Limit Theorems for a Space-Homogeneous Data-Diffusion with Nonlinear Reactions

早稲田大学大学院 基幹理工学研究科　数学応用数理専攻
渡辺　樹 (Itsuki WATANABE)

Abstract

We discuss the difference between two mathematical models of data-diffusion; the deterministic and stochastic models. These are given by a nonlinear reaction-diffusion equation and a multi-dimensional jump Markov processes. In this paper, we classify the limiting behavior of difference between two models, and show that the law of large numbers and central limit theorem holds for these models.

1 Introduction.

Data spread on large network attracts many researchers, and there are many theoretical analysis using SIS, SIR or SIRS models mainly analyzed by the system of ordinary differential equations [9]. They were modeled by the transitions of the node states; (1) having no information (denoted by S), (2) spreading information (denoted by I), and (3) having information but not spreading (denoted by R). There are two problems for these models. First, although these models can be applied to data-diffusion in complex networks, the node has only limited couple of states. Second, these analysis assume that, even though the network size increases, the transition rate is always inversely proportional to its size because they use the mean-field approach. We see many stochastic events such as blog flamings, which could not be modeled by deterministic model. Our study presents a solution for these problems by building a space-jump Markov processes for large network, and we find two regimes; one leads the deterministic and one leads stochastic limits.

The first study of a Markov model converging to its differential equation was introduced by Kurtz [7, 8]. He compared a sequence of Markov pure jump processes with the deterministic model which is given by ordinary differential equations. In 1980, Arnold and Theodosopulu [1] constructed a space-time jump Markov process $X^N(t, r)$, which is the stochastic model of chemical reaction, as follows; a simple random walk on discrete torus $\mathbb{Z}/N\mathbb{Z}$ with diffusion rate proportional to N^2, where N is the number of sites the random walk can visit, and dividing the number of particles by the particle size parameter $l > 0$ is proportional to the initial number of particles placed in each sites. Moreover, they compared it with the deterministic model $u(t, r)$ given by a heat equation $\partial_t u(t, r) = \Delta u(t, r)$, and proved that the law of large numbers holds for these models.

This work was advanced by Blount [2, 3] with the periodic boundary condition. He added the reaction term as birth and death processes (see Either and Kurtz[5]) to the above models, and compared the two models by some norms, in particular the norm on the Hilbert distribution space H_α. For a linear reaction case, it was proved that law of large numbers (LLN) and central limit theorem (CLT) hold in the particular regimes of index α and parameter l in Blount [2]. The nonlinear reaction case was also proved in Blount [3].

From Hutson et.al [6], we consider the following reaction-diffusion equation which is a space-homogeneous,

$$\frac{d}{dt}u(t,r) = \int_0^1 u(t,r')dr' - u(t,r) + R(u(t,r)) \tag{1.1}$$

where $R(x)$ denote a polynomial with degree $m \geq 1$. Let $X^N(t,r)$ be the stochastic model of data-diffusion which is composed by the simple random walk. It can move everywhere on discrete torus $\mathbb{Z}/N\mathbb{Z}$ with the rate proportional to N^{-1}. For a linear case $(m = 1)$, we could prove that

$$\sup_{[0,T]} \|X^N(t) - u(t)\|_\infty \to 0$$

in probability if $\log(N/l) \to 1$ as $N \to \infty$. Also, the rescaled difference

$$\sqrt{l}\big(X^N(t) - u^N(t)\big)$$

converges to the Ornstein-Uhlenbeck process in distribution on skorokhod space $D([0,T];L^2)$ if $\sqrt{l}/N \to 0$ as $N \to \infty$. These results implies that, depending on the growth speed of transition rate with respect to the network size, we may differentiate the use of deterministic reaction-diffusion equation and stochastic differential equation. In this paper, we extend this result to a nonlinear case $m \geq 2$. For $T > 0$, under the assumptions of initial value and parameter l, we show that

$$\sup_{[0,T]} l^{1/4}\|X^N(t) - u(t)\|_2 \to 0$$

in probability if $l^{1/4}/N \to 0$ as $N \to \infty$, where $\|\cdot\|_2$ is the $L^2[0,1]$-norm, and the same result as linear case holds, that is,

$$\sqrt{l}\big(X^N(t) - u(t)\big)$$

converges to the Ornstein-Uhlenbeck process in distribution on $D([0,T];L^2)$ if $l^{1/4}/N \to 0$ as $N \to \infty$.

The rest of the paper is organized as follows. In Section 2, we introduce the deterministic model of data-diffusion given by the (1.1) with a periodic boundary condition. In Section 3, we construct the stochastic model of data-diffusion on a discrete torus $\mathbb{Z}/N\mathbb{Z}$. Finally, in Section 4, we introduce the main results.

2 The deterministic model.

For $x \in \mathbb{R}$, let $b(x)$ and $d(x)$ be polynomials with nonnegative coefficients, degree less than m, and $d(0) = 0$. The functions $b(x)$ and $d(x)$ represent birth and death rate of data-particles, respectively.

Assume $m \geq 2$. We consider the following reaction-diffusion equation

$$\begin{cases} \dfrac{d}{dt}u(t,r) = \displaystyle\int_0^1 u(t,r')dr' - u(t,r) + R\left(u(t,r)\right), \\ u(t,0) = u(t,1), \\ 0 \leq u(0,r) = u_0(r) \in C\big([0,1]\big), \end{cases} \tag{2.1}$$

where $R(x) = b(x) - d(x) = \sum_{j=0}^{m} c_j x^j$ with $c_1 < 0$ and $c_j \geq 0$ if $j \neq 1$. For sufficiently large ρ, suppose that $R(x) < 0$ for $x > \rho$ and $R'(\rho) < 0$. Then, by the basic semigroup theorys (cf. Pazy [10]), there exists a mild solution $u(t,r) \in C\big([0,T]; C\big([0,1]\big)\big)$ such that $0 < u(t,r) \leq \rho$ for $t \in [0,T]$.

3 The stochastic model.

Given $1 \leq N \in \mathbb{N}$ and $l > 0$ as the network size and data-particle size. Let H^N be the space of real valued step-functions on $[0,1]$ which are constant on $[kN^{-1}, (k+1)N^{-1})$, $0 \leq k \leq N-1$, and define an operator A_N by $A_N f(r) = \frac{1}{N}\sum_{i=0}^{N-1} f(r + iN^{-1}) - f(r)$ for $f \in H^N$. Note that this operator is obtained by discretizing the torus in (2.1).

Let $n_k(t)$ be the number of data-particles in k-th site on $\mathbb{Z}/N\mathbb{Z}$ at time t and let $n(t) = (n_0, \cdots, n_{N-1}(t))$ be the state of a multi-dimensional Markov chain at time t, and transition rates given by

$$\begin{cases} n \to n_{(i:+1,k:-1)} = (\cdots, n_i + 1, \cdots, n_k - 1, \cdots) & \text{at rate } n_k N^{-1}, \\ n \to n_{(k:+1)} = (\cdots, n_k + 1, \cdots) & \text{at rate } l b(n_k l^{-1}), \\ n \to n_{(k:-1)} = (\cdots, n_k - 1, \cdots) & \text{at rate } l d(n_k l^{-1}), \end{cases}$$

for $i \in \{0, 1, \cdots, k-1, k+1, \cdots, N-1\}$. Here $n_N = n_0$ and $n_{-1} = n_{N-1}$. We set $n(t)$ to be a cádlág process defined on some probability space.

Let $\mathscr{F}_t^N$ be the completion of the σ-algebra $\sigma(n(s); s \leq t)$ and let $\Lambda n(t) = n(t) - n(t-)$ be the jump at time t. Suppose that τ is an $\mathscr{F}_t^N$-stopping time such that

$$\sup_{[0,T]} \sup_{k} \mathbb{1}_{\{\tau > 0\}} n_k(t \wedge \tau) \leq M(T, N, l) < \infty,$$

for all $T > 0$, then we obtain as in [2];

Lemma 3.1.

$$n_k(t \wedge \tau) - n_k(0) - \int_0^{t \wedge \tau} \frac{1}{N} \sum_{i=0}^{N-1} (n_i(s) - n_k(s)) ds - \int_0^{t \wedge \tau} l R(n_k(s) l^{-1}) ds$$

is a mean 0 martingale. This formula is the amount of data change accumulated in $[0, t \wedge \tau]$ at k-th site.

Let $X^N(t)$ be the H^N-valued Markov process defined as

$$X^N(t,r) = n_k(t) l^{-1} \quad \text{for} \quad r \in [kN^{-1}, (k+1)N^{-1}).$$

Then from lemma 3.1, we can write

$$X^N(t) = X^N(0) + \int_0^t A_N X^N(s) ds + \int_0^t R\left(X^N(s)\right) ds + Z^N(t),$$

where $Z^N(t \wedge \tau)$ is an H^N-valued martingale.

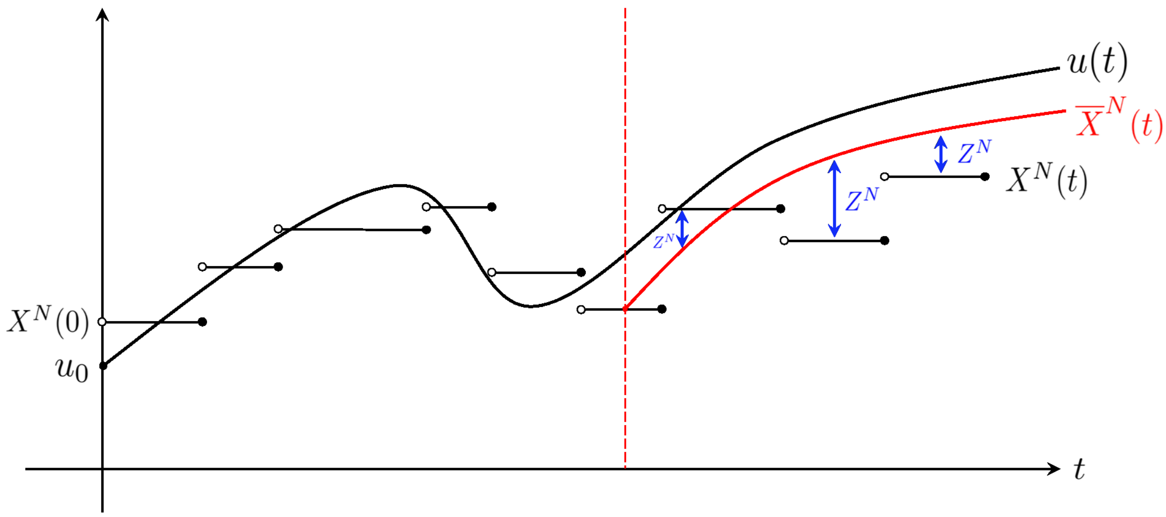

Figure1 Difference of two models and simple interpretation of notations; $\overline{X}^N$ represents the function stating from a value of X^N at some time t, and moves deterministically afterwards.

4 Main results.

In this section, we compare the deterministic model $u(t)$ and stochastic model $X^N(t)$. We give some notation. Let $V^N(t) = \sqrt{l}(X^N(t) - u(t))$, and let $M^N(t) = \sqrt{l}Z^N(t)$ (see Figure1). Define the operator A as $Af(r) = \int_0^1 f(r')dr' - f(r)$ for $f \in L^2[0,1]$. From Curtain and Pritcchard [4], we can construct the evolution operator $U(t,s)$ generated by time dependent operator $A + R'(u(t))$. The evolution operator is one of method for solving the differential equations which has time dependent coefficients. The followings are main results in this talk;

Theorem 4.1. Assume
(a) $l^{1/4}/N \to 0$,
(b) $l^{1/4}\|X^N(0) - u(0)\|_2 \to 0$ in probability as $N \to \infty$.
Then for $T > 0$,

$$\sup_{[0,T]} l^{1/4}\|X^N(t) - u(t)\|_2 \to 0 \quad \text{in probability.}$$

Theorem 4.2. In addition to the assumption of Theorem 4.1, assume $V^N(0) \to V_0$ in distribution on L^2. Then there exists a unique Gaussian process $M(t)$ such that
(i) $V^N(t)$ converges to the Ornstein-Uhlenbeck process $V(t)$ in distribution on $D_{L^2}[0,T]$ where

$$V(t) = U(t,0)V_0 + \int_0^t U(t,s)dM(s)$$

is the mild solution of the stochastic differential equation

$$dV(t) = \Big(A + R'(u(t))\Big)V(t)dt + dM(t), \quad V(0) = V_0.$$

(ii) $(V^N(0), M^N) \to (V_0, M)$ in distribution on $L^2[0,1] \times D_{L^2}[0,\infty)$.

Through the above theorems, data-diffusion can be distinguished the cases when they are modeled by either the reaction-diffusion equation or stochastic differential equation.

Acknowledgements

The author would like to thank the organizers of the 16th Mathematics Conference for Young Researchers for the opportunity to give a presentation.

Refferences

[1] Arnold, L. and Theodosopulu, M. (1980). Deterministic limit of the stochastic model of chemical reactions with diffusion, Adv. in Appl. Probab. **12** 367-379

[2] Blount, D. (1991). Comparison of Stochastic and Deterministic Models of a Linear Chemical Reaction with Diffusion, Ann. Probab. **19** 1440-1462

[3] Blount, D. (1993). Limits theorems for a sequence of nonlinear reaction-diffusion systems, Stochastic Processes and their Applications, **45** 193-207

[4] Curtain, R.F. and Pritcchard, A.J. (1978). Infinite Dimensional Linear Systems Theory, Springer-Verlag, Berlin, Heidelberg, New York

[5] Either, S. and Kurtz, T. (1986). Markov processes, Characterization and Convergence, Wiley, New York

[6] Hutson, V., Martinez, S., Mischaikow, K. and Vickers, G.T. (2003). The evolution of dispersal, J. Math. Biol. **47** 483-517

[7] Kurtz, T. (1970). Solutions of ordinary differential equations as limits of pure jump Markov processes, J. Appl. Probab. **7** 49-58

[8] Kurtz, T. (1971). Limit Theorems for Sequences of Jump Markov Approximating Ordinary Differential Processes, J. Appl. Probab. **8** 344-356

[9] Pastor-Satorras, R., Castellano, C., Mieghem, P.V. and Vespignani, A. (2015). Epidemic processes in complex networks, Rev. Modern Phys. **87** 925-979

[10] Pazy, A. (1983). Semigroups of Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, Berlin