BOUNDEDNESS OF COMPOSITION OPERATORS ON MORREY SPACES

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1. Introduction

In this talk, we investigate the boundedness of composition operators on Morrey spaces.

Let $L^0(\mathbb{R}^n)$ be the set of all measurable functions on $\mathbb{R}^n$. We provide a precise definition of the composition operators induced by a measurable map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 1.1 (Composition operator)**.** Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable map, and assume that φ is nonsingular, namely, $|\varphi^{-1}(E)| = 0$ for each measurable null set E. The *composition operator* C_φ is defined by

$$C_\varphi f \equiv f \circ \varphi.$$

Subsequently, we employ the result obtained by Singh [2] for the boundedness of the composition operator on the Lebesgue space $L^p(\mathbb{R}^n)$. The Lebesgue space $L^p(\mathbb{R}^n)$ is the set of all measurable functions f defined on $\mathbb{R}^n$ with the finite norm

$$\|f\|_{L^p} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Singh [2] provided the following necessary and sufficient condition for the map φ to generate a bounded mapping acting on Lebesgue spaces:

Theorem 1.2 ([2])**.** *Let $0 < p < \infty$. Then, the composition operator C_φ induced by $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is bounded on the Lebesgue space $L^p(\mathbb{R}^n)$ if and only if there exists a constant $K = K(\varphi)$ such that for all measurable sets E in $\mathbb{R}^n$,*

$$|\varphi^{-1}(E)| \le K|E|.$$

The study is to investigate a necessary and sufficient condition on the boundedness of the composition operator C_φ on Morrey spaces.

Let $\chi_A : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be an indicator function for a subset $A \subset \mathbb{R}^n$, which is defined as $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$, otherwise.

Now, we recall the definition of Morrey spaces on $\mathbb{R}^n$.

Definition 1.3 (Morrey space)**.** Let $0 < q \le p < \infty$. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is a quasi-Banach space defined by

$$\mathcal{M}^p_q(\mathbb{R}^n) \equiv \{ f \in L^0(\mathbb{R}^n) : \|f\|_{\mathcal{M}^p_q} < \infty \},$$

endowed with the quasi-norm

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{p} - \frac{1}{q}} \|f \chi_Q\|_{L^q},$$

where $\mathcal{Q}$ denotes the family of all cubes parallel to the coordinate axis in $\mathbb{R}^n$.

Definition 1.4. $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **Lipschitz** if there exists $L \geq 1$ such that for all $x, \tilde{x} \in \mathbb{R}^n$,

$$|\varphi(x) - \varphi(\tilde{x})| \leq L|x - \tilde{x}|.$$

Definition 1.5. $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **bi-Lipschitz** if there exists $L \geq 1$ such that for all $x, \tilde{x} \in \mathbb{R}^n$,

$$L^{-1}|x - \tilde{x}| \leq |\varphi(x) - \varphi(\tilde{x})| \leq L|x - \tilde{x}|.$$

We now state the main results of the present paper. The following theorem provides a sufficient condition on the boundedness of the composition operator C_φ on the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$.

Theorem 1.6 ([1])**.** *Let $0 < q \leq p < \infty$. Then, the composition operator C_φ induced by $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is bounded on the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$, if φ is a Lipschitz map that satisfies the volume estimate*

$$|\varphi^{-1}(E)| \leq K|E|,$$

for all measurable sets E in $\mathbb{R}^n$ and some constant K independent of E.

Conversely, as stated in the following theorem, if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, then the $\mathcal{M}^p_q(\mathbb{R}^n)$-boundedness of the composition operators C_φ and $C_{\varphi^{-1}}$ indicates that φ is bi-Lipschitz. Note that any bi-Lipschitz mapping satisfies the assumption of Theorem 1.6.

Theorem 1.7 ([1])**.** *Let $n \in \mathbb{N}$, and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism in the sense that φ and its inverse φ^{-1} are differentiable. Suppose $0 < q < p < \infty$, or $q = p$ and $n = 1$. If the composition operators C_φ and $C_{\varphi^{-1}}$ induced by maps φ and φ^{-1}, respectively, are bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$, then φ is bi-Lipschitz.*

References

[1] N. Hatano, M. Ikeda, I. Ishikawa and Y. Sawano, Boundedness of composition operators on Morrey spaces and weak Morrey spaces, arXiv:2008.12464. [2]

[2] R. K. Singh, Composition operators induced by rational functions, Proc. Amer. Math. Soc. 59 (1976), no. 2, 329–333. [1]