Geometric Relations between the Bott-Virasoro Group and the Space of Equicentroaffine Curves

N.A.

Boshu Ding

1 Background

1.1 The Bott-Virasoro Group

For M a smooth manifold with metric $(-,-)$ and $c : [a,b] \to M$ a smooth curve, the energy of c is given by

$$\mathcal{E}(c) := \frac{1}{2}\int_a^b (\dot{c}(t), \dot{c}(t))dt, \tag{1}$$

A variation of c is defined to be a smooth map $\tilde{c} : [a,b] \times (-\varepsilon, \varepsilon) \to M$ with $\tilde{c}(-,0) = c$ and a curve c is called a geodesic if for every variation $\tilde{c}$ with endpoints fixed, we have

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{E}(\tilde{c}(-,s)) = 0. \tag{2}$$

When M is a Lie group G with inner product $\langle -,- \rangle$ on its Lie algebra $\mathfrak{g}$, there is a left-invariant metric $(-,-)$ on G by

$$(X_g, Y_g) := \langle l_{g^{-1}*} X_g, l_{g^{-1}*} Y_g \rangle.$$

where $X_g, Y_g \in T_g G$. Letting $A : \mathfrak{g} \to \mathfrak{g}^*$ be the natural isomorphism induced from the inner product, we define a related curve of c in the dual Lie algebra $m : [a,b] \to \mathfrak{g}^*$ given by

$$m(t) := A(l_{c(t)^{-1}*} \dot{c}(t)).$$

In this case, the condition (2) is equivalent to the so-called Euler equation:

$$\dot{m}(t) = -\mathrm{ad}^*_{A^{-1}(m(t))} m(t). \tag{3}$$

Now, we restrict G to the Bott-Virasoro group Vir. It is defined to be a set $\mathrm{Diff}(S^1) \times_B \mathbb{R}$, with multiplication

$$(\varphi, a)(\psi, b) = (\varphi \circ \psi, a + b + B(\varphi, \psi)),$$

where $B : \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) \to \mathbb{R}$ is the Bott cocycle given by

$$B(\varphi, \psi) := \frac{1}{2}\int_{S^1} \log(\varphi' \circ \psi) d \log \psi'.$$

The dual space of its Lie algebra $\mathfrak{vir}^*$ is identified with the vector space $\mathfrak{X}(S^1)^* \times \mathbb{R}$, where $\mathfrak{X}(S^1)^* = \{udx \otimes dx | u \in C^\infty(S^1)\}$ consists of the quadratic forms on S^1. Assume that $(udx \otimes dx, a) : [a,b] \to \mathfrak{vir}^*$ is the curved defined in (3) for the Bott-Virasoro group. Then, it is known that the Euler equation amounts to

$$\dot{u} = -au''' - 3uu'. \tag{4}$$

which is nothing but the KdV equation. (See Vizman [13] for details).

1.2 The Space of Equicentroaffine Curves

On the other hand, the KdV equation can be derived in the following way. Let $\mathcal{M}$ denote the connected component of the space of equicentroaffine curves containing the unit circle c. An element in $\mathcal{M}$ is a plain curve $\gamma : S^1 \to \mathbb{R}^2$ with $\det(\gamma, \gamma') = 1$. Given $\gamma \in \mathcal{M}$, its equicentroaffine curvature $\kappa : S^1 \to \mathbb{R}$ is determined by

$$\gamma'' + \kappa\gamma = 0.$$

A tangent vector $X \in T_\gamma\mathcal{M}$ on $\mathcal{M}$ is identified with a vector field along γ, expressed in the form

$$X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma',$$

where $\lambda : S^1 \to \mathbb{R}$ is some function on S^1. In [5], Fujioka and Kurose studied two presymplectic forms $\hat{\omega}_0$ and $\hat{\omega}_1$ on $\mathcal{M}$, where $\hat{\omega}_0$ was first created by Pinkall [11]. For $X = -\frac{1}{2}\lambda'\gamma + \lambda\gamma', Y = -\frac{1}{2}\mu'\gamma + \mu\gamma' \in T_\gamma\mathcal{M}$, $\hat{\omega}_0$ is defined by

$$\hat{\omega}_0(X,Y) := \int_{S^1} \lambda\mu' dx,$$

which we will call the Pinkall 2-form, and $\hat{\omega}_1$ by

$$\hat{\omega}_1(X,Y) := \int_{S^1} \lambda(\frac{1}{2}\mu''' + 2\kappa\mu' + \kappa'\mu) dx,$$

the Fujioka-Kurose 2-form. Fujioka and Kurose also studied a Hamiltonian function H on $\mathcal{M}$, whose Hamiltonian vector field X_H with respect to $\hat{\omega}_1$ is given by

$$X_H(\gamma) = \frac{1}{2}\kappa'\gamma - \kappa\gamma',$$

where κ is the equicentroaffine curvature of γ. In terms of this vector field, let $\tilde{\gamma} : \mathbb{R} \to \mathcal{M}$ be an integral curve of X_H, and $\tilde{\kappa}(-,t) : S^1 \to \mathbb{R}$ the corresponding equicentroaffine curvature of $\tilde{\gamma}(t)$, which we will call the equicentroaffine curvature flow. Pinkall proved that such equicentroaffine curvature flow $\tilde{\kappa}$ must satisfy the KdV equation:

$$\dot{\tilde{\kappa}} = -\frac{1}{2}\tilde{\kappa}''' - 3\tilde{\kappa}'\tilde{\kappa}. \tag{5}$$

It is very interesting that the KdV equation appears in such two completely different objects, the Bott-Virasoro group and the space of equicentroaffine curves. We think that there may exist some mathematical connection behind them, and this becomes the motivation for this study.

2 Main Results

Note that as the space of equicentroaffine curves, the KdV equation (4) derived from the Bott-Virasoro group is also associated to a 2-form - the canonical symplectic form $d\Theta$ on the cotangent bundle $G \ltimes \mathfrak{g}^*$. With respect to $d\Theta$, there is a Hamiltonian function E on $G \ltimes \mathfrak{g}^*$ such that the $\mathfrak{g}^*$-part of an integral curve (φ, ξ) of its Hamiltonian vector field X_E satisfies the Euler equation, i.e.,

$$\dot{\xi}(t) = -\mathrm{ad}^*_{A^{-1}(\xi(t))}\xi(t).$$

This means that by setting G to be the Bott-Virasoro group Vir, we can obtain the KdV equation in this way. In a word, we have related the KdV equation (4) derived from the Bott-Virasoro group to a symplectic from $d\Theta$ and a Hamiltonian vector field X_E with Hamiltonian function E. On the other hand, we have already seen that KdV equation (5) is related to the presymplectic form $\hat{\omega}_1$ and the vector field X_H with Hamiltonian function H

$$dH = -i_{X_H}\hat{\omega}_1.$$

Thus, it is reasonable to ask if there are some relations between these two forms, Hamiltonian vector fields, and Hamiltonian functions. As a result, it does, and we have proved that both the Fujioka-Kurose 2-form $\hat{\omega}_1$, and the Pinkall 2-form $\hat{\omega}_0$ are the pullbacks of the canonical symplectic form $d\Theta$ under certain maps. Note that for any $\gamma \in \mathcal{M}$, there exists $\psi \in \mathrm{Diff}(S^1)$ such that

$$\gamma = c \cdot \psi = \frac{c \circ \psi}{\sqrt{\psi'}},$$

where the action on the right is called Pinkall's right action. Now, we shall state the theorems:

Theorem 1. *Given $\gamma \in \mathcal{M}$, let $\psi \in \mathrm{Diff}(S^1)$ such that $\gamma = c \cdot \psi$, and define $\sigma_0 : \mathcal{M} \to \mathrm{Vir} \ltimes \mathfrak{vir}^*$ by*

$$\sigma_0(\gamma) := ((\psi^{-1}, 0), (-\frac{1}{2}(\psi^{-1})'^2 dx \otimes dx, 0)).$$

Then, we have

$$\sigma_0^* d\Theta = \hat{\omega}_0,$$

where $d\Theta$ is the canonical symplectic form on $\mathrm{Vir} \ltimes \mathfrak{vir}^*$ and $\hat{\omega}_0$ the Pinkall 2-form on $\mathcal{M}$. □

Theorem 2. *Given $\gamma \in \mathcal{M}$, let $\psi \in \mathrm{Diff}(S^1)$ such that $\gamma = c \cdot \psi$ and define $\sigma_1 : \mathcal{M} \to \mathrm{Vir} \ltimes \mathfrak{vir}^*$ by*

$$\sigma_1(\gamma) := ((\psi, 0), (-\kappa dx \otimes dx, -\frac{1}{2})),$$

where κ is the equicentroaffine curvature of γ. Then, we have

$$\sigma_1^* d\Theta = \hat{\omega}_1,$$

where $\hat{\omega}_1$ is the Fujioka-Kurose 2-form on $\mathcal{M}$. □

Remark 3. *In Theorem 1 and Theorem 2, it turns out that the $\mathfrak{vir}^*$-part of σ_0 and σ_1 are momentum maps with respect to certain actions of* Vir *on $\mathcal{M}$. This is one of the mains results in Fujioka, Kurose and Moriyoshi* [6].

By making use of σ_1, we can also obtain the following relation between X_H and X_E

$$\sigma_{1*}(X_H(\gamma)) = X_E(\sigma_1(\gamma)) + X \tag{6}$$

where $X \in T_{\sigma_1(\gamma)}(\mathrm{Vir} \ltimes \mathfrak{vir}^*)$ is a tangent vector such that

$$d\Theta(X_{\sigma_1(\gamma)}, \sigma_{1*}Z) = 0 \tag{7}$$

for all $Z \in T_\gamma\mathcal{M}$. The existence of X in (6) may keep us from getting an explicit relation between X_H and X_E. But we have proved

Theorem 4. *Let $\gamma \in \mathcal{M}$ be an element in $\mathcal{M}$. Suppose that $X \in T_{\sigma_1(\gamma)}(G \ltimes \mathfrak{g}^*)$ is the tangent vector in (6) which satisfies (7). Then, X has the form*

$$X = (K, (0dx \otimes dx, 0))$$

where K represents a tangent vector over G.

Theorem 4 means that the $\mathfrak{vir}^*$-part of X vanishes, and this will kill the obstruction produced by X. Then, we can finally provide an explanation, or say another proof, of why the equicentroaffine curvature flow $\tilde{\kappa}$ must satisfies the KdV equation:

Corollary 5. *Let $\tilde{\gamma}$ be an integral curve of X_H. Then, we have*

$$\dot{\tilde{\kappa}} = -\frac{1}{2}\tilde{\kappa}''' - 3\tilde{\kappa}'\tilde{\kappa},$$

where $\tilde{\kappa}$ is the equicentroaffine curvature of $\tilde{\gamma}$.

References

[1] Alekseevsky D., Grabowski J., Marmo G., Michor P. W., *Poisson structures on the cotangent bundle of a Lie group or a principle bundle and their reductions*, J. Math. Phys. 35, 4909-27 (1994).

[2] ArnoldV. I., Givental A. V., *Symplectic geometry, In V. I. Arnold and S. P. Novikov (eds.), Dynamical Systems IV*, (1985), pp. 1–136, Berlin, Springer.

[3] Burgers, J. M., *A mathematical model illustrating the theory of turbulence*, Adv. Appl. Mech., 1 (1948), pp. 171-199.

[4] Chou K. S., Qu C. Z., *The KdV equation and motion of plane curves*, J. Phys. Soc. Japan 70 (2001) 1912–1916.

[5] Fujioka A., Kurose T., *Multi-Hamiltonian structures on spaces of closed equicentroaffine plane curves associated to higher KdV flows*, SIGMA 10 (2014), 048, 11pp.

[6] Fujioka A., Kurose T., Moriyoshi H. In Preparation.

[7] Hasimoto H., *A soliton on a vortex filament*, J. Fluid Mech. 51, (1972), 477–485.

[8] Lamb G. L., *Solitons and the motion of helical curves*, Phys. Rev. Lett. 37 (1976), 235–237.

[9] McDuff D., Salamon D. *Introduction to Symplectic Topology*, Oxford University Press. (2017).

[10] Ovsienko V. Y., Khesin B. A., *Korteweg–de Vrieß superequations as an Euler equation*, Funct. Anal. Appl. 21 (1987), 329–331.

[11] Pinkall U., *Hamiltonian flows on the space of star-shaped curves*, Results Math. 27, 328–332 (1995)

[12] Roger C., *Extensions centrales d'algebres et de groupes de Lie de dimension infinie, algebres de Virasoro et generalisations*, Rep. Math. Phys., 35 (1995), 225–266.

[13] Vizman C., *Geodesic equations on diffeomorphism groups*, SIGMA 4, 030 (2008), 22 pp