Hyperbolic Regularization of Nonlinear Black-Scholes Equations

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Abstract

We study a nonlinear generalization of the Black-Scholes equation for pricing European options in the presence of transaction costs. The volatility coefficient of this model is a nonlinear function of the underlying asset price and the Gamma of the option. We show the existence of mild solutions, under minimal regularity conditions, applying the method of hyperbolic regularization. The backward parabolic equation is replaced by a dissipative hyperbolic equation with a small parameter and studied through the inverse function theorem and energy inequality.

1 Introduction

The celebrated Black-Scholes model (1973) in [1] provides a useful option pricing formula for financial markets without transaction costs. An example is the European stock option which certifies the holder's right, but not obligation, to buy (sell) a specific amount of an underlying stock for a fixed price E at a fixed future time T called maturity or expiry. A key assumption of the Black-Scholes analysis is that buying and selling can take place continuously. In real markets with proportional transaction costs, this process will accumulate infinite expenses and render the whole analysis inadequate.

The first model that allows trades only at discrete times was presented by Leland (1985) [7], [3]. He replaced the constant volatility coefficient σ in the classical Black-Scholes model by the nonlinear volatility function

$$\sigma(S\partial_S^2 V) = \sigma_0 \left(1 + \mu(S\partial_S^2 V)^{1/3}\right)^{1/2}, \tag{1.1}$$

where S is the underlying asset price, while V is the option price. The two parameters $\sigma_0 > 0$ and $\mu > 0$ are empirically determined. Leland's model leads to the backward parabolic problem

$$\begin{cases} \frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \quad \tau < T \\ V(S,T) = \max(S-E, 0), \end{cases} \tag{1.2}$$

where r and q are the interest rate and dividend yield, respectively.

The existence, uniqueness and approximation of solutions is the main goal of our work. The fully nonlinear equation (1.2) is usually transformed into a quasilinear parabolic equation, called the Gamma equation [4], for

$$H(S,\tau) = S\partial_S^2 V(S,\tau).$$

There is a well-developed theory for the existence of smooth solutions based on Schauder type estimates [6]. Various numerical methods have also been proposed most of which utilize finite-difference schemes [2], [5], [10].

We follow a different approach to the problems of existence, uniqueness and numerical approximation. Introducing a second-order term with

$$\varepsilon^2 \partial_\tau^2 H(S,\tau), \quad \varepsilon > 0,$$

we changes equation (1.2) into the one-dimensional wave equation with a nonlinear damping. This is known as hyperbolic relaxation or regularization [8]. The new equation will be equivalent, in the class of C^2-functions, to an integral equation of Volterra type over a finite domain in two variables. We can readily establish the existence of C^2-solutions. Moreover, we can use energy estimates for dissipative hyperbolic equations to show that approximations convergence as $\varepsilon \to 0$ to a solution in $C([0,\infty), H^2(\mathbb{R}))$.

2 Hyperbolic Regularization

Let us begin with the standard transformation of equation (1.2) into a quasilinear parabolic equation. The new variables are

$$t = T - \tau, \quad x = \ln\frac{S}{E}.$$

Recall that T is the time of maturity and E is the exercise price. Then

$$S\frac{\partial}{\partial S} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial \tau} = -\frac{\partial}{\partial t}.$$

If we define $H = S\frac{\partial^2 V}{\partial S^2}$ and apply $S\frac{\partial^2}{\partial S^2}$ to the Black-Scholes equation, we obtain the so-called Gamma equation

$$\begin{cases} \frac{\partial H}{\partial t} - \frac{\partial^2 \beta(H)}{\partial x^2} - \frac{\partial \beta(H)}{\partial x} - (r-q)\frac{\partial H}{\partial x} + qH = 0, \quad t > 0 \\ H(x,0) = H_0(x). \end{cases} \tag{2.1}$$

The nonlinearity $\beta(H) = \frac{1}{2}\sigma^2(H)H$, with σ defined in (1.1), is smooth at $H > -\mu^{-3}$. In addition, there exists a smooth inverse α, such that

$$H = \alpha(\beta(H)). \tag{2.2}$$

We will construct a mild solutions to the above problem which is defined as follows: $H \in C([0,\infty), H^2(\mathbb{R}))$ and satisfies the integral equation

$$H(x,t) - \int_0^t \left(\frac{\partial^2 \beta(H)}{\partial x^2} - \frac{\partial \beta(H)}{\partial x} - (r-q)\frac{\partial H}{\partial x} + qH \right) ds = H(x,0).$$

Let us now introduce the hyperbolic relaxation of (2.1). We have

$$\varepsilon^2 \frac{\partial^2 \beta(H_\varepsilon)}{\partial t^2} + \frac{\partial H_\varepsilon}{\partial t} - \frac{\partial^2 \beta(H_\varepsilon)}{\partial x^2} - \frac{\partial \beta(H_\varepsilon)}{\partial x} - (r-q)\frac{\partial H_\varepsilon}{\partial x} + qH_\varepsilon = 0,$$

where $\varepsilon > 0$ is a small parameter. It is convenient to rename the unknown $u_\varepsilon = \beta(H_\varepsilon)$ and express the other terms through the inverse function of β, that is use $H_\varepsilon = \alpha(u_\varepsilon)$ from (2.2). Then u_ε will be a solution to

$$\begin{cases} \varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial t^2} + \frac{\partial \alpha(u_\varepsilon)}{\partial t} - \frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{\partial u_\varepsilon}{\partial x} - (r-q)\frac{\partial \alpha(u_\varepsilon)}{\partial x} + q\alpha(u_\varepsilon) = 0, \quad t > 0 \\ u_\varepsilon(x,0) = a(x) \\ \frac{\partial_t u(x,0)}{\partial t} = b(x) \end{cases}$$

with suitable $a(x)$ and $b(x)$ determined from $H_0(x)$. This wave equation with nonlinear damping is studied in the next section.

3 Existence of Regularized Solutions

We combine a classical theorem about homeomorphisms of Banach spaces from [9] with the simple observation that α is sub-linear. In fact, we have

$$|\alpha'(u)| \le c_1(1+|u|)^{-1/4}, \quad u > -\frac{\mu^{-3}}{2}. \tag{3.1}$$

Let $T > 0$ and restate the initial value problem as the equation

$$F(u_\varepsilon(x,t)) = f(x,t), \tag{3.2}$$

where the right hand side is

$$\begin{aligned} f(x,t) &= \frac{1}{2}\left(a(x+t/\varepsilon) + a(x-t/\varepsilon)\right) + \frac{\varepsilon}{2}\int_{x-t/\varepsilon}^{x+t/\varepsilon} b(y)\,dy \\ &\quad + \frac{\varepsilon}{2}\int_{x-t/\varepsilon}^{x+t/\varepsilon} \alpha(a(y))\,dy \end{aligned}$$

and the map $F : C_b(\mathbb{R}\times[0,T]) \to C_b(\mathbb{R}\times[0,T])$ is

$$
\begin{aligned}
&F(u(x,t))\\
&= u(x,t) + q\frac{\varepsilon}{2}\int_0^t \left(\int_{x-(t-s)\varepsilon}^{x+(t-s)/\varepsilon} \alpha(u(y,s))\,dy\right) ds\\
&-\frac{\varepsilon}{2}\int_0^t (u(x+(t-s)\varepsilon, s) - u(x-(t-s)/\varepsilon, s))\,ds\\
&-\frac{(r-q)\varepsilon}{2}\int_0^t (\alpha(u(x+(t-s)/\varepsilon, s)) - \alpha(u(x-(t-s)/\varepsilon, s)))\,ds\\
&+\frac{1}{2}\int_0^t (\alpha(u(x+(t-s)/\varepsilon, s)) + \alpha(u(x-(t-s)/\varepsilon, s)))\,ds.
\end{aligned}
$$

The norm in the space of continuous functions $C_b(\mathbb{R}\times[0,T])$ is defined by

$$\|f\| = \sup_{(x,t)\in\mathbb{R}\times[0,T]} |f(x,t)|.$$

In the case of C^2-functions f, the integral equation and initial value problem are equivalent. Our main result in this section is the following.

Lemma 3.1. *Let $a, b \in C_0(\mathbb{R})$. Equation (3.2) has a unique solution $u_\varepsilon \in C_b(\mathbb{R}\times[0,T])$ for every $T > 0$. Moreover, $u_\varepsilon \in C_b^2(\mathbb{R}\times[0,T])$ if the initial data $a \in C_0^2(\mathbb{R})$ and $b \in C_0^1(\mathbb{R})$.*

Outline of proof: According to [9], $F : C_b(\mathbb{R}\times[0,T]) \longrightarrow C_b(\mathbb{R}\times[0,T])$ is a homeomorphism if

(a) $[F'(u)]^{-1}$ exists for every u;

(b) $\|[F'(u)]^{-1}\| \le c(1+\|u\|)$ for some constant c.

The existence of $F'(u)$ is evident. To verify condition (a), we show that

$$F'(u)v = 0 \;\Rightarrow v = 0.$$

This is a simple consequence of Gronwall's lemma. The existence of v, such that $F'(u)v = g$ for every $g \in C_b(\mathbb{R}\times[0,T])$ also follows from Gronwall's estimate: we define

$$v = \sum_{n=0}^{\infty} (-1)^n (F'(u) - I)^n g$$

and show that $v \in C_b(\mathbb{R}\times[0,T])$ using that $F'(u) - I$ is a Volterra type integral operator. This completes the proof of (a).

Estimate (b) is a consequence of the sub-linear growth of α in (3.1).

Finally, the higher regularity of $u_\varepsilon = F^{-1}(f)$ for suitable f is a well-known result. We can now study the eventual convergence of subsequence u_ε, which will give a solution to the original problem (2.1).

4 Convergence of Approximate Solutions

This part is based on the following energy estimates.

Lemma 4.1. *Let v be a solution to*

$$\varepsilon^2 \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + c(x,t)\frac{\partial v}{\partial t} = f, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

If $c(t,x) \geq c_0$ for $c_0 > 0$, then

$$\begin{aligned} \varepsilon^2 \|\partial_t v(\cdot,t)\|_2^2 + \|\partial_x v(\cdot,t)\|_2^2 &\leq \varepsilon^2 \|\partial_t v(\cdot,0)\|_2^2 + \|\partial_x v(\cdot,0)\|_2^2 \\ &\quad + \frac{1}{c_0}\int_0^t \|f(\cdot,s)\|_2^2 \, ds. \end{aligned}$$

Here and $\|\cdot\|_2$ means the L^2-norm $\|\cdot\|_{L^2}$. Moreover,

$$\begin{aligned} c_0 \int_0^t \|\partial_s v(\cdot,s)\|_2^2 \, ds &\leq \varepsilon^2 \|\partial_t v(\cdot,0)\|_2^2 + \|\partial_x v(\cdot,0)\|_2^2 \\ &\quad + \frac{1}{c_0}\int_0^t \|f(\cdot,s)\|_2^2 \, ds. \end{aligned}$$

Another useful result is the Sobolev embedding estimate below.

Lemma 4.2. *If $\|\cdot\|_\infty$ and $\|\cdot\|_2$ mean $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{L^2}$, respectively, then*

$$\begin{aligned} \|v(\cdot,t)\|_\infty^2 &\leq 2\|v(\cdot,t)\|_2 \|\partial_x v(\cdot,t)\|_2, \\ \|\partial_t v(\cdot,t)\|_\infty^2 &\leq 2\|\partial_t v(\cdot,t)\|_2 \|\partial_t \partial_x v(\cdot,t)\|_2. \end{aligned}$$

A combination of Lemma 4.1 for u_ε and $\partial_t u_\varepsilon$, together with Lemma 4.2, establishes the solvability of nonlinear Black-Scholes equation.

Theorem 4.3. *Let $(a,b) \in H_0^2(\mathbb{R}) \times H_0^1(\mathbb{R})$. There exists a sequence $\varepsilon \to 0$, such that $u_\varepsilon \to u$ in $C([0,T], H^2(\mathbb{R}))$ for every $T > 0$. Moreover, $H = \alpha(u)$ is a mild solution to the Black-Scholes equation (2.1).*

Outline of proof: The solution constructed in Lemma 3.1 satisfies

$$\varepsilon^2 \frac{\partial u_\varepsilon}{\partial t} + \alpha(u_\varepsilon) - \int_0^t \left(\frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{\partial u_\varepsilon}{\partial x} - (r-q)\frac{\partial \alpha(u_\varepsilon)}{\partial x} + q\alpha(u_\varepsilon) \right) ds$$
$$= \varepsilon^2 \frac{\partial u_\varepsilon(x,0)}{\partial t} + \alpha(u_\varepsilon(x,0))$$

We can show that $\varepsilon^2 \frac{\partial u_\varepsilon(x,t)}{\partial t} \to 0$ as $\varepsilon \to 0$, while the remaining terms have strong limits in $C([0,\infty), L^2(\mathbb{R}))$. This completes the proof.

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