Quantitative estimates for the Bakry-Ledoux isoperimetric inequality

Osaka University, Department of Mathematics
MAI Cong Hung (joint work Shin-ichi Ohta)

1 Introduction

Our work studied a quantitative isoperimetric inequality under Ricci curvature bound condition on (non-compact) weighted Riemannian manifolds.

Curvatures is an important concept in geometry as they are the geometric measures of the non flatness of geometric shapes. From an analytic point of view, differential equations that describe the physical phenomena in curved spaces are affected by the distortion of geometric quantities that arose from the non flatness of the underlying spaces. Therefore, curvatures are important in the study of functional analysis, differential equations, and the analysis of curved spaces. Among various notions of curvatures, Ricci curvature is of interest. Ricci curvature can be characterized by the measurement of how a shape is deformed as one moves along geodesics. The lower bound of Ricci curvature allows one to extract global geometric and topological information by comparison with a model space of constant curvature. For instance, positive Ricci curvature bound implies the bound of diameter and degree of volume distortion of a ball with respect to its radius.

In recent years, the theory developed by Lott, Sturm, and Villani has shed new light on studies of lower bounds on Ricci curvature by connecting differential geometry with Wasserstein geometry and optimal transport. Ricci curvature can be used to measure the distortion of optimal transports between subsets in curved spaces, hence the study of Ricci curvature is connected to the theory of optimal transport. Weighted Ricci curvature is modified to include the appearance of density and based on the generalization of Brunn–Minkowsky inequality, which extracts the relation between optimal transport and geometry of lower Ricci curvature bound. In Riemannian setting, we can study this relation in the sense of weighted manifolds.

In weighted manifolds and metric measure spaces, the weighted Ricci curvature Ric_N involving a parameter N called the *effective dimension*. On a weighted Riemannian manifold, the weighted Ricci curvature bound $\mathrm{Ric}_N \geq K$ is equivalent to the curvature dimension condition $\mathrm{CD}(K, N)$ in the theory of Lott, Sturm, and Villani. The parameters K and N are interpreted as "a lower bound of the Ricci curvature" and "an upper bound of the dimension", respectively. It is interesting that the condition $\mathrm{Ric}_N \geq K > 0$ would be satisfied by non-compact manifolds when $N = \infty$. In this work, I would study the stability problem of the isoperimetric inequality in Riemannian manifolds of $\mathrm{Ric}_\infty \geq K > 0$ or the Bakry–Ledoux isoperimetric inequality.

The rigidity (to characterize a space attaining equality or model space) and stability (to show that the space is close to the model space when equality nearly holds) problems are important subjects in geometric analysis as they are connected with the theory of convergence of spaces. Theses tasks in non-compact weighted manifolds are challenging because one might not be able use traditional approach like maximal diameter argument. But a breakthrough named *needle decomposition* was introduced by Klartag in ([Kl]) provided an efficient approach to the rigidity and stability problems in non-compact weighted manifolds. Needle decomposition is a localization technique to reduce high-dimensional inequalities into their one-dimensional counterpart which enable one to use real analysis to verify.

In [Ma2], the author gave an alternative proof of the equality case of the Bakry–Ledoux isoperimetric inequality via needle decomposition induced from the L1-optimal transport of the isoperimetric minimizer. By reviewing this process, one might obtained the respected rigidity result. It stated that if a weighted manifold satisfies the equality in isoperimetric inequality, it must be isometric to the product of Gausian space and a hypersurface. This observation comes from the fact that distance functions (or guiding function of the needle decomposition) on needles playing the role of an eigenfunction in the sharp spectral gap inequality in [CZ].

Based on this observation, we studied a quantitative estimate of Bakry–Ledoux inequality on weighted Riemannian manifolds with $\mathrm{Ric}_\infty \geq K > 0$. We followed the proof in [Ma2] to obtain an upper bound of the volume of the symmetric difference between a Borel set and a sub-level set of the guiding function arising in the needle decomposition, in terms of the deficit in Bakry–Ledoux's Gaussian isoperimetric inequality. This work is the first quantitative isoperimetric inequality on non-compact spaces besides Euclidean and Gaussian spaces so far. Our main result is the following theorem.

Main Theorem (Theorem 7.5) *Let $(M, g, \mathfrak{m})$ be a complete weighted Riemannian manifold such that $\mathrm{Ric}_\infty \geq 1$ and $\mathfrak{m}(M) = 1$. Fix $\theta \in (0,1) \setminus \{1/2\}$ and $\varepsilon \in (0,1)$, take a Borel set $A \subset M$ with $\mathfrak{m}(A) = \theta$, and assume that $\mathsf{P}(A) \leq \mathcal{I}_{(\mathbb{R},\gamma)}(\theta) + \delta$ holds for sufficiently small $\delta > 0$ (relative to θ and ε). Then, for the guiding function u associated with A such that $\int_M u \, d\mathfrak{m} = 0$, we have*

$$\min\Big\{\mathfrak{m}\big(A \triangle \{u \leq a_\theta\}\big), \mathfrak{m}\big(A \triangle \{u \geq a_{1-\theta}\}\big)\Big\} \leq C(\theta, \varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)}.$$

We would explain the notions and geometric meaning of this result on the following sections.

2 Preliminaries

A *weighted Riemannian manifold* means a triple $(M, g, \mathfrak{m})$, where (M, g) be a connected, complete $\mathcal{C}^\infty$-Riemannian manifold of dimension $n \geq 2$ without boundary and $\mathfrak{m} = \mathrm{e}^{-\Psi} \mathrm{vol}_g$ is a measure modifying the volume measure vol_g of (M, g) with a $\mathcal{C}^\infty$ *weight function* Ψ. In a weighted Riemannian manifold, the Ricci curvature is modified as follow:

Definition 2.1 (Weighted Ricci curvature) Given $v \in T_xM$ and $N \in \mathbb{R} \setminus \{n\}$, the *weighted Ricci curvature* $\mathrm{Ric}_N(v)$ is defined by

$$\mathrm{Ric}_N(v) := \mathrm{Ric}_g(v) + \mathrm{Hess}\,\Psi(v,v) - \frac{\langle \nabla\Psi(x), v\rangle^2}{N-n}.$$

As the limits of $N \to \infty$ and $N \downarrow n$, we also define

$$\mathrm{Ric}_\infty(v) := \mathrm{Ric}_g(v) + \mathrm{Hess}\,\Psi(v,v),$$
$$\mathrm{Ric}_n(v) := \begin{cases} \mathrm{Ric}_g(v) + \mathrm{Hess}\,\Psi(v,v) & \text{if } \langle \nabla\Psi(x), v\rangle = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

where $\mathrm{Ric}_g(v)$ is the original Ricci curvature. We note that $\mathrm{Ric}_N(cv) = c^2 \,\mathrm{Ric}_N(v)$ for all $c \in \mathbb{R}$. We also write $\mathrm{Ric}_N \geq K$ for $K \in \mathbb{R}$ when $\mathrm{Ric}_N(v) \geq K|v|^2$ holds for all $v \in TM$.

Remark 2.2 (i) By definition Ric_N enjoys the monotonicity

$$\mathrm{Ric}_n(v) \leq \mathrm{Ric}_N(v) \leq \mathrm{Ric}_\infty(v) \leq \mathrm{Ric}_{N'}(v)$$

for $N \in [n,\infty)$ and $N' \in (-\infty, n)$. Therefore, $\mathrm{Ric}_\infty \geq K$ is weaker than $\mathrm{Ric}_N \geq K$ with $N \in [n,\infty)$.

(ii) The range $N \in [n,\infty]$ has been intensively studied by Bakry and his collaborators via Γ-*calculus* (see [BaGL]), including the Bakry–Ledoux isoperimetric inequality under $\mathrm{Ric}_\infty \geq K > 0$ ([BL]).

(iii) The curvature bound $\mathrm{Ric}_N \geq K$ is equivalent to the *curvature-dimension condition* $\mathrm{CD}(K,N)$ in the sense of Lott–Sturm–Villani, see [LV2, St1, St2, Vi].

We also define the Laplacian associated with $\mathfrak{m}$.

Definition 2.3 (Weighted Laplacian) The *weighted Laplacian* acting on $u \in \mathcal{C}^\infty(M)$ is defined by

$$\Delta_{\mathfrak{m}} u := \Delta u - \langle \nabla u, \nabla \Psi \rangle,$$

where Δ is the Laplacian with respect to g.

The Green formula also holds for $\mathfrak{m}$:

$$\int_M \phi \Delta_{\mathfrak{m}} u \, d\mathfrak{m} = -\int_M \langle \nabla \phi, \nabla u \rangle \, d\mathfrak{m},$$

for $\phi \in \mathcal{C}^\infty(M)$ with compact support.

The condition $\mathrm{Ric}_\infty \geq K > 0$ implies $\mathfrak{m}$ has a Gaussian decay and $\mathfrak{m}(M) < \infty$ holds ([St1, Theorem 4.26]). Without loss of generality, we can normalize $\mathfrak{m}$ as $\mathfrak{m}(M) = 1$ since adding a constant to Ψ does not change Ric_∞. From $\mathrm{Ric}_\infty \geq K > 0$ we also have the lower bound of the first nonzero eigenvalue of $-\Delta_{\mathfrak{m}}$ as $\lambda_1 \geq K$, equivalent to the sharp *Poincaré inequality*.

$$\mathrm{Var}_{(M,\mathfrak{m})}(u) := \int_M u^2 \, d\mathfrak{m} - \left(\int_M u \, d\mathfrak{m} \right)^2 \leq \frac{1}{K} \int_M |\nabla u|^2 \, d\mathfrak{m}. \tag{2.1}$$

The rigidity problem was investigated in [CZ] as a counterpart to the celebrated Obata rigidity theorem ([Ob]).

Theorem 2.4 (Rigidity of spectral gap) *Let $(M,g,\mathfrak{m})$ be a complete weighted Riemannian manifold satisfying $\mathfrak{m}(M) = 1$ and $\mathrm{Ric}_\infty \geq K > 0$. If equality $\lambda_1 = K$ is achieved with an eigenfunction u, then:*

(i) *$(M,g,\mathfrak{m})$ is isometric to the product space $\mathbb{R} \times \Sigma$ as weighted Riemannian manifolds, where $\Sigma = u^{-1}(0)$ and $(\Sigma, g_\Sigma, \mathfrak{m}_\Sigma)$ is an $(n-1)$-dimensional weighted Riemannian manifold of $\mathrm{Ric}_\infty \geq K$, and $\mathbb{R}$ is equipped with the Gaussian measure $\sqrt{K/(2\pi)} \mathrm{e}^{-Kx^2/2} \, dx$.*

(ii) *The function u is constant on $\{t\} \times \Sigma$ for each $t \in \mathbb{R}$, and we can moreover choose as $u(t,x) = t$.*

3 Isoperimetric inequalities

To state the isoperimetric inequality, we define the *perimeter* of a Borel set $A \subset M$ with $\mathfrak{m}(A) < \infty$ as

$$\mathsf{P}(A) := \inf_{\{\phi_i\}_{i \in \mathbb{N}}} \liminf_{i \to \infty} \int_M |\nabla \phi_i| \, d\mathfrak{m}, \tag{3.1}$$

where $\{\phi_i\}_{i \in \mathbb{N}}$ runs over all sequences of Lipschitz functions converging to the indicator function χ_A in $L^1(\mathfrak{m})$. When $\mathsf{P}(A) < \infty$, we have $\mathsf{P}(X \setminus A) = \mathsf{P}(A)$.

Normalizing $\mathfrak{m}(M)=1$, we define the *isoperimetric profile* as

$$\mathcal{I}_{(M,\mathfrak{m})}(\theta) := \inf\{\mathsf{P}(A) \mid A \subset M,\, \mathfrak{m}(A)=\theta\}$$

for $\theta \in (0,1)$, where A runs over all Borel sets with $\mathfrak{m}(A)=\theta$. An isoperimetric inequality under the condition $\mathrm{Ric}_\infty \geq K > 0$ was first shown by Bakry–Ledoux [BL] in the same form in Gaussian spaces. Under a combination of $\mathrm{Ric}_N \geq K$ and the diameter bound $\mathrm{diam}(M) \leq D$, Milman [Mi1, Mi2] studied the following isoperimetric inequality:

Theorem 3.1 (Isoperimetric inequalities) *Let $(M,g,\mathfrak{m})$ be a complete weighted Riemannian manifold satisfying $\mathfrak{m}(M)=1$, $\mathrm{diam}(M) \leq D$ with $D \in (0,\infty]$, and $\mathrm{Ric}_N \geq K$ for $N \in (-\infty,0) \cup [n,\infty]$ and $K \in \mathbb{R}$. Then we have $\mathcal{I}_{(M,\mathfrak{m})}(\theta) \geq \mathcal{I}_{(K,N,D)}(\theta)$ for all $\theta \in (0,1)$, where $\mathcal{I}_{(K,N,D)}$ depends only on K, N and D.*

This inequality is sharp in all the parameters K, N and D, and $\mathcal{I}_{(K,N,D)}$ is independent from the dimension n of M. For the precise formulas of the function $\mathcal{I}_{(K,N,D)}$, we refer to [Mi1, Mi2]. Here we give the formula in the case $K>0$ and $N=\infty$ as in Bakry–Ledoux inequality. Without the diameter bound ($D=\infty$), the model space is the Gaussian space and

$$\mathcal{I}_{(K,\infty,\infty)}(\theta) = \sqrt{\frac{K}{2\pi}}\mathrm{e}^{-Ka_\theta^2/2}, \qquad \text{where } \sqrt{\frac{K}{2\pi}}\int_{-\infty}^{a_\theta} \mathrm{e}^{-Kt^2/2}\,dt = \theta.$$

For $D \in (0,\infty)$, we have

$$\mathcal{I}_{(K,\infty,D)}(\theta) = \inf_{\xi \in [-D,0]} f_{\xi,D}(\theta)$$

with

$$f_{\xi,D}(\theta) := \frac{\mathrm{e}^{-Kb_{\theta,\xi,D}^2/2}}{\int_\xi^{\xi+D} \mathrm{e}^{-Kt^2/2}\,dt}, \qquad \text{where } \frac{\int_\xi^{b_{\theta,\xi,D}} \mathrm{e}^{-Kt^2/2}\,dt}{\int_\xi^{\xi+D} \mathrm{e}^{-Kt^2/2}\,dt} = \theta.$$

The following lemma gives a look on how the diameter influences the isoperimetric profile.

Lemma 3.2 (Difference between $\mathcal{I}_{(K,\infty,D)}$ and $\mathcal{I}_{(K,\infty,\infty)}$) *Let $K, D \in (0,\infty)$. For $\theta \in (0,1)$, we have*

$$\mathcal{I}_{(K,\infty,D)}(\theta) - \mathcal{I}_{(K,\infty,\infty)}(\theta) > \frac{\sqrt{K}}{\pi}\frac{\mathrm{e}^{-KD^2}}{\sqrt{K}D+1}.$$

Moreover, the profile $\mathcal{I}_{(K,\infty,\infty)}$ is strictly concave by the following lemma.

Lemma 3.3 (Concavity of $\mathcal{I}_{(K,\infty,\infty)}$) *For $\theta \in (0,1)$, we have*

$$\mathcal{I}''_{(K,\infty,\infty)}(\theta) = -\frac{K}{\mathcal{I}_{(K,\infty,\infty)}(\theta)}.$$

The rigidity result of Bakry–Ledoux inequality was provided Morgan [Mo, Theorem 18.7] and an alternative proof based on the needle decomposition was introduced in [Ma2, Section 3].

Theorem 3.4 (Rigidity of isoperimetric inequality) *Let $(M,g,\mathfrak{m})$ be a complete weighted Riemannian manifold satisfying $\mathfrak{m}(M)=1$ and $\mathrm{Ric}_\infty \geq K$ for some $K>0$. If $\mathfrak{m}^+(A) = \mathcal{I}_{(K,\infty,\infty)}(\theta)$ holds for some $A \subset M$ with $\theta = \mathfrak{m}(A) \in (0,1)$, then we have the following.*

(i) *$(M,g,\mathfrak{m})$ is isometric to the product space $\mathbb{R} \times \Sigma$ as weighted Riemannian manifolds, where $(\Sigma, g_\Sigma, \mathfrak{m}_\Sigma)$ is an $(n-1)$-dimensional weighted Riemannian manifold of $\mathrm{Ric}_\infty \geq K$, and $\mathbb{R}$ is equipped with the Gaussian measure $\sqrt{K/(2\pi)}\mathrm{e}^{-Kx^2/2}\,dx$.*

(ii) *The set A is a half-space in this product structure, in the sense that A coincides with $(-\infty, a_\theta] \times \Sigma$ or $[a_{1-\theta}, \infty) \times \Sigma$.*

Our motivation is to study a quantitative version of this theorem.

4 Needle decomposition

Our main approach on this study is the localization method called *needle decomposition* (also called the *localization*), introduced by Klartag [Kl]. Via the needle decomposition one can reduce an inequality on a high-dimensional space to those on geodesics.

At first, we define transport rays associated with a 1-Lipschitz function.

Definition 4.1 (Transport rays) Let u be a 1-Lipschitz function on M. We say that $X \subset M$ is a *transport ray* associated with u if $|u(x) - u(y)| = d(x, y)$ holds for all $x, y \in X$ and if, for all $z \notin X$, there exists $x \in X$ such that $|u(x) - u(z)| < d(x, z)$.

Any transport ray is a closed set and necessarily the image of a minimal geodesic. Hence each transport ray is equipped with the natural distance in real analysis. The next theorem is the needle decomposition ([Kl, Theorems 1.2, 1.5]) with distance-like *guiding function* u acting as a 'guide' of the decomposition.

Theorem 4.2 (Needle decomposition) *Let $(M, g, \mathfrak{m})$ be a complete weighted Riemannian manifold satisfying $\mathrm{Ric}_N \geq K$, and take a function $f \in L^1(\mathfrak{m})$ such that $\int_M f \, d\mathfrak{m} = 0$ and $\int_M |f(x)| d(x_0, x) \, \mathfrak{m}(dx) < \infty$ for some $x_0 \in M$. Then there exists a 1-Lipschitz function u on M, a partition $\{X_q\}_{q \in Q}$ of M, a measure ν on Q and a family of probability measures $\{\mathfrak{m}_q\}_{q \in Q}$ on M satisfying the following.*

(i) *For any measurable set $A \subset M$, we have $\mathfrak{m}(A) = \int_Q \mathfrak{m}_q(A) \, \nu(dq)$. Moreover, for ν-almost every $q \in Q$, we have $\mathrm{supp}(\mathfrak{m}_q) = X_q$.*

(ii) *For ν-almost every $q \in Q$, X_q is a transport ray associated with u. Moreover, if X_q is not a singleton, then the weighted Ricci curvature of $(X_q, |\cdot|, \mathfrak{m}_q)$ satisfies $\mathrm{Ric}_N \geq K$.*

(iii) *For ν-almost every $q \in Q$, we have $\int_{X_q} f \, d\mathfrak{m}_q = 0$.*

Our approach on quantitative isoperimetric inequalities based on Klartag's proof in [Kl] of the isoperimetric inequality (Theorem 3.1) via the needle decomposition. Here we give an outline of this proof as we would give the quantitative estimate based on it.

Let $(M, g, \mathfrak{m})$ satisfy $\mathrm{Ric}_N \geq K$, $\mathrm{diam}(M) \leq D$ and $\mathfrak{m}(M) = 1$. Given the volume parameter $\theta \in (0, 1)$, we fix an arbitrary Borel set $A \subset M$ such that $\mathfrak{m}(A) = \theta$. We consider the needle decomposition associated with the function $f(x) := \chi_A(x) - \theta$. Then we find $\int_M f \, d\mathfrak{m} = 0$, and obtain (Q, ν) and $\{(X_q, \mathfrak{m}_q)\}_{q \in Q}$ associated with f as in Theorem 4.2. (iii) in Theorem 4.2 implies $\mathfrak{m}_q(A) = \theta$ for ν-almost every $q \in Q$. Moreover, $(X_q, |\cdot|, \mathfrak{m}_q)$ preserving the curvature condition $\mathrm{CD}(K, N)$ by (ii) and $\mathrm{diam}(X_q) \leq D$ trivially. Therefore, the 1-dimensional isoperimetric inequality yields $\mathsf{P}(A \cap X_q) \geq \mathcal{I}_{(K,N,D)}(\theta)$ for ν-almost every $q \in Q$, where $\mathsf{P}(A \cap X_q)$ denotes the perimeter of $A \cap X_q$ in $(X_q, |\cdot|, \mathfrak{m}_q)$. By Lemma 6.2 in the later section, we obtain

$$\mathsf{P}(A) \geq \int_Q \mathsf{P}(A \cap X_q) \, \nu(dq) \geq \mathcal{I}_{(K,N,D)}(\theta).$$

Taking the infimum in A gives $\mathcal{I}_{(M,\mathfrak{m})}(\theta) \geq \mathcal{I}_{(K,N,D)}(\theta)$.

5 One-dimensional analysis

From this section, we normalize as $K = 1$ without loss of generality. In this section, we consider 1-dimensional spaces enjoying $\mathrm{CD}(1, \infty)$ represent needles in the needle decomposition. Let $I \subset \mathbb{R}$ be a closed interval equipped with a measure $\mathfrak{m} = \mathrm{e}^{-\psi} \, dx$, where dx denotes the Lebesgue measure and ψ is locally Lipschitz. If $(I,|\cdot|,\mathfrak{m})$ satisfying $\mathrm{CD}(1,\infty)$ then ψ is 1-convex in the following sense:

$$\psi\big((1-t)x+ty\big)\le(1-t)\psi(x)+t\psi(y)-\frac{(1-t)t}{2}|x-y|^2$$

for any $x,y\in I$ and $t\in(0,1)$.

The following lemma in ([Bo, Proposition 2.1]) enables one to simplify the calculation on isoperimetric minimizer to 1-dimensional halfspaces.

Lemma 5.1 *Let $\mathfrak{m}=\mathrm{e}^{-\psi}\,dx$ be a probability measure on a closed interval $I\subset\mathbb{R}$ such that ψ is convex. Then the minimum of $\mathsf{P}(A)$ on the class of all Borel sets $A\subset I$ with $\mathfrak{m}(A)=\theta$ coincides with the minimum on the subclass consisting of the (semi-infinite) intervals $(-\infty,a]\cap I$ and $[b,\infty)\cap I$.*

The next proposition is the core estimate in 1-dimensional analysis as we shall compare $\mathfrak{m}$ on I with the Gaussian measure $\boldsymbol{\gamma}$ on $\mathbb{R}$ with mean 0 and variance 1, denoted by

$$\boldsymbol{\gamma}:=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,dx=\mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}(x)}\,dx,\qquad \boldsymbol{\psi}_{\mathrm{g}}(x):=\log\big(\sqrt{2\pi}\big)+\frac{1}{2}x^2.$$

Proposition 5.2 (Difference of weight functions) *Let $I\subset\mathbb{R}$ be a closed interval equipped with a probability measure $\mathfrak{m}=\mathrm{e}^{-\psi}\,dx$ such that ψ is 1-convex. Fix $\theta\in(0,1)$ and assume that*

$$\int_{I\cap(-\infty,a_\theta]}\mathrm{e}^{-\psi}\,dx=\theta \tag{5.1}$$

and that

$$\mathrm{e}^{-\psi(a_\theta)}\le\mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}(a_\theta)}+\delta \tag{5.2}$$

holds for sufficiently small $\delta>0$ (relative to θ). Then we have

$$\psi(x)-\boldsymbol{\psi}_{\mathrm{g}}(x)\ge\big(\psi'_+(a_\theta)-a_\theta\big)(x-a_\theta)-\omega(\theta)\delta \tag{5.3}$$

for every $x\in I$, and

$$\psi(x)-\boldsymbol{\psi}_{\mathrm{g}}(x)\le\big(\psi'_+(a_\theta)-a_\theta\big)(x-a_\theta)+\omega(\theta)\sqrt{\delta} \tag{5.4}$$

for every $x\in[S,T]$ such that $\lim_{\delta\to0}S=-\infty$ and $\lim_{\delta\to0}T=\infty$, where ψ'_+ denotes the right derivative of ψ and $\omega(\theta)$ is a constant depending only on θ.

With the help of Proposition 5.2, we obtain the following result as a quantitative version of Lemma 5.1 in 1-dimensional spaces. It states that a set with small isoperimetric deficit would have small symmetric difference with a half space.

Proposition 5.3 (Small symmetric difference) *Let $I\subset\mathbb{R}$ be a closed interval equipped with a probability measure $\mathfrak{m}=\mathrm{e}^{-\psi}\,dx$ such that ψ is 1-convex. Fix $\theta\in(0,1)$ and assume that, for a Borel set $A\subset I$ with $\mathfrak{m}(A)=\theta$,*

$$\mathsf{P}(A)\le\mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}(a_\theta)}+\delta$$

holds for a sufficiently small $\delta>0$. Then we have

$$\min\Big\{\mathfrak{m}\big(A\,\triangle\,(-\infty,r^-_{\mathfrak{m}}(\theta)]\big),\mathfrak{m}\big(A\,\triangle\,[r^+_{\mathfrak{m}}(\theta),\infty)\big)\Big\}\le\frac{\mathsf{P}(A)-\mathcal{I}_{(I,\mathfrak{m})}(\theta)}{C_1(\theta,\delta)}, \tag{5.5}$$

where $r^-_{\mathfrak{m}}(\theta),r^+_{\mathfrak{m}}(\theta)\in I$ are defined by

$$\mathfrak{m}\big(I\cap(-\infty,r^-_{\mathfrak{m}}(\theta)]\big)=\mathfrak{m}\big(I\cap[r^+_{\mathfrak{m}}(\theta),\infty)\big)=\theta,$$

and $\lim_{\delta\to0}C_1(\theta,\delta)=\infty$.

6 Reverse Poincare inequality

To study the well-behavior of the guiding function when the isoperimetric deficit is small, we consider the reverse direction of the Poincaré inequality as the guiding function of the needle decomposition plays the role of eigenfunction in the equality case of sharp Poincaré inequality. In 1-dimensional space with a small isoperimetric deficit, we obtain:

Proposition 6.1 (Reverse Poincaré inequality on needles) *Let $I \subset \mathbb{R}$ be a closed interval equipped with a probability measure $\mathfrak{m} = \mathrm{e}^{-\psi}\,dx$ such that ψ is 1-convex. Fix $\theta \in (0,1)$ and assume (5.1) and $\mathrm{e}^{-\psi(a_\theta)} \le \mathrm{e}^{-\psi_{\mathrm{g}}(a_\theta)} + \delta$. Then, given $\varepsilon \in (0,1)$, if $\delta > 0$ is sufficiently small (relative to θ and ε), we have*

$$\mathrm{Var}_{(I,\mathfrak{m})}(u) := \int_I u^2\,d\mathfrak{m} - \left(\int_I u\,d\mathfrak{m}\right)^2 \ge \frac{1}{\Lambda(\theta,\varepsilon,\delta)} \int_I |u'|^2\,d\mathfrak{m} \tag{6.1}$$

for every affine function $u(x) = ax + b$ with $a, b \in \mathbb{R}$, where $\Lambda(\theta,\varepsilon,\delta) \le (1 - C_2(\theta,\varepsilon)\delta^{(1-\varepsilon)/2})^{-1}$ and, in particular, $\lim_{\delta\to 0}\Lambda(\theta,\varepsilon,\delta) = 1$.

Put $f := \chi_A - \theta$ and denote by (Q,ν) and $\{(X_q,\mathfrak{m}_q)\}_{q\in Q}$ the elements of the needle decomposition as in Theorem 4.2 that we used to prove the isoperimetric inequality. $(X_q,\mathfrak{m}_q)$ enjoys $\mathrm{Ric}_\infty \ge 1$ (or $\mathrm{CD}(1,\infty)$) for ν-almost every $q \in Q$ and we define $A_q := A \cap X_q$ for $q \in Q$.

To extend the reverse Poincaré inequality to the high-dimensional manifold, we would make use of the following lemma which gives the nice decomposition of the isoperimetric deficit.

Lemma 6.2 (Decomposition of deficit) *We have*

$$\mathsf{P}(A) - \mathcal{I}_{(\mathbb{R},\gamma)}(\theta) \ge \int_Q \big(\mathsf{P}(A_q) - \mathcal{I}_{(\mathbb{R},\gamma)}(\theta)\big)\,\nu(dq),$$

where $\mathsf{P}(A_q)$ denotes the perimeter of A_q in $(X_q,\mathfrak{m}_q)$.

Let $u : M \longrightarrow \mathbb{R}$ be the guiding function associated with $f = \chi_A - \theta$ above. A reverse inequality is obtained by integrating (6.1) on needles.

Theorem 6.3 (Reverse Poincaré inequality) *Let $(M,g,\mathfrak{m})$ be a complete weighted Riemannian manifold such that $\mathrm{Ric}_\infty \ge 1$ and $\mathfrak{m}(M) = 1$. Fix $\theta, \varepsilon \in (0,1)$ and take a Borel set $A \subset M$ with $\mathfrak{m}(A) = \theta$ and $\mathsf{P}(A) \le \mathcal{I}_{(\mathbb{R},\gamma)}(\theta) + \delta$ for sufficiently small $\delta > 0$ (relative to θ and ε). Then the guiding function u associated with $f = \chi_A - \theta$ satisfies*

$$\mathrm{Var}_{(M,\mathfrak{m})}(u) \ge \frac{1}{\Lambda'(\theta,\varepsilon,\delta)} = \frac{1}{\Lambda'(\theta,\varepsilon,\delta)} \int_M |\nabla u|^2\,d\mathfrak{m},$$

where $\Lambda'(\theta,\varepsilon,\delta) \le (1 - C_2'(\theta,\varepsilon)\delta^{(1-\varepsilon)/(3-\varepsilon)})^{-1}$ and in particular $\lim_{\delta\to 0}\Lambda'(\theta,\varepsilon,\delta) = 1$.

7 Quantitative Bakry–Ledoux inequality

In this section we would give an outline of approach to our main result. Let $(M,g,\mathfrak{m})$ be a weighted Riemannian manifold with $\mathrm{Ric}_\infty \ge 1$ and $\mathfrak{m}(M) = 1$, fix $\theta \in (0,1)$ and take a Borel set $A \subset M$ with $\mathfrak{m}(A) = \theta$. We consider the needle decomposition associated with $f := \chi_A - \theta$ and obtain the guiding function u with $\int_M u\,d\mathfrak{m} = 0$, a partition $(X_q,\mathfrak{m}_q)_{q\in Q}$ and a probability measure ν on Q. Set $A_q := A \cap X_q$ as in the previous section.

We set $\delta(A) := \mathsf{P}(A) - \mathcal{I}_{(\mathbb{R},\gamma)}(\theta)$ and define the 'long' needles (as small deficit implies large diameter):

$$Q_\ell := \left\{ q \in Q \,\middle|\, \mathsf{P}(A_q) - \mathcal{I}_{(\mathbb{R},\gamma)}(\theta) < \sqrt{\delta(A)} \right\} \tag{7.1}$$

The decomposition of deficit in Lemma 6.2 implies the following.

Lemma 7.1 (Q_ℓ is large) *We have $\nu(Q_\ell) \geq 1 - \sqrt{\delta(A)}$.*

To study the behavior of long needles, we define two sets of different directions of the isoperimetric minimizers:

$$\begin{aligned} Q_\ell^- &:= \left\{ q \in Q_\ell \,\middle|\, \mathfrak{m}_q\big(A_q \triangle (-\infty, r^-_{\mathfrak{m}_q}(\theta)]\big) \leq \sqrt{\delta(A)} \right\}, \\ Q_\ell^+ &:= \left\{ q \in Q_\ell \,\middle|\, \mathfrak{m}_q\big(A_q \triangle [r^+_{\mathfrak{m}_q}(\theta), \infty)\big) \leq \sqrt{\delta(A)} \right\}, \end{aligned} \tag{7.2}$$

where X_q is parametrized by u and $r^{\pm}_{\mathfrak{m}_q}(\theta) \in X_q$ are defined by

$$\mathfrak{m}\big(X_q \cap (-\infty, r^-_{\mathfrak{m}_q}(\theta)]\big) = \mathfrak{m}\big(X_q \cap [r^+_{\mathfrak{m}_q}(\theta), \infty)\big) = \theta$$

as in Proposition 5.3.

The next lemma is a consequence of Proposition 5.3.

Lemma 7.2 ($Q_\ell^- \cup Q_\ell^+$ is large) *If $\delta(A)$ is sufficiently small, then we have*

$$\nu\big(Q_\ell \setminus (Q_\ell^- \cup Q_\ell^+)\big) \leq \sqrt{\delta(A)}.$$

Via the reverse Poincaré inequality, we obtain the following proposition which shows the guiding function u behaves well when the deficit is small.

Proposition 7.3 (u is nearly centered on most needles) *If $\delta(A)$ is sufficiently small, then there exists a measurable set $Q_c \subset Q$ such that $\nu(Q_c) \geq 1 - \delta(A)^{(1-\varepsilon)/(9-3\varepsilon)}$ and*

$$\max\left\{ |a_\theta - r^-_{\mathfrak{m}_q}(\theta)|, |a_{1-\theta} - r^+_{\mathfrak{m}_q}(\theta)| \right\} \leq C_3(\theta, \varepsilon)\delta(A)^{(1-\varepsilon)/(9-3\varepsilon)} \tag{7.3}$$

holds for every $q \in Q_c \cap Q_\ell$.

By the well-behavior of the guiding function, one might obtain that one of Q_ℓ^- and Q_ℓ^+ has a small volume or most of needles have the same direction of the isoperimetric minimizers. We need an additional assumption $\theta \neq 1/2$ as a technical condition.

Proposition 7.4 (One of Q_ℓ^- and Q_ℓ^+ is small) *Assume $\theta \neq 1/2$. Then we have*

$$\min\{\nu(Q_\ell^-), \nu(Q_\ell^+)\} \leq C_4(\theta)\delta(A)^{(1-\varepsilon)/(9-3\varepsilon)},$$

provided that $\delta(A)$ is sufficiently small.

Now we would give the proof of the main theorem by applying the previous results on this section.

Theorem 7.5 (Quantitative isoperimetry) *Let $(M, g, \mathfrak{m})$ be a complete weighted Riemannian manifold such that $\mathrm{Ric}_\infty \geq 1$ and $\mathfrak{m}(M) = 1$. Fix $\theta \in (0,1) \setminus \{1/2\}$ and $\varepsilon \in (0,1)$, take a Borel set $A \subset M$ with $\mathfrak{m}(A) = \theta$, and assume that $\mathsf{P}(A) \leq \mathcal{I}_{(\mathbb{R},\gamma)}(\theta) + \delta$ holds for sufficiently small $\delta > 0$ (relative to θ and ε). Then, for the guiding function u associated with A such that $\int_M u \, d\mathfrak{m} = 0$, we have*

$$\min\Big\{ \mathfrak{m}\big(A \triangle \{u \leq a_\theta\}\big), \mathfrak{m}\big(A \triangle \{u \geq a_{1-\theta}\}\big) \Big\} \leq C(\theta, \varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)}. \tag{7.4}$$

Proof. By Proposition 7.4, we first assume $\nu(Q_\ell^+) \le C_4(\theta)\delta^{(1-\varepsilon)/(9-3\varepsilon)}$ without loss of generality. From Lemmas 7.1 and 7.2,

$$\nu(Q \setminus Q_\ell^-) = \nu(Q \setminus Q_\ell) + \nu\big(Q_\ell \setminus (Q_\ell^- \cup Q_\ell^+)\big) + \nu(Q_\ell^+) \le 2\sqrt{\delta} + C_9(\theta)\delta^{(1-\varepsilon)/(9-3\varepsilon)}.$$

Therefore we obtain

$$\begin{aligned}
&\mathfrak{m}\big(A \bigtriangleup \{u \le a_\theta\}\big) \\
&\le \int_{Q_\ell^-} \mathfrak{m}_q\big(A_q \bigtriangleup (-\infty, a_\theta]\big)\,\nu(dq) + \nu(Q \setminus Q_\ell^-) \\
&\le \int_{Q_\ell^-} \mathfrak{m}_q\big((-\infty, a_\theta] \bigtriangleup (-\infty, r_{\mathfrak{m}_q}^-(\theta)]\big)\,\nu(dq) + \int_{Q_\ell^-} \mathfrak{m}_q\big(A_q \bigtriangleup (-\infty, r_{\mathfrak{m}_q}^-(\theta)]\big)\,\nu(dq) \\
&\quad + \nu(Q \setminus Q_\ell^-) \\
&\le \int_{Q_\ell^-} \mathfrak{m}_q\big((-\infty, a_\theta] \bigtriangleup (-\infty, r_{\mathfrak{m}_q}^-(\theta)]\big)\,\nu(dq) + 3\sqrt{\delta} + C_4(\theta)\delta^{(1-\varepsilon)/(9-3\varepsilon)}. \qquad (7.5)
\end{aligned}$$

Proposition 7.3 implies that $|a_\theta - r_{\mathfrak{m}_q}^-(\theta)| \le C_3(\theta,\varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)}$ for $q \in Q_c \cap Q_\ell$. Hence,

$$\begin{aligned}
\mathfrak{m}_q\big((-\infty, a_\theta] \bigtriangleup (-\infty, r_{\mathfrak{m}_q}^-(\theta)]\big) &= \mathfrak{m}_q\big(\min\{a_\theta, r_{\mathfrak{m}_q}^-(\theta)\}, \max\{a_\theta, r_{\mathfrak{m}_q}^-(\theta)\}\big) \\
&\le C(\theta,\varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)}
\end{aligned}$$

for $q \in Q_c \cap Q_\ell$. Substituting into (7.5), we obtain

$$\begin{aligned}
&\mathfrak{m}\big(A \bigtriangleup \{u \le a_\theta\}\big) \\
&\le C(\theta,\varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)} + \nu(Q_\ell^- \setminus Q_c) + 3\sqrt{\delta} + C_4(\theta)\delta^{(1-\varepsilon)/(9-3\varepsilon)} \\
&\le C(\theta,\varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)}.
\end{aligned}$$

□

References

[AGS] L. Ambrosio, N. Gigli and G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below. Duke Math. J. **163** (2014), 1405–1490.

[BL] D. Bakry and M. Ledoux, Lévy–Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator. Invent. Math. **123** (1996), 259–281.

[BaGL] D. Bakry, I. Gentil and M. Ledoux, Analysis and geometry of Markov diffusion operators. Springer, Cham, 2014.

[BBJ] M. Barchiesi, A. Brancolini and V. Julin, Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality. Ann. Probab. **45** (2017), 668–697.

[Bo] S. Bobkov, Extremal properties of half-spaces for log-concave distributions. Ann. Probab. **24** (1996), 35–48.

[CMM] F. Cavalletti, F. Maggi and A. Mondino, Quantitative isoperimetry à la Levy–Gromov. Comm. Pure Appl. Math. **72** (2019), 1631–1677.

[CM] F. Cavalletti and A. Mondino, Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. Invent. Math. **208** (2017), 803–849.

[CZ] X. Cheng and D. Zhou, Eigenvalues of the drifted Laplacian on complete metric measure spaces. Commun. Contemp. Math. **19** (2017), 1650001, 17 pp.

[CFMP] A. Cianchi, N. Fusco, F. Maggi and A. Pratelli, On the isoperimetric deficit in Gauss space. Amer. J. Math. **133** (2011), 131–186.

[Kl] B. Klartag, Needle decompositions in Riemannian geometry. Mem. Amer. Math. Soc. **249** (2017), no. 1180.

[LV2] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport. Ann. of Math. **169** (2009), 903–991.

[Ma1] C. H. Mai, On Riemannian manifolds with positive weighted Ricci curvature of negative effective dimension. Kyushu J. Math. **73** (2019), 205–218.

[Ma2] C. H. Mai, Rigidity for the isoperimetric inequality of negative effective dimension on weighted Riemannian manifolds. Geom. Dedicata **202** (2019), 213–232.

[MO] C. H. Mai and S. Ohta, Quantitative estimates for the Bakry-Ledoux isoperimetric inequality. To appear in *Comment. Math. Helv.*

[Mi1] E. Milman, Sharp isoperimetric inequalities and model spaces for the curvature-dimension-diameter condition. J. Eur. Math. Soc. **17** (2015), 1041–1078.

[Mi2] E. Milman, Beyond traditional curvature-dimension I: new model spaces for isoperimetric and concentration inequalities in negative dimension. Trans. Amer. Math. Soc. **369** (2017), 3605–3637.

[Mo] F. Morgan, Geometric Measure Theory. A beginner's guide. Fifth edition. Elsevier/Academic Press, Amsterdam, 2016.

[Ob] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere. J. Math. Soc. Japan **14** (1962), 333–340.

[Oh] S. Ohta, Needle decompositions and isoperimetric inequalities in Finsler geometry. J. Math. Soc. Japan, **70** (2018), 651–693.

[St1] K.-T. Sturm, On the geometry of metric measure spaces. I. Acta Math. **196** (2006), 65–131.

[St2] K.-T. Sturm, On the geometry of metric measure spaces. II. Acta Math. **196** (2006), 133–177.

[Vi] C. Villani, Optimal transport, old and new. Springer-Verlag, Berlin, 2009.