CLASSIFICATION OF COMBINATORIAL WEAVING DIAGRAMS

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ABSTRACT. Weavings are historically well-known structures in materials science, and have more recently become a very active research topic in mathematics. We will first attempt to present a formal mathematical definition of a weave, as a three-dimensional entangled network embedded into the thickened Euclidean plane. Next, we will also introduce a new construction method to build and classify a class of weaving diagrams - which are projections of weaves as in knot theory – using topological and combinatorial arguments.

1. WEAVES AND THEIR DIAGRAMS

A weave W is defined as a three-dimensional object embedded in the topological ambient space $\mathbb{X}^3 = \mathbb{E}^2 \times \mathrm{I}$, with $\mathrm{I} = [-1,1]$. Here, we will only consider a class of *two-periodic* Euclidean weaves and we will study their two-dimensional projections by a map $\pi : \mathbb{X}^3 \to \mathbb{E}^2$, $(x,y,z) \mapsto (x,y,0)$. Such a class is particular in that the weaves are constructed from straight lines which entangle in a cyclic pattern of overcrossings and undercrossings. We usually call such a structure a doubly periodic untwisted (p,q)-weave, but we will simply use the term weave from now on for simpler notation.

Definition 1.1. *A weave* W *is an embedding in* $\mathbb{X}^3$ *of infinitely many threads belonging to* $N \geq 2$ *disjoint sets of threads* $T_1,\dots,T_N$*, entangled to each other with respect to a set of crossing sequences* $\{C_{i,j} \mid i,j \in (1,...,N), i \neq j\}$ *such that,*

- *a thread t is a set homeomorphic to* $\mathbb{R}$*, embedded in* $\mathbb{X}^3$ *as a geodesic ;*
- *two threads belong to different sets of threads if their respective projections onto* $\mathbb{E}^2$ *by* π *intersect only once ;*
- *a crossing c is an intersection between the projections of two distinct threads onto* $\mathbb{E}^2$ *by* π *with an over or under information ;*
- *a crossing sequence* $C_{i,j} = (p,q)$ *indicates that for any thread of the set* T_i*, there are cyclically p crossings in which this thread is over the other strands of* T_j*, followed by q crossings in which it is under.*

Then, a weave W in a general position on $\mathbb{X}^3$ can be projected onto the Euclidean plane by the map π, as in knot theory [1]. This projection leads to a planar quadrivalent connected graph W_0, meaning that all the vertices have a degree four, and like in [2]., in the particular case of a doubly periodic structure, any unit cell can be seen as a link in the thickened torus as described in Figure 1.

Definition 1.2. *The projection* W_0 *of a weave* W *onto* $\mathbb{X}^2$ *by the map* $\pi : \mathbb{X}^3 \to \mathbb{X}^2$, $(x,y,z) \mapsto (x,y,0)$ *is called a regular projection, and once an over or under information is given at each vertex of* W_0*, we say that this structure is an infinite weaving diagram* D_{W_0}*. Moreover, if* D_{W_0} *is doubly periodic, then any unit cell contains non-trivial simple closed curve components and is called a torus diagram.*

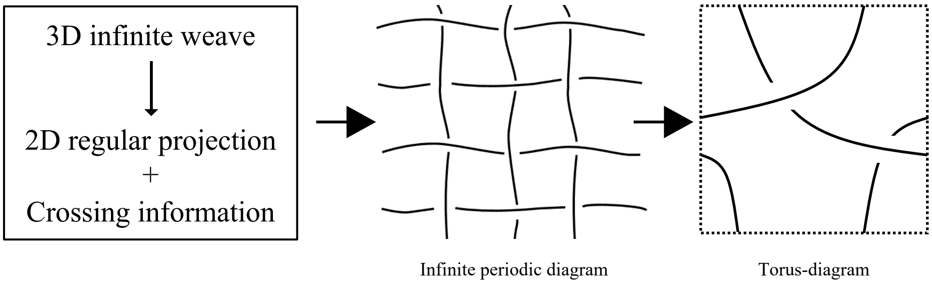

FIGURE 1. Weaving Diagram

2. EQUIVALENCE CLASSES OF WEAVES

Weaves are mainly characterized by a number N of disjoint sets of threads and a set of crossing sequences Σ, and as seen above, it is convenient to study their planar diagrams. Given a graph Γ satisfying the definition of a regular projection and a set Σ, it is possible to build a weaving diagram by assigning an over or under information to each vertex. However, this attribution is not unique as illustrated by some examples of existing woven materials showing that the weaving diagrams of two distinct weaves can be reconstruct from the same pair (Γ, Σ). The simplest cases are the diagrams related to the *basket weave (2,2)* and the *twill weave (2,2)*, showed in the Figure 2 below. Nevertheless, these two woven materials have distinct physical properties, such as strength or stiffness, and it is important to characterize these differences from a mathematical point of view.

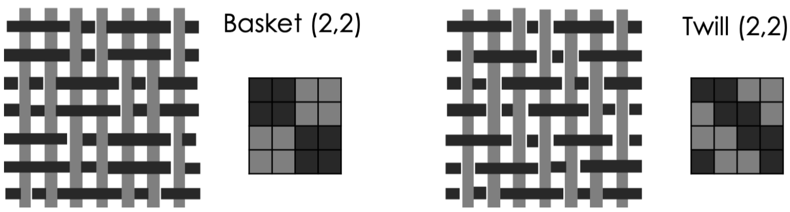

FIGURE 2. Twill and Basket $(2,2)$ square weaving diagram with their associate design.

This observation motivated the study of equivalence classes of weaves and the development of a new parameter Π, such that any weaving diagram constructed from the triple (Γ, Σ, Π) would be *unique*, up to equivalence. The notion of equivalence has been defined by S. Grishanov et al. by an extension of the classic *Reidemeister Theorem* for the case of doubly periodic weaves that can be studied from their torus diagrams [2].

Theorem 2.1. ***(Reidemeister Theorem for Weaves*** [2]***)***
Two weaves in $\mathbb{X}^3$ are ambient isotopic if and only if their torus diagrams can be obtained from each other by a sequence of Reidemeister moves Ω_1, Ω_2, and Ω_3, isotopies on the surface of the torus, and torus-twists.

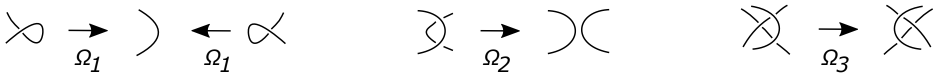

FIGURE 3. Reidemeister moves

To create our new parameter, we were inspired by the concept of *design* used for the classification of weaves in the textile industry. A design characterizes on a square grid the configuration of the crossings on a torus diagram according to the crossing sequence, for the restricted case of $N = 2$. More information on designs from a mathematical viewpoint can be found in the work of Grünbaum and Shephard in [3], and an example is given in Figure 3 for the basket and twill weaves mentioned above. To read them, consider that the rows represent the horizontal strands on a unit cell, the columns the vertical strands, and each square represents a crossing. Here, a gray square corresponds to a crossing where the horizontal strand is under the vertical strand, and conversely for a black square. Notice in Figure 2 that the gray and black squares have a *diagonal* organization for the case of twill weave, while they have a *bloc* organization for the case of basket weave. This illustrates the fact that there are different possibilities of assigning the crossing information to a same graph Γ, given a (set of) crossing sequence(s), and this justifies the different physical properties of the corresponding woven materials.

Our purpose is to generalize this concept to weaving diagrams with $N \geq 2$ sets of threads, so that it can describe the organization of crossings for each pair of sets of threads on a unit cell. Our approach consist in creating a set of *crossing matrices* associated with a flat torus diagram which would make it possible to distinguish the structures characterized by the same pair (Γ, Σ), and thus become a *weaving invariant* for the infinite diagram D_{W_0}. Our concept of crossing matrices is directly related to the crossing sequences of a weaving diagram D_{W_0}, which means that each matrix is associated with a pair of distinct sets of threads of the diagrams. The elements of a crossing matrix are the symbols $+1$ representing an overcrossing, or -1 representing an undercrossing. Each matrix encodes the crossing configuration between two sets of threads, from the point of view of one of them. This means that at an arbitrary crossing between two threads $t_1 \in T_1$ and $t_2 \in T_1$, with T_1 and T_2 two disjoint sets of threads, t_1 is over (or under) t_2, if we analyze the position of the threads of T_1 with respect to the threads of T_2, or conversely.

Definition 2.2. *Let $C_{i,j} = (p,q)$ be the crossing sequence of the sets T_i and T_j of a weaving diagram D_{W_0} with $N \geq 2$ sets of threads, $i, j \in \{1, \cdots, N\}$. Let $M_{i,j}$ be a $(m \text{ x } m)$ matrix consisting of symbols $+1$ and -1, where $m = p + q$ is called the module of $M_{i,j}$. Then, $M_{i,j}$ is called the crossing matrix of D_{W_0} associated with $C_{i,j}$, if each row and each column of $M_{i,j}$ simultaneously contains p symbol $+1$ and q symbols -1, considering cyclic and countercyclical permutations of the rows and columns of the matrix.*

This concept of crossing matrix makes it possible to distinguish two weaving diagrams characterized by the same pair (Γ, Σ), by assigning them a fixed sequence of crossing matrices $\Pi = \{M_{i,j} \setminus i, j \in \{1, \cdots, N\}\}$, and such that the sets of threads are indexed on W_0 in order to compare the strands having the same direction on the two diagrams. So now, it is possible to characterize the notion of equivalence classes for weaving diagrams of $\mathbb{E}^2$, using the triple (Γ, Σ, Π). Recall that the Reidemeister Theorem for Weaves defines the notion of equivalent weaves through the equivalence of their corresponding torus diagrams, so we will show the invariance by the Reidemeister moves and the torus' twists.

Theorem 2.3. *Let D_{W_1} and D_{W_2} be two torus-diagrams with $N \geq 2$ sets of threads, indexed such that their regular projections are identical, and with the same set of crossing sequences. Then, they are equivalent if and only if their crossing-matrices are pairwise equivalent, meaning that if at least one of the two conditions is satisfied:*

- *all the matrices of D_{W_2} are equivalent to the respective matrices of D_{W_1}, up to a same cyclic or countercyclical permutations of all the rows and/or columns;*
- *all the matrices of D_{W_2} are equivalent to the respective matrices of D_{W_1}, up to a same clockwise of counterclockwise rotation of $\frac{\pi}{4}$ or $\frac{\pi}{2}$, together with an inversion of all its symbols.*

One of the other great interests of these crossing matrices for the study of equivalence classes of weaving diagrams is that by defining any arbitrary weaving diagram by a triple (Σ', Σ, Π), we can find non equivalent structures with the same pair (Σ', Σ) just by using non-equivalent crossing matrices. This is very useful for the construction an classification of our diagrams.

3. Acknoledgement

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