Poisson Algebas II

Maram Alossaimi

The University of Sheffield

Abstract

The concept of Poisson algebras is one of the most important concepts in mathematics that make a link between commutative and non-commutative algebra. Poisson algebras can be defined as Lie algebras that satisfy the Leibniz rule. Our research is about classifying a large Poisson algebra class $\mathcal{A} = K[t][x, y]$, that is a Poisson polynomial algebra in two variables x and y with coefficients on the Poisson polynomial algebra $K[t]$, where K is an algebraic closure field with zero characteristic. There are three main cases of the classification of the Poisson algebra class $\mathcal{A}$. We are interested in the Poisson spectrum of $\mathcal{A}$, minimal and maximal Poisson ideals of $\mathcal{A}$. I presented the first case of the classification in the poster called 'Poisson Algebras I' and in this poster, I will present a part of the second case of the classification and its Poisson spectrum.

1. Introduction

A commutative algebra D over a field K is called a *Poisson algebra* if there exists a bilinear product $\{\cdot, \cdot\} : D \times D \to D$, called a *Poisson bracket*, such that

1. $\{a, b\} = -\{b, a\}$ for all $a, b \in D$ (anti-commutative),
2. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in D$ (Jacobi identity), and
3. $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in D$ (Leibniz rule).

Definition. Let D be a Poisson algebra. An ideal I of the algebra D is a *Poisson ideal* of D if $\{D, I\} \subseteq I$. Moreover, a Poisson ideal P of the algebra D is a *Poisson prime ideal* of D provided

$$IJ \subseteq P \Rightarrow I \subseteq P, \quad \text{or} \quad J \subseteq P$$

where I and J are Poisson ideals of D. A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

Definition. Let D be a Poisson algebra over a field K. A K-linear map $\alpha : D \to D$ is a *Poisson derivation* of D if α is a K-derivation of D and

$$\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\} \quad \text{for all } a, b \in D.$$

A set of all Poisson derivations of D is denoted by $\mathrm{PDer}_K(D)$.

2. How do we get our Poisson algebra class $\mathcal{A}$?

Lemma. [Oh3] *Let D be a Poisson algebra over a field K, $c \in K$, $u \in D$ and $\alpha, \beta \in \mathrm{PDer}_K(D)$ such that*

$$\alpha\beta = \beta\alpha \quad \textit{and} \quad \{d, u\} = (\alpha + \beta)(d)u \quad \textit{for all } d \in D. \tag{1}$$

Then the polynomial ring $D[x,y]$ becomes a Poisson algebra with Poisson bracket

$$\{d,y\} = \alpha(d)y, \quad \{d,x\} = \beta(d)x \quad \textit{and} \quad \{y,x\} = cyx + u \quad \textit{for all } d \in D. \tag{2}$$

The Poisson algebra $D[x,y]$ with Poisson bracket (2) is denoted by $(D;\alpha,\beta,c,u)$.

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3. How do we classify $\mathcal{A}$?

We aim to classify all the Poisson algebra's $\mathcal{A} = (K[t];\alpha,\beta,c,u)$, where K is an algebraically closed field of characteristic zero and $K[t]$ is the polynomial Poisson algebra (with necessarily trivial Poisson bracket, i.e. $\{a,b\} = 0$ for all $a,b \in K[t]$). Notice that, it follows from the second part of equality (1) that

$$0 = \{d,u\} = (\alpha+\beta)(d)u \quad \text{for all } d \in K[t],$$

which implies that precisely one of the three cases holds:

(Case I: $\alpha+\beta = 0$ and $u = 0$), (Case II: $\alpha+\beta = 0$ and $u \neq 0$) or (Case III: $\alpha+\beta \neq 0$ and $u = 0$) .

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4. What have we done so far?

The next lemma states that in order to complete the classification of Poisson algebra class $\mathcal{A}$. This lemma describes all commuting pairs of derivations of the polynomial Poisson algebra $K[t]$.

Lemma. *Let $K[t]$ be the polynomial Poisson algebra with trivial Poisson bracket and $\alpha,\beta \in$ $\text{PDer}_K = \text{Der}_K(K[t]) = K[t]\partial_t$ such that $\alpha = f\partial_t$ and $\beta = g\partial_t$, where $f,\ g \in K[t]\backslash\{0\},\ \partial_t = d/dt$ then*

$$\alpha\beta = \beta\alpha \quad \textit{if and only if} \quad g = \frac{1}{\lambda}f \quad \textit{for some} \quad \lambda \in K^{\times} := K\backslash\{0\}. \tag{3}$$

By using the previous lemma, we can assume that $\alpha = f\partial_t$, $\beta = \frac{1}{\lambda}f\partial_t$, $c \in K$, $u \in K[t]$, where $f \in K[t]$ and $\lambda \in K^{\times}$. Then we have the class of Poisson algebras $\mathcal{A} = K[t][x,y] = (K[t];\alpha = f\partial_t, \beta = \frac{1}{\lambda}f\partial_t, c, u)$ with Poisson bracket defined by the rule:

$$\{t,y\} = fy, \qquad \{t,x\} = \frac{1}{\lambda}fx \quad \text{and} \quad \{y,x\} = cyx + u. \tag{4}$$

The first case of the classification

The first case (Case I) of the Poisson algebra class $\mathcal{A}$ has two main subcases: Case I.1 and Case I.2. The results were indicated in these six subcases $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_9$ and $\mathcal{A}_{10}$. Also, we presented some of their Poisson spectrum in diagrams in the poster called '*Poisson Algebras I*', see the diagram 1.

Diagram 1: `The 'Poisson Algebras I' poster`

The first part of second case (Case II) of the classification is presented in this poster and the next diagram shows the second case structure.

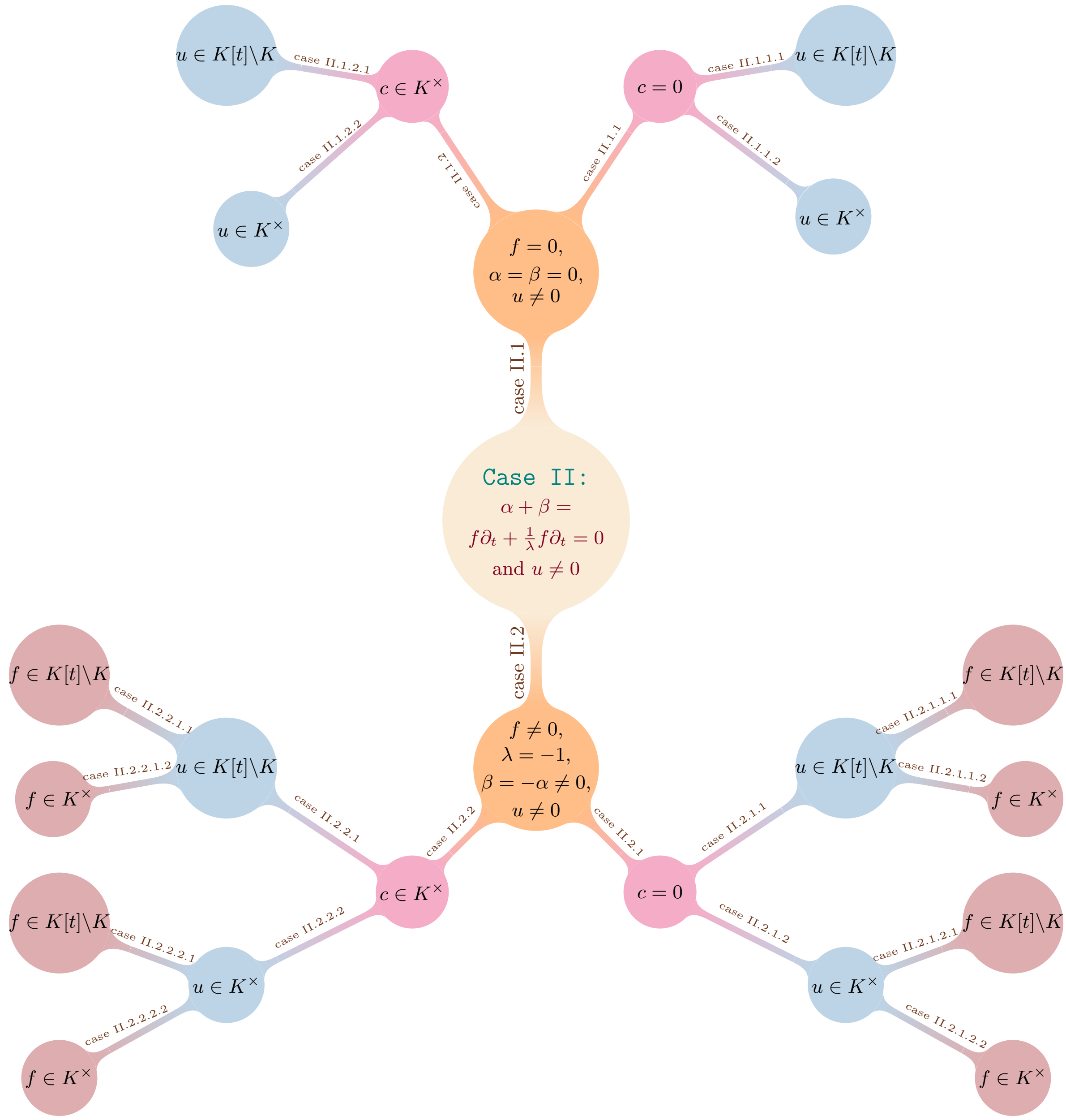

Diagram 2: `Structure of the second case of Poisson algebra class` $\mathcal{A}$

Case II: $\alpha+\beta=f\partial_t+\frac{1}{\lambda}f\partial_t=(1+\frac{1}{\lambda})f\partial_t=0$ and $u\neq 0$

Case II.1:

If $f=0$, i.e. $\alpha=\beta=0$ and $u\in K[t]\backslash\{0\}$ then we have the Poisson algebra $\mathcal{A}_{11}=(K[t];0,0,c,u)$

with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \text{ and } \{y,x\}=cyx+u. \tag{5}$$

There are two subcases: $c=0$ and $c\in K^{\times}$.

Case `II.1.1:` If $c=0$ then we have the Poisson algebra $\mathcal{A}_{12}=(K[t];0,0,0,u)$ with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \text{ and } \{y,x\}=u. \tag{6}$$

There are two subcases: $u\in K[t]\backslash K$ and $u\in K^{\times}$.

Case `II.1.1.1:`
If $u\in K[t]\backslash K$ and $R_u=\{\lambda_1,\dots,\lambda_s\}$ is the set of distinct roots of u then $\mathcal{A}_{13}=(K[t];0,0,0,u)$ is a Poisson algebra with Poisson bracket (6), we found PSpec($\mathcal{A}_{14}$), see diagram 3.

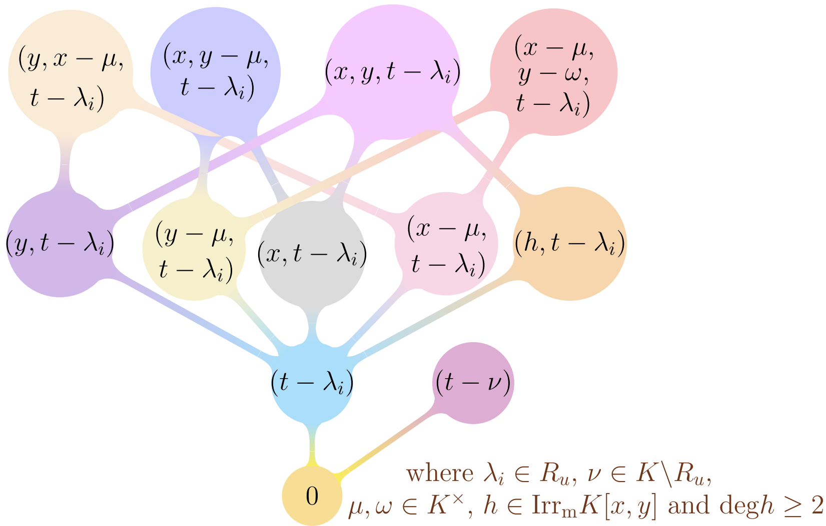

Diagram 3: `The containment information between Poisson prime ideals of` $\mathcal{A}_{13}$

Case `II.1.1.2:`
If $u=a\in K^{\times}$, i.e. $R_a=\emptyset$ then we have the Poisson algebra $\mathcal{A}_{14}=(K[t];0,0,0,a)$ with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \text{ and } \{y,x\}=a. \tag{7}$$

The PSpec($\mathcal{A}_{14}$) $=\{\mathfrak{p}\otimes K[x,y]\mid \mathfrak{p}\in \mathrm{Spec}(K[t])\}\subseteq$ PSpec($\mathcal{A}_{13}$).

Case `II.1.2:` If $c\in K^{\times}$ then we have the Poisson algebra $\mathcal{A}_{15}=(K[t];0,0,c,u)$ with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \text{ and } \{y,x\}=cyx+u:=\rho. \tag{8}$$

There are two subcases: $u\in K[t]\backslash K$ and $u\in K^{\times}$.

Case `II.1.2.1:`
If $u\in K[t]\backslash K$ and $R_u=\{\lambda_1,\dots,\lambda_s\}$ is the set of distinct roots of u then $\mathcal{A}_{16}=(K[t];0,0,c,u)$ is a Poisson algebra with Poisson bracket

$$\{t,y\}=0,\quad \{t,x\}=0 \text{ and } \{y,x\}=cyx+u. \tag{9}$$

It follows that the element $\rho=cyx+u$ is an irreducible polynomial in $\mathcal{A}_{16}$. Moreover, we found PSpec($\mathcal{A}_{16}$), see diagram 4

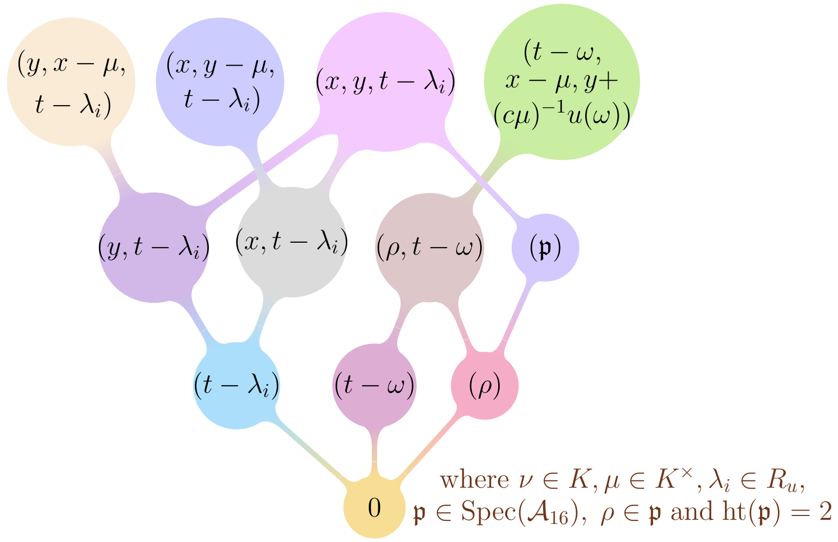

Diagram 4: The containment information between Poisson prime ideals of $\mathcal{A}_{16}$

Case II.1.2.2:
If $u = a \in K^\times$, i.e. $R_a = \emptyset$ then we have the Poisson algebra $\mathcal{A}_{17} = (K[t]; 0, 0, c, a)$ with Poisson bracket

$$\{t, y\} = 0, \quad \{t, x\} = 0 \ \text{ and } \ \{y, x\} = cyx + a. \tag{10}$$

It follows that $\mathcal{A}_{17} = K[t] \otimes K[x, y]$ is a tensor product of the trivial Poisson algebra $K[t]$ and the Poisson algebra $K[x, y]$ with $\{y, x\} = \rho$. The element $\rho = cyx + a$ is an irreducible polynomial in $\mathcal{A}_{17}$. Moreover, we found PSpec($\mathcal{A}_{17}$), see diagram 5.

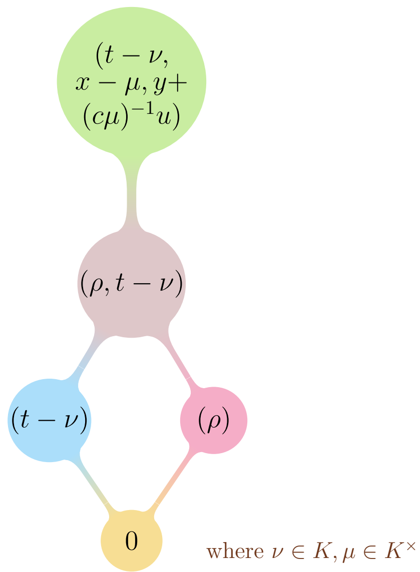

Diagram 5: The containment information between Poisson prime ideals of $\mathcal{A}_{17}$

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5. Conclusion / Future research

A classification of Poisson prime ideals of $\mathcal{A}$ was obtained in 10 cases out of 22. We will complete the classification of $\mathcal{A}$. Then we aim to classify some simple finite dimension modules over the class $\mathcal{A}$.

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References

[Bav6] V. V. Bavula, The Generalized Weyl Poisson algebras and their Poisson simplicity criterion. *Letters in Mathematical Physics*, **110** (2020), $105 - 119$.

[Bav7] V. V. Bavula, The PBW Theorem and simplicity criteria for the Poisson enveloping algebra and the algebra of Poisson differential operators, submitted, arxiv.2107.00321.

[GoWa] K. R. Goodearl and R. B. Warfield. *An introduction to noncommutative noetherian rings*. 2nd ed. New York: Cambridge University Press. (2004), pages $1 - 85$, $105 - 122$ and $166 - 186$.

[Oh3] S.-Q. Oh, Poisson polynomial rings. *Communications in Algebra*, **34** (2006), $1265 - 1277$.