Time Global Bounds for System of Partial Differential Equations with Ambrosetti-Rabinowitz Nonlinearity

Osaka University, Graduate School of Engineering Science, Department of Systems Innovation
Evan William Chandra

Abstract

We are concerned with L^∞-global bounds for global solutions of a system of some semilinear heat equations with subcritical Ambrosetti-Rabinowitz condition. For single equation case with polynomial term, it is known that global solutions have L^∞-global bounds under appropriate conditions. In particular, we will show that the global solutions to our problem has L^∞-global bounds which extends the result from single semilinear parabolic equation case to a system of semilinear parabolic equations.

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1 Introduction

1.1 Background and motivation

There have been a lot of studies on semilinear parabolic PDEs (see e.g. [2], [4], [8], [12], [15], [16], [17], etc). However, most of the results are related to semilinear parabolic PDEs with (Fujita type) polynomial nonlinearity. As an example, let $N \in \mathbb{N}$, $a \in \mathbb{R}$, $\theta \in (2, 2^*)$, $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain in $\mathbb{R}^N$, and consider

$$\begin{cases} \partial_t u = \Delta u + au + u|u|^{\theta-2} & \text{in } \Omega \times (0,\infty), \\ u(\,\cdot\,,t) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\,\cdot\,,0) = u_0 & \text{in } \Omega, \end{cases} \tag{P0}$$

where $u_0 \in L^\infty(\Omega) \cap H^1_0(\Omega)$ (for simplicity), and recall critical Sobolev exponent $2^* := \frac{2N}{N-2}$ when $N \geq 3$ and $2^* := \infty$ when $N = 1, 2$. When $a = 0$, we have the usual Fujita type equation.

In general, results for local existence of solution for (P0) when $a = 0$ has been proven by Weissler in [19]. Moreover, Weissler also shows that global existence to Problem (P0) for small non-negative initial data in [18]. In fact, Cazenave, Dickstein, and Weissler show that time global sign-changing solutions to Problem (P0) exist for some initial data in [3].

Next, we will briefly discuss some known results related to the behavior of global solution to Dirichlet problem (P0). First, we recall that for $\theta \in (2, 2^*)$ (subcritical case), Cazenave and Lions show that any global solution of (P0) with initial data in $C^2(\overline{\Omega})$ has an L^∞-global bound in [5]. However, when $\theta \geq \frac{6N+4}{3N-4}$, the dependence of any given initial data in $C^2(\overline{\Omega})$ for time global bounds is unclear. Some related results are also given by Giga in [11]. Here, the results of Cazenave and Lions in [5] are extended to full subcritical nonlinearity provided $a = 0$ and the solution is nonnegative. Moreover, the results can be extended even further by removing the assumption of nonnegativity of global solutions as shown by Ishiwata in [13]. In fact, the existence of time global bounds of global solutions to (P) can be obtained under some appropriate conditions for the energy functional of (P0).

Both necessary and sufficient conditions for the existence of time global bounds of (P0) have been obtained by Ishiwata in [13]. Namely, a global solution of (P0) has an L^∞-global bound if and only if the energy functional associated with (P0) satisfies the *Palais-Smale* condition along the trajectory of the solution itself.

In particular, we would like to study the behavior of global solutions to a system of semilinear parabolic equations with more general nonlinearity compared to the usual polynomial nonlinearity in Fujita Type equation. For instance, our result here covers logarithmic functions (1) as principle nonlinearity. Since we impose condition (N2) to the nonlinear term, it is essentially still a subcritical nonlinear term. Therefore, we expect that global solutions to (P) have time global bounds as in single equation case with polynomial nonlinearity. Our main result confirms that our global solutions indeed have time global bounds.

1.2 Problem

Let $N \in \mathbb{N}$, $a_1, a_2, b \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Moreover, we denote $2^* := \frac{2N}{N-2}$ when $N \geq 3$ and $2^* := \infty$ when $N = 1, 2$ as the critical Sobolev exponent. We consider the following system of semilinear heat equations:

$$\begin{cases} \partial_t u = \Delta u + a_1 u + bv + f(u,v) & \text{in } \Omega \times (0,\infty), \\ \partial_t v = \Delta v + bu + a_2 v + g(u,v) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\,\cdot\,,0) = u_0 & \text{in } \Omega, \\ v(\,\cdot\,,0) = v_0 & \text{in } \Omega, \end{cases} \tag{P}$$

where u_0, $v_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$. Moreover, we also impose the following conditions for f and g:

(N1) f, $g : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz in $\mathbb{R}$ and $f(0,0) = g(0,0) = 0$.

(N2) There exists $\alpha > 1$, $\beta > 1$ so that $\alpha + \beta = p \in [2 + \frac{2}{N}, 2^*)$, and $C_{N2} > 0$ for any $(u,v) \in \mathbb{R}^2$ with $|f(u,v)| \leq C_{N2}|u|^{\alpha-1}|v|^\beta$ and $|g(u,v)| \leq C_{N2}|u|^\alpha|v|^{\beta-1}$.

(AR) There exists $F : \mathbb{R}^2 \to \mathbb{R}$ which satifies the following conditions:

- (AR0) For any $(u,v) \in \mathbb{R}^2$, $\nabla F(u,v) = (f(u,v), g(u,v))^\mathsf{T}$.
- (AR1) F is positive in $\mathbb{R}^2 \setminus \{(0,0)\}$.
- (AR2) There exists $\mu > 2$ for any $(u,v) \in \mathbb{R}^2$ with $\mu F(u,v) \leq \frac{uf(u,v)+vg(u,v)}{2}$.

In particular, we are concerned with a system of semilinear heat equations with subcritical general nonlinearity.

For instance, we can consider an example when $\Omega \subset \mathbb{R}^3$ with $\alpha = 2$, $\beta = 2.2$ and $\mu = 2.1$ as follows:

$$\begin{cases} f(u,v) & = 2u\log(1+|u|^2) + \frac{2u^3|v|^{2.2}}{1+|u|^2}, \\ g(u,v) & = 2.2u^2\log(1+|u|^2)v|v|^{0.2}, \end{cases} \tag{1}$$

which satisfies (N1), (N2), and

$$F(u,v) = u^2\log(1+|u|^2)|v|^{2.2} \tag{2}$$

satisfies (AR) where $\nabla F(u,v) = (f(u,v), g(u,v))^\mathsf{T}$. Clearly, we can see that (1) is not a polynomial nonlinearity.

Next, we define

$$A := \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} \text{ with eigenvalue } \mu_1 \text{ and } \mu_2. \tag{3}$$

Without loss of generality, we may assume $\mu_1 \le \mu_2$. Then, we denote μ_* as the first eigenvalue of the eigenvalue problem $(-\Delta, H_0^1(\Omega))$ and assume

(AN) A is real symmetric matrix and $\mu_2 < \mu_*$.

Here, we only consider (time) global classical solutions with respect to (P). Furthermore, we can see that the growth condition of f and g is controlled by (N2) whereas (AR) is the so called Ambrosetti-Rabinowitz condition. We also put a remark here that the assumption $p \in [2+\frac{2}{N}, 2^*)$ is crucial for scaling invariance and to preserve subcritical nonlinearity.

Next, we would like to clarify the definition of L^∞-global bound for our problem as follows.

Definition 1.1. *Let* (u,v) *be a global classical solution of* (P). (u,v) *is said to have an* L^∞*-global bound (time global bound) provided that*

$$\sup_{t\in[0,\infty)} (\|u(t)\|_\infty + \|v(t)\|_\infty) < \infty.$$

2 Main result

Our goal is to establish the existence of L^∞-global bounds (time global bounds) of (u,v) which satisfies (P).

Theorem 2.1 (Main Theorem)**.** *Assume* (u,v) *as the global classical solution of* (P). *Then,* (u,v) *has an* L^∞*-global bound.*

3 Preliminaries

In this section, we prepare some basic tools necessary to obtain our main result. We mainly use the inherent properties of energy functionals related to (P) and scaling invariance property. First, we need to define the energy functionals related to (P) as follows:

$$I[u,v] := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2}\int_\Omega (u,v)A(u,v)^\mathsf{T} dx - \int_\Omega F(u,v)dx \tag{I1}$$

and

$$J[u,v] := \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - \int_\Omega (u,v)A(u,v)^\mathsf{T} dx - \int_\Omega (uf(u,v) + vg(u,v))dx \tag{J1}$$

for any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$. For convenience, we will denote $H_0^1(\Omega) \times H_0^1(\Omega)$ as H.

Next, we consider the following energy equalities.

Lemma 3.1 (Energy equalities)**.** *We have the following energy equalities.*

$$\frac{d}{dt} I[u(t), v(t)] = -(\|\partial_t u(t)\|_2^2 + \|\partial_t v(t)\|_2^2) \tag{I2}$$

and

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_2^2 + \|v(t)\|_2^2) = -J[u(t), v(t)]. \tag{J2}$$

Next, we observe that the energy functional I is nonnegative by using concavity argument (see e.g. [16, pp.373, Theorem I]).

Proposition 3.2 (Nonnegativity of energy functional)**.** *Let* (u, v) *be a global classical solution of* (P)*. Then, we have*

$$I[u(t), v(t)] \geq 0 \text{ for any } t \in [0, \infty). \tag{4}$$

Corollary 3.3. $I[u(t), v(t)] \to L$ *as* $t \to \infty$ *for some* $L \geq 0$*.*

By (I2) and Proposition 3.2, I is non-increasing and bounded from below. Therefore, Corollary 3.3 holds true. Next, we will prove some basic results related to time sequence along global solutions. Let $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ with $t_n \to \infty$ as $n \to \infty$. Here, we define the following two conditions for our energy functional I:

(I-L) $I[u(t_n), v(t_n)] \to L$ as $n \to \infty$.

(I-FD) $(dI)_{u(t_n),v(t_n)} \to 0$ as $n \to \infty$ in H^* where $(dI)_{u(t_n),v(t_n)}$ is the Fréchet derivative of I at $(u(t_n), v(t_n))$ in H and H^* is the dual space of H.

We proceed to prove the following proposition.

Proposition 3.4 (Compactness of a certain type of time sequence in H)**.** *Assume that* (u, v) *is a global solution of* (P)*. Then, for any time sequence* $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ *with* $t_n \to \infty$ *as* $n \to \infty$*, if* I *satisfies (I-L) and (I-FD), we have* $((u_n, v_n))_{n\in\mathbb{N}}$ *is compact in* H*.*

This proposition is essential to show that our global solution is uniformly bounded in H. Next, we proceed to clarify the scaling invariance. Let $\lambda > 0$, $x_* \in \mathbb{R}^N$, $t_* \in [0, \infty)$, and define the following variable transformations

$$\begin{cases} y & := \lambda(x - x_*), \\ s & := \lambda^2(t - t_*), \\ \tilde{u}_\lambda(y, s) & := \lambda^{\frac{-2}{p-2}} u(x, t), \\ \tilde{v}_\lambda(y, s) & := \lambda^{\frac{-2}{p-2}} v(x, t). \end{cases} \tag{5}$$

Then, by using (P) and (5), we will obtain

$$\begin{cases} \partial_s \tilde{u}_\lambda = \Delta \tilde{u}_\lambda + \lambda^{-2}(a_1 \tilde{u}_\lambda + b\tilde{v}_\lambda) + \lambda^{\frac{-2}{p-2}-2} f(\lambda^{\frac{2}{p-2}} \tilde{u}_\lambda, \lambda^{\frac{2}{p-2}} \tilde{v}_\lambda), \\ \partial_s \tilde{v}_\lambda = \Delta \tilde{v}_\lambda + \lambda^{-2}(b\tilde{u}_\lambda + a_2 \tilde{v}_\lambda) + \lambda^{\frac{-2}{p-2}-2} g(\lambda^{\frac{2}{p-2}} \tilde{u}_\lambda, \lambda^{\frac{2}{p-2}} \tilde{v}_\lambda). \end{cases} \tag{P1}$$

Next, we set $p_0 := \frac{N}{2}(p - 2)$ and we will have the following scaling invariance.

Lemma 3.5. *(Scaling invariance) The* L^{p_0}*-norm of* $\tilde{u}$ *and* $\tilde{v}$ *are invariant under* (5)*; i.e.,*

$$\|u(t)\|_{p_0} = \|\tilde{u}_\lambda(s)\|_{p_0,\Omega_\lambda} \text{ and } \|v(t)\|_{p_0} = \|\tilde{v}_\lambda(s)\|_{p_0,\Omega_\lambda} \text{ for any } t \in [0, \infty).$$

The proof of this lemma is rather similar to the proof of Lemma 3.3 (b) in [13]. Finally, we will show the following proposition.

Proposition 3.6 (Convergence in Hölder space)**.** *Assume* $(w_n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N \times [-1, 0]))$ *for any* $\alpha' \in (0, 1)$*. Then, we have*

$$w_n \to w \text{ in } C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N \times [-1, 0]) \text{ as } n \to \infty \tag{6}$$

up to a subsequence (which is denoted by the same notation).

4 Proof of main result

4.1 Compactness of orbit in $L^q(\Omega) \times L^q(\Omega)$

In this subsection, we begin by showing that our global solution for Problem (P) is uniformly bounded in H. This leads us to the following proposition.

Proposition 4.1 (Boundedness in H)**.** *Let (u, v) be a global solution of* (P)*. Then, it is uniformly bounded in H.*

The boundedness of global solution in H will be crucial for the proof of our main theorem since it enables us to use Rellich-Kondrachov Theorem. Then, by recalling Proposition 4.1, we deduce the weak convergence (up to a subsequence) of orbit in H. Next, we apply Rellich-Kondrachov Theorem to deduce that for any time sequence $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ with $t_n \to \infty$ as $n \to \infty$, there exists a subsequence (still denoted by the same notation) and $(u_*, v_*) \in H$ such that

$$(u_n, v_n) \to (u_*, v_*) \text{ in } L^q(\Omega) \times L^q(\Omega) \text{ as } n \to \infty \tag{7}$$

for any $q \in [1, 2^*)$.

The compactness of our global solution in $L^q(\Omega) \times L^q(\Omega)$ space is crucial to prove Theorem 2.1. Now, we are ready to prove our main result.

4.2 Proof of Theorem 2.1

We will prove this theorem by using contradiction argument which is similar to the one given in [13, pp.1030, Proof of Proposition 4.2] for single equation case. However, we discuss system of PDEs instead of a single equation here which is not covered in [13]. Assume the conclusion does not hold, then we have a time sequence $(t'_n)_{n\in\mathbb{N}}$ such that $t'_n \to \infty$ as $n \to \infty$ and $\|u(t'_n)\|_\infty + \|v(t'_n)\|_\infty \to \infty$ as $n \to \infty$. Without loss of generality, we may assume

$$\|u(t'_n)\|_\infty \geq \|v(t'_n)\|_\infty \text{ and set } M(t'_n) := \|u(t'_n)\|_\infty \text{ for any } n \in \mathbb{N}.$$

However, this means that we can guarantee the existence of time sequence (by taking the subsequence of $(t'_n)_{n\in\mathbb{N}}$ if necessary) $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ which satisfies

$$\begin{cases} t_n \to \infty \text{ as } n \to \infty, \\ M(t_n) \to \infty \text{ as } n \to \infty, \\ \sup\limits_{t\in[0,t_n]} M(t) = M(t_n) \text{ for any } n \in \mathbb{N}. \end{cases} \tag{8}$$

Moreover, by continuity of (u, v), we can find $(x_n)_{n\in\mathbb{N}}$ (and take a subsequence if necessary) and $x_0 \in \overline{\Omega}$ such that

$$\begin{cases} x_n \to \infty \text{ as } n \to \infty, \\ \frac{1}{2}M(t_n) \leq |u(x_n, t_n)| \text{ for any } n \in \mathbb{N}. \end{cases} \tag{9}$$

Let us take $\lambda^{\frac{2}{p-2}} = M(t_n) =: \lambda_n^{\frac{2}{p-2}}$, $x_* = x_n$, and $t_* = t_n$ to apply (5) to our original problem (P). Then, we will have the following equation:

$$\begin{cases} \partial_s \tilde{u}_n = \Delta \tilde{u}_n + \lambda_n^{-2}(a_1 \tilde{u}_n + b \tilde{v}_n) + \lambda_n^{\frac{-2}{p-2}-2} f(\lambda_n^{\frac{2}{p-2}} \tilde{u}_n, \lambda_n^{\frac{2}{p-2}} \tilde{v}_n), \\ \partial_s \tilde{v}_n = \Delta \tilde{v}_n + \lambda_n^{-2}(b \tilde{u}_n + a_2 \tilde{v}_n) + \lambda_n^{\frac{-2}{p-2}-2} g(\lambda_n^{\frac{2}{p-2}} \tilde{u}_n, \lambda_n^{\frac{2}{p-2}} \tilde{v}_n). \end{cases} \tag{Pwn}$$

Notice that by using (8) and (9), we also have

$$\begin{cases} \frac{1}{2} \leq \tilde{u}_n(0_y, 0_s) \text{ for any } n \in \mathbb{N}, \\ \|\tilde{v}_n\|_{L^\infty(-2,0;L^\infty)} \leq \|\tilde{u}_n\|_{L^\infty(-2,0;L^\infty)} \leq 1 \text{ for any } n \in \mathbb{N} \text{ (sufficiently large).} \end{cases} \tag{10}$$

Furthermore, we will have two cases as follows:

(Case 1) $\limsup_{n\to\infty} \lambda_n \mathrm{dist}(x_n, \partial\Omega) = \infty$.
(Case 2) $\limsup_{n\to\infty} \lambda_n \mathrm{dist}(x_n, \partial\Omega) < \infty$.

For simplicity, we denote $\gamma = \limsup_{n\to\infty} \lambda_n \mathrm{dist}(x_n, \partial\Omega)$ here and proceed to (Case 1) first.

We begin by assuming $\gamma = \infty$ and may assume further that $\lambda_n(\Omega - x_n) \to \mathbb{R}^N$ as $n \to \infty$ and $x_0 \in \Omega$. Then, we apply L^p-estimate for parabolic operators (see e.g. [15, pp. 172, Theorem 7.13]) to deduce that $(\tilde{u}_n)_{n\in\mathbb{N}}$ and $(\tilde{v}_n)_{n\in\mathbb{N}}$ are bounded sequences in $W^{2,1}_{q,loc}(\mathbb{R}^N \times (-1,0))$ for sufficiently large $q > 1$ by using (10). Thus, $(\tilde{u}_n)_{n\in\mathbb{N}}$ and $(\tilde{v}_n)_{n\in\mathbb{N}}$ are also bounded sequences in $C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N \times [-1,0]))$ by recalling Sobolev embedding $W^{2,1}_{q,loc} \hookrightarrow C^{0,\alpha';0,\alpha'/2}_{loc}$ (see e.g. [17, pp. 80, Lemma 3.3]) for any $\alpha' \in (0,1)$. Next, we will show that the following lemma holds true.

Lemma 4.2. *Let $((u_n, v_n))_{n\in\mathbb{N}}$ be a solution of* (Pwn). *Then, we have*

$$\|\partial_s \tilde{u}_n\|^2_{L^2(-1,0;L^2)} + \|\partial_s \tilde{v}_n\|^2_{L^2(-1,0;L^2)} \to 0 \text{ as } n \to \infty \tag{11}$$

Then, we can take $(w_n)_{n\in\mathbb{N}}$ as $(\tilde{u}_n)_{n\in\mathbb{N}}$ or $(\tilde{v}_n)_{n\in\mathbb{N}}$ in Proposition 3.6 to deduce that

$$\tilde{u}_n \to \tilde{u} \text{ and } \tilde{v}_n \to \tilde{v} \text{ in } C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N \times [-1,0]) \text{ as } n \to \infty \tag{12}$$

up to a subsequence. Moreover, by using (11), we deduce the following corollary.

Corollary 4.3. *Assume that* (11) *holds true. Then, we have*

$$\int_{-1}^{0}\int_{\mathbb{R}^N} \tilde{u}\partial_s\varphi dyds = 0 \text{ and } \int_{-1}^{0}\int_{\mathbb{R}^N} \tilde{v}\partial_s\varphi dyds = 0 \tag{13}$$

for any test function $\varphi \in C_0^\infty(\mathbb{R}^N \times [-1,0])$.

In other words, the weak derivative of $\tilde{u}$ and $\tilde{v}$ with respect to s are equal to zero. Thus, we conclude that $\tilde{u}$ and $\tilde{v}$ are independent of s since their weak derivatives are zero and they are locally uniformly continuous functions.

Next, by using (10), we can find $r > 0$ small enough so that

$$\|\tilde{u}\|_{L^{p_0}(B_y(0,r))} > 0. \tag{14}$$

Similarly, we let $\varepsilon > 0$ and define

$$B_x(x_0;\varepsilon) := \{x \in \mathbb{R}^N \mid |x - x_0| < \varepsilon\}$$
$$B_x(x_n;\frac{r}{\lambda_n}) := \left\{x \in \mathbb{R}^N \mid |x - x_n| < \frac{r}{\lambda_n}\right\}.$$

Note that for ε small enough, $B_x(x_0;\varepsilon) \subset \Omega$ which is well defined and $B_x(x_n;\frac{r}{\lambda_n}) \subset B_x(x_0;\varepsilon)$ for sufficiently large n. Thus, by using Proposition 3.6 (uniform convergence) and Lemma 3.5 (scaling invariance), we see that

$$\begin{aligned} \|\tilde{u}\|_{L^{p_0}(B_y(0;r))} &= \|\tilde{u}_n(0_s)\|_{L^{p_0}(B_y(0;r))} + o(1) \\ &= \|u_n\|_{L^{p_0}(B_x(x_n;\frac{r}{\lambda_n}))} + o(1). \end{aligned}$$

Moreover, recall that $p_0 \in [1, 2^*)$ since $p \in [2 + \frac{2}{N}, 2^*)$. Hence, we can use (7) to see that

$$\begin{aligned} \|\tilde{u}\|_{L^{p_0}(B_y(0;r))} &\leq \|u_n\|_{L^{p_0}(B_x(x_0;\varepsilon))} + o(1) \\ &= \|u_*\|_{L^{p_0}(B_x(x_0;\varepsilon))} \to 0 \text{ as } \varepsilon \downarrow 0 \end{aligned}$$

which is impossible since (14) holds true. Therefore, this is a contradiction.

Now, we asssume that $\gamma < \infty$ with $x_0 \in \partial\Omega$ and $\lambda_n(\Omega - x_n) \to \mathbb{R}^N_+$ (using coordinate rotation if necessary so that $x_0^N < x^N$ for any $x \in \overline{\Omega}$) as $n \to \infty$. For clarity, here we define

$$\mathbb{R}^N_+ := \{y \in \mathbb{R}^N \mid y^N > -\gamma\}.$$

Next, we simply repeat the procedure as before to see that $(\tilde{u}_n)_{n\in\mathbb{N}}$ and $(\tilde{v}_n)_{n\in\mathbb{N}}$ are uniformly bounded in $W^{2,1}_{q,q;loc}(\mathbb{R}^N_+ \times (-1, 0))$ for any $q > 1$ sufficiently large. Hence, we can deduce that they are uniformly bounded in $C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N_+ \times [-1, 0])$. Since (P) satisfies zero Dirichlet boundary condition, we can deduce that $\gamma > 0$ (see e.g. [7, pp.477, Case (ii)] and [11, pp. 418, Case 2 (iii)]). Then, we only need to take $r = \frac{\gamma}{2}$ to deduce a contradiction by following similar steps as in (Case 1). Thus, we finish the proof for (Case 2) and complete the proof of Theorem 2.1.

References

[1] Ambrosetti, A., Rabinowitz, P.H. *Dual variational methods in critical point theory*, J. Funct. Anal. 14 (1973), 349-381.

[2] Cazenave, H., Brezis, H. *A nonlinear heat equation with singular initial data*, Journal D'Analyse Mathématique, Vol. 68 (1996), 277–304.

[3] Cazenave, H., Dickstein, F., Weissler, F. *Global existence and blowup for sign-changing solutions of the nonlinear heat equation*, J. Differential Equations 246 (2009), 2669–2680.

[4] Cazenave, H., Haraux, A. *An Introduction to Semilinear Evolution Equations.* Translated from the 1990 French original by Yvan Martel and revised by the authors. Oxford Lecture Series in Mathematics and its Applications, 13. The Clarendon Press, Oxford University Press, New York, 1998.

[5] Cazenave, H., Lions, P.L. *Solutions globales d'equations de la chaleur semilineaires*, Comm. Partial Differential Equations 9 (1984), 935–978.

[6] Chandra, E.W., Ishiwata, M. *On bounds for global solutions of seliminear parabolic eequations system with critical nonlinearity*, preprint.

[7] Fila, M., Souplet, P. *The blow-up rate for semilinear parabolic problems on general domains*, NoDEA Nonlinear Differential Equations Appl., 8 (2001), 473–480.

[8] Galaktinov, V.A., King, J.R. *Continuation of blowup solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Appl. Math. 50 (1997), 1–67.

[9] Giga, Y., Kohn, R. *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math. 42 (1989), no. 6, 845–884.

[10] Giga, Y., Kohn, R. *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. 36 (1987), no. 1, 1–40.

[11] Giga, Y. *A bound for global solutions of semilinear heat equations*, Comm. Math. Phys., 103 (1986), 415–421.

[12] Ikehata, R., Suzuki, T. *Stable and unstable sets for evolution equations of parabolic and hyperbolic type*, Hiroshima Math. J. 26 (1996), no. 3, 475–491.

[13] Ishiwata, M. *On bounds for global solutions of semilinear parabolic equations with critical and subcritical Sobolev exponent*, Differential and Integral Equations 20 (2007), no. 9, 1021–1034.

[14] Ishiwata, M. *On a potential-well type result and global bounds of solutions for semilinear parabolic equation involving critical Sobolev exponents.* Notes. Private communication.

[15] Lieberman, G.M. *Second Order Parabolic Differential Equations.* World Scientific Publishing, River Edge, 1996.

[16] Levine, H.A. *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + \mathscr{F}(u)$, Arch. Rational Mech. Anal. 51 (1973), 371–386.

[17] Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N. *Linear and Quasilinear Equations of Parabolic Type.* Amer. Math Soc., Providence, 1968.

[18] Weissler, F. *Existence and non-existence of global solutions for semilinear heat equation*, Israel Journal of Mathematics, Vol. 38, Nos. 1–2, 1981, 29–40.

[19] Weissler, F. *Local existence and nonexistence for semilinear parabolic equations in* L^p, Indiana Univ. Math. J. 29 (1980), 79–102.

[20] Willem, M. *Minimax theorems.* Progress in Nonlinear Differential Equations and their Applications, 24. Birkhauser Boston, Inc., Boston, MA, 1996.