Inverse source problems for diffusion/wave equations with time-fractional derivatives by Carleman estimates

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概要

時間非整数階微分方程式は、異常拡散現象及び光音響イメージングのモデルとして認識される。観測できるデータからソースの情報などを決めるという逆問題は、汚染源推定や非破壊検査など応用に役立ち、有意義な問題である。本講演は、解析学理論の立場から、時間非整数階拡散・波動方程式を考察し、Carleman 評価（一種の L^2 重み付き評価）に基づき、境界における解の時系列の観測データによるソース項の空間成分を決める逆問題の安定性評価を確立する。

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary (e.g., of $\mathcal{C}^2$-class). Denote $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$ with arbitrarily fixed $T > 0$. Throughout the article, we use the notations ∂_t for the time derivative and ∂_{x_i}, $i = 1, \ldots, n$ for the spatial derivative with respect to the i-th component. Moreover, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\Delta = \nabla \cdot \nabla = \sum_{i=1}^n \partial_{x_i}^2$.

In this article, we consider the following initial-boundary value problem for the diffusion/wave equation with time-fractional derivatives

$$\begin{cases} \partial_t^K u + q(x)\partial_t^\alpha u - \Delta u = F(x,t), & (x,t) \in Q, \\ u(x,t) = 0, & (x,t) \in \Sigma, \\ u(x,0) = 0, & x \in \Omega, \\ \partial_t u(x,0) = 0, & x \in \Omega, \text{ if } K = 2 \end{cases} \tag{1}$$

where $K = 1, 2$, $0 < \alpha < 1$, $q \in W^{1,\infty}(\Omega)$ and the source F is in some suitable space which will be fixed later. Here and henceforth, $W^{k,p}(X)$, $k \in \mathbb{N}$, $p \in \mathbb{N} \cup \{\infty\}$ denotes the Sobolev space on $X = (0,T)$ (or $X = \Omega$) and $H^k(X) = W^{k,2}(X)$, $k \in \mathbb{N}$, $L^p(X) = W^{0,p}(X)$, $p \in \mathbb{N} \cup \{\infty\}$. Moreover, ∂_t^α denotes the Caputo fractional derivative defined by

$$\partial_t^\alpha g(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s g(s) ds, \ t > 0.$$

Although there are several different uses of terminology, here in this article, we call it diffusion equation with time-fractional derivatives for the case $K = 1$ while we say wave equation with time-fractional derivatives for the case $K = 2$. Furthermore, we may call it time-fractional diffusion/wave equation for a more general case

$$\partial_t^\beta u - \Delta u = F(x,t)$$

with $0 < \beta \leq 2$.

1.1 Background

As we know, fractional calculus is a quite classical topic. Katugampola mentioned in his paper [9] that the history of fractional calculus should go back to seventeenth century, when in 1695 the derivative of order $1/2$ was described by Leibnitz in his letter to L'Hospital.

Within the last few decades, an abundance of anomalous processes was confirmed by experiments in several different application areas including physics, engineering and biology, see e.g., [5], [16], [18] and the references therein. As models of such anomalous processes, time-fractional diffusion/wave equations have drawn increasing attention in the recent years. Here we also refer to Schumer and Benson [15], Patch and Haltmeier [13] and Szabo [17], which are closely related to the governing equation (1) discussed in this article.

1.2 Known results

For the unique existence of solutions to initial-boundary value problems, we may refer to e.g., Li, Liu and Yamamoto [10] and Luchko [12], while we mention e.g., [3] for the asymptotic behavior. As for a comprehensive theoretical introduction, we can refer to the recent book [7].

Moreover, for the inverse source problems for time-fractional diffusion/wave equations, we suggest the survey paper [11] and the references therein. As for other inverse problems, we refer to the later chapters in the same book of [11].

In particular, one of the important approaches for solving inverse problems, especially to derive the uniqueness and the stability estimates, is the Bukhgeim-Klibanov method [1] based on so-called Carleman estimates which are weighted L^2-estimates for the solutions to partial differential equations with large parameter(s). As for this approach, we refer to the books [2, 6], the survey papers [8, 19] and the references therein.

2 Main results

In this article, we investigate the inverse problems of determining the spatial varying factor in the source term F from a single boundary measurement of the solution, both in the case $K = 1$ and in the case $K = 2$. More precisely, we consider the following inverse source problem

Inverse source problem
Let $F(x,t) = R(x,t)f(x)$, $x \in \Omega, t \in (0,T)$. For given q and R, let u be the solution to the initial-boundary value problem (1). Determine spatial component f of the source by the

measured Neumann data of the solution on some suitable sub-boundary Γ over the time span $(0,T)$:

$$\partial_\nu u|_{\Gamma\times(0,T)} \longrightarrow f|_\Omega.$$

Here and henceforth, ν denotes the outward normal vector of the boundary $\partial\Omega$ and $\partial_\nu u = \nabla u\cdot\nu$ is the normal derivative on $\partial\Omega$. In this article, we focus on the theoretical part of the above inverse source problem and give some stability estimates.

2.1 Inverse source problem for $K = 1$

We first introduce the following notations

$$\mathcal{R}_{1,M} := \{R; \|D_t^\gamma R\|_{L^\infty(Q)} \le M, 0<\gamma\le \frac{3}{2}\},$$
$$\mathcal{U}_1 := \{u \in H^1(0,T;H_0^1(\Omega)\cap H^2(\Omega))\cap H^2(0,T;L^2(\Omega))\}.$$

Here D_t^γ denotes the Riemann-Liouville fractional derivative defined by

$$D_t^\gamma g(t) := \frac{1}{\Gamma(m-\gamma)}\frac{d^m}{dt^m}\int_0^t (t-s)^{m-\gamma-1}g(s)ds,$$

where $m = \lceil\gamma\rceil$, i.e., m is the smallest integer which is larger than or equal to γ.

Now we are ready to state the first result.

Theorem 2.1 (Lipschitz stability for case K=1) Let $\alpha = \frac{m}{k} \le \frac{3}{4}$, $k\in\mathbb{N}, m = 1, ..., k-1$ and $0 < t_0 < T$, sub-boundary $\Gamma \subset \partial\Omega$ be arbitrarily fixed. Assume that $f \in L^2(\Omega)$ and $R \in \mathcal{R}_{1,M}$ satisfies

$$|R(\cdot,t_0)| \ge r_0 \quad on\ \overline{\Omega}$$

for some constant $r_0 > 0$. Furthermore, $u \in \mathcal{U}_1$ satisfies the initial-boundary value problem (1) with $K = 1$. Then there exists a constant $C > 0$, depending on M, k, T and the coefficients, such that

$$\|f\|_{L^2(\Omega)} \le C\left(\|u(\cdot,t_0)\|_{H^2(\Omega)} + \|D_t^{\frac{3}{2}}(\partial_\nu u)\|_{L^2(\Gamma\times(0,T))}\right).$$

Remark 2.1 The constant C tends to infinity as k goes to infinity. So we could not easily apply the density of rational numbers in real numbers and it is still open whether the theorem holds true for all real $\alpha\in(0,\frac{3}{4}]$ or not.

Moreover, due to some technical reason, we need additional measurement of the solution at $t = t_0$. Also one could find such additional data while one considers the inverse source problem for the diffusion equation (e.g., [19, Theorem 6.2]).

The proof is similar to [4, Theorem 5] where the authors proved a conditional stability of Hölder type.

2.2 Inverse source problem for $K = 2$

We introduce the following notations

$$\mathcal{R}_{2,M} := \{R; \|R\|_{W^{1,\infty}(0,T;L^\infty(\Omega))} \le M\},$$
$$\mathcal{U}_2 := \{u \in H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))\}.$$

Then we state the second main result.

Theorem 2.2 (Lipschitz stability for case K=2) Let $0 < \alpha \le 1$ and $T \ge T_0$. Assume that $f \in L^2(\Omega)$ and $R \in \mathcal{R}_{2,M}$ satisfies

$$|R(\cdot,0)| \ge r_0 \quad on\ \overline{\Omega}$$

for some $r_0 > 0$. Furthermore, $u \in \mathcal{U}_2$ satisfies the initial-boundary value problem (1) with $K = 2$. Then we can find some sub-boundary $\Gamma_0 \subset \partial\Omega$ such that there exists constant $C > 0$, depending on M, T and the coefficients, fulfilling

$$\|f\|_{L^2(\Omega)} \le C\|\partial_t(\partial_\nu u)\|_{L^2(\Gamma_0\times(0,T))}.$$

Remark 2.2 In the statement of Theorem 2.2, $T_0 > 0$ is some sufficient large number which depends on the size and shape of Ω. This comes from the finite propagation speed while one consider the wave equations.

Moreover, actually the sub-boundary $\Gamma_0 \subset \partial\Omega$ could be characterized by the following relation

$$\{x \in \partial\Omega;\ (x - x_0)\cdot\nu \ge 0\} \subset \Gamma_0 \tag{2}$$

where $x_0 \in \mathbb{R}^n$ could be an arbitrarily fixed point satisfying $x_0 \notin \overline{\Omega}$.

The proof follows the idea of [2, Theorem 5.1] where the inverse source problem for the wave equation is considered. In comparison, here we should use a key lemma to estimate the additional time-fractional derivative.

3 Technical arguments

In this section, we propose some key arguments which are connected to our main results. Firstly we give the key Carleman estimates for (1).

First of all, we introduce the weight function $e^{s\varphi}$ with

$$\varphi(x,t) = e^{\lambda\psi(x,t)}, \quad \psi(x,t) = d(x) - \beta(t-t_0)^2, \quad (x,t) \in \overline{Q} \tag{3}$$

where $s, \lambda > 0$ are large parameters and positive constants $\beta, t_0 > 0$ as well as function $d \in C^2(\overline{\Omega})$ will be fixed in the following contexts. This is one of the well-known choices of the weight functions in the Carleman estimates. One could find other choices in e.g., [8, 19] and the references therein which could yield suitable weighted estimates that help one solve different types of inverse problems.

3.1 Carleman estimate for the the case $K = 1$

In the case $K = 1$, since the highest order of time derivative is one, we could guess that the estimate is similar to the counterpart of diffusion equation (see e.g., [19]).

Hence in (3) we assume

$$\beta > 0, \quad 0 < t_0 < T \tag{4}$$

and $d \in C^2(\overline{\Omega})$ satisfies

$$d > 0 \text{ in } \Omega, \quad |\nabla d| > 0 \text{ on } \overline{\Omega}, \quad d = 0 \text{ on } \partial\Omega \setminus \Gamma. \tag{5}$$

where $\Gamma \subset \partial\Omega$ is a given relatively open sub-boundary. Then we have

Theorem 3.1 (Carleman estimate for case K=1) Let φ satisfy (3) with (4), (5), and $\alpha = \frac{m}{k} \in (0, \frac{3}{4}]$ be a rational number. Suppose $D_t^{\frac{j}{k}} F \in L^2(Q)$, $j = 0, ..., k-1$. Then for an arbitrarily fixed relatively open sub-boundary $\Gamma \subset \partial\Omega$, there exists a constant $C > 0$, independent of s, such that

$$\int_Q \sum_{j=0}^{2k-1} s^{-\frac{4}{k}j+3} |D_t^{\frac{j}{k}} u|^2 e^{2s\varphi} dxdt \le C \int_Q \sum_{j=0}^{k-1} |D_t^{\frac{j}{k}} F|^2 e^{2s\varphi} dxdt + Ce^{Cs} \int_{\Gamma \times (0,T)} |\partial_t(\partial_\nu u)|^2 dSdt$$

for all sufficiently large $s > 0$ and all u smooth enough satisfying (1) with $K = 1$ and $u(\cdot, 0) = u(\cdot, T) = 0$ in Ω.

The proof is a combination of the well-known Carleman estimate for diffusion equation and the technical argument in the subsection 3.3.

3.2 Carleman estimate for the the case $K = 2$

In the case $K = 2$, since the highest order of time derivative is two, it is natural to guess the estimate is similar to the counterpart of wave equation (see e.g., [2]).

Hence in (3) we assume

$$0 < \beta < 1, \quad t_0 = 0 \tag{6}$$

and $d \in C^2(\overline{\Omega})$ satisfies

$$d = |x - x_0|^2 \tag{7}$$

where $x_0 \in \mathbb{R}^n$ is an arbitrarily fixed point satisfying $x_0 \notin \overline{\Omega}$. Then we have

Theorem 3.2 (Carleman estimate for case K=2) Let φ satisfy (3) with (6), (7), and $0 < \alpha \le 1$. Suppose $F \in L^2(Q)$. Then we can find sub-boundary $\Gamma_0 \subset \partial\Omega$ fulfilling (2), such that there exists a constant $C > 0$ satisfying

$$\int_Q (s|\partial_t u|^2 + s|\nabla u|^2 + s^3|u|^2) e^{2s\varphi} dxdt \le C \int_Q |F|^2 e^{2s\varphi} dxdt + Ce^{Cs} \int_{\Gamma_0 \times (0,T)} |\partial_\nu u|^2 dSdt$$

for all sufficiently large $s > 0$ and all u smooth enough satisfying (1) with $K = 2$ and $u(\cdot, 0) = u(\cdot, T) = 0$ in Ω.

The proof is a combination of the well-known Carleman estimate for wave equation and the technical argument in the subsection 3.4.

3.3 Reduction of governing equation to a parabolic system in the case $K = 1$

Here we show the idea how to deal with the term $\partial_t^\alpha u$ of time-fractional derivative in the case $K = 1$. Recall the following facts on the fractional calculus:

(i) $\partial_t^m g = D_t^m g$, $m = 1, 2, \ldots$;

(ii) $D_t^\alpha(\partial_t g) = \partial_t(D_t^\alpha g) = D_t^{\alpha+1} g$, $\alpha > 0$ provided that $g(0) = 0$;

(iii) $D_t^\alpha(D_t^\beta g) = D_t^{\alpha+\beta} g$, $m - 1 < \alpha < m$, $m = 1, 2, \ldots$, $0 < \beta < 1$ provided that $g(0) = 0$;

(iv) $D_t^\alpha g = \partial_t^\alpha g$, $0 < \alpha < 1$ provided that $g(0) = 0$.

The above facts can be justified by formal calculations and we refer to Podlubny [14] for example.

In order to clarify the essence of the idea, here we consider a special case $\alpha = \frac{2}{3}$. For the general rational number α, we refer to [4, Appendix]. According to (iv), one can rewrite the first equation of (1) by

$$\partial_t u - \Delta u = F - q D_t^{\frac{2}{3}} u. \tag{8}$$

Then we apply Riemann-Liouville fractional derivatives $D_t^{\frac{1}{3}}$ and $D_t^{\frac{2}{3}}$ respectively to (8) and obtain

$$\partial_t(D_t^{\frac{1}{3}} u) - \Delta(D_t^{\frac{1}{3}} u) = D_t^{\frac{1}{3}} F - q D_t^1 u, \tag{9}$$

$$\partial_t(D_t^{\frac{2}{3}} u) - \Delta(D_t^{\frac{2}{3}} u) = D_t^{\frac{2}{3}} F - q D_t^{\frac{4}{3}} u, \tag{10}$$

Here we used the facts (ii) and (iii). Let $v = D_t^{\frac{1}{3}} u$ and $w = D_t^{\frac{2}{3}} u$. Hence again by (ii), we derive from (8)–(10) a coupled parabolic system, that is,

$$\begin{cases} \partial_t u - \Delta u = F - q w, \\ \partial_t v - \Delta v = D_t^{\frac{1}{3}} F - q \partial_t u, \\ \partial_t w - \Delta w = D_t^{\frac{2}{3}} F - q \partial_t v. \end{cases}$$

Thus, Theorem 3.1 can be derived by employing Carleman estimate for parabolic equations ([19, Theorem 3.2]) with respect to u, v and w respectively.

3.4 Estimation of time-fractional derivative in the case $K = 2$

We deal with the term $\partial_t^\alpha u$ of time-fractional derivative in the case $K = 2$. Although we could use similar idea as the above subsection to construct a coupled system of hyperbolic equations, we propose another way to overcome this term.

We should mention that the following argument works because the weight function for the case $K = 2$ could be an decreasing function with respect to t in the interval $[0, T]$ (see (3), (6) and (7)) and the fractional order α is sufficiently small compared to K, i.e., $\alpha \le K - 1$.

In fact, we have the following lemma.

Lemma 3.1 Let φ satisfy (3) with (6) and (7). Then there exists a constant $C = C(T) > 0$, independent of s, such that

$$\int_Q |\partial_t^\alpha u|^2 e^{2s\varphi} dxdt \le C \int_Q |\partial_t u|^2 e^{2s\varphi} dxdt.$$

Proof. This is a direct calculation according to the definition of time-fractional derivative. By noting that φ is decreasing in the interval $(0, T)$ and thus $e^{2s\varphi}$ is decreasing in the same interval, we obtain

$$\begin{aligned}
\int_Q |\partial_t^\alpha u|^2 e^{2s\varphi} dxdt &= \int_Q \left| \frac{e^{s\varphi(x,t)}}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \partial_\tau u(x,\tau) d\tau \right|^2 dxdt \\
&\le \int_Q \frac{1}{(\Gamma(1-\alpha))^2} \left| \int_0^t (t-\tau)^{-\alpha} |\partial_\tau u(x,\tau)| e^{s\varphi(x,t)} d\tau \right|^2 dxdt \\
&\le \int_Q \frac{1}{(\Gamma(1-\alpha))^2} \left| \int_0^t (t-\tau)^{-\alpha} |\partial_\tau u(x,\tau)| e^{s\varphi(x,\tau)} d\tau \right|^2 dxdt \\
&\le \int_Q \left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \right)^2 \left| \partial_t u(x,t) e^{s\varphi(x,t)} \right|^2 dxdt \\
&\le C \int_Q \left| \partial_t u(x,t) e^{s\varphi(x,t)} \right|^2 dxdt.
\end{aligned}$$

Here we used Young's convolution inequality in the last line.

Thus, Theorem 3.2 can be derived by employing Carleman estimate for hyperbolic equations ([2, Theorem 4.2]) and Lemma 3.1 above. Actually from Lemma 3.1, we will have the additional term $C\|e^{s\varphi}(\partial_t u)\|^2_{L^2(Q)}$ on the right-hand side of the estimate. Whereas, on the left-hand side (see [2, Theorem 4.2]), we have $s\|e^{s\varphi}(\partial_t u)\|^2_{L^2(Q)}$ and thus we can absorb the additional into the left-hand side by taking the parameter $s > 0$ sufficiently large.

4 Concluding remarks

In this article, we investigate the inverse problems of determining the spatial component of the source for the diffusion/wave equation (1) with time-fractional derivative. Our main results are the theoretical Lipschitz stability estimates for both the case $K = 1$ and the case $K = 2$. The proofs of the stability estimates are based on the BK method by using Carleman estimates. The key point is to establish suitable Carleman estimates for (1) where we meet the difficulty of dealing with the additional fractional derivatives. This can be overcome by applying some technical arguments that we introduced in Section 3.

It is clear from the proofs that the governing equation (1) could be generalized to the following one

$$\partial_t^K u + \sum_{j=1}^{N} q_j(x)\partial_t^{\alpha_j} u + \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + \sum_{j=1}^{n} b_j(x)\partial_{x_j} u + c(x)u = F(x,t)$$

where $A = (a_{ij})$ is a symmetric strictly elliptic operator, i.e., $a_{ij} = a_{ji}$, $1 \le i, j \le n$ satisfies

$$a_0|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le a_1|\xi|^2, \quad x \in \overline{\Omega},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$$

for some constants $a_0, a_1 > 0$ and the coefficients a_{ij}, q_j, b_j, c, $1 \le i, j \le n$ are smooth enough.

However, with the approach by Carleman estimates, we could not prove the stability for all $0 < \alpha < K$. The inverse source problem still remains open for

- $\alpha \notin \mathbb{Q}$ and $3/4 < \alpha < 1$ in the case $K = 1$ and
- $1 < \alpha < 2$ in the case $K = 2$.

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