-c\rho}\,\mathrm{d}\rho s^{-1/\alpha}\,\mathrm{d}s \right\|_{L^2(|\xi|\le\epsilon t^{-1/\alpha})} \\
&\ge \left\| \xi_1\xi_2 \int_1^t e^{-\epsilon} C s^{-1/\alpha}\,\mathrm{d}s \right\|_{L^2(|\xi|\le\epsilon t^{-1/\alpha})} \\
&\ge \begin{cases} Ct^{1-\frac{4}{\alpha}}\epsilon^3, & \text{for } \alpha\in(1,2], \\ Ct^{-3}\epsilon^3\ln(t), & \text{for } \alpha=1. \end{cases}
\end{aligned}$$

Finally, by our setting of supp $\hat{\theta}_0$, we obtain

$$\|(R)\|_{L^2(|\xi|\le\epsilon t^{-1})} \le Ct^{-3}\epsilon^4\ln(t),$$

by estimations of integrals similar to before. Thus we obtain (3.9). □

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