T1 theorem for Calderón-Zygumnd type operators bounded from L^p to L^q

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Abstract

The T1 theorem is the necessary and sufficiently condition for L^2 boundedness of Calderón-Zygmund operators, given by G. David and J.L. Journé in 1984 [3]. In this talk, we shall consider the group of operators $CZO_\alpha(\delta)$ which includes Calderón-Zygmund operators, and we give a necessary and sufficiently condition for $T \in CZO_\alpha(\delta)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

1 Introduction

In this paper, we denote $A \lesssim B$ as there exists a constant $C > 0$ and it satisfies $A \leq CB$. For a linear integral operator T, we say $T \in CZO(\delta)$ if T satisfies the following conditions:

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \qquad (x \notin \operatorname{supp}(f)) \tag{1.1}$$

for all $f \in L^2(\mathbb{R}^n)$ with compact support, where $K(x,y)$ is a standard kernel, that is, a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ and satisfying the following conditions for all $x, x_0, y, y_0 \in \mathbb{R}^n$:

$$(1)\ |K(x,y)| \lesssim \frac{1}{|x-y|^n}, \tag{1.2a}$$

$$(2)\ |K(x,y) - K(x,y_0)| \lesssim \frac{|y-y_0|^\delta}{|x-y|^{n+\delta}} \qquad \left(|y-y_0| < \frac{|x-y|}{2}\right), \tag{1.2b}$$

$$(3)\ |K(x,y) - K(x_0,y)| \lesssim \frac{|x-x_0|^\delta}{|x-y|^{n+\delta}} \qquad \left(|x-x_0| < \frac{|x-y|}{2}\right). \tag{1.2c}$$

Here $\delta \in (0,1]$ is a positive constant. Researching boundedness of $T \in CZO(\delta)$ plays an important role in harmonic analysis, moreover in viewing of differential equation theory. For example, Riesz transform $\mathcal{R}_j$ and Hilbert transform H satisfy $\mathcal{R}_j, H \in CZO(1)$.

More generally, for a linear integral operator T, we say $T \in CZO_\alpha(\delta)$ if it satisfies (1.1), (1.2a) and the following conditions:

$$(1)\ |K(x,y) - K(x,y_0)| \lesssim \frac{|y-y_0|^{\delta+\alpha}}{|x-y|^{n+\delta}} \qquad \left(|y-y_0| < \frac{|x-y|}{2}\right), \tag{1.2b'}$$

$$(2)\ |K(x,y) - K(x_0,y)| \lesssim \frac{|x-x_0|^{\delta}}{|x-y|^{n+\delta}} \qquad \left(|x-x_0| < \frac{|x-y|}{2}\right). \tag{1.2c'}$$

Here $\delta \in (0,1]$ is a positive constant, again.

Next, we define the function space $\mathrm{BMO}(\mathbb{R}^n)$ and Campanato spaces $\mathcal{L}_\alpha(\mathbb{R}^n)$, which are classical and important spaces in function analysis. Here and after, we let $\mathcal{P}$ be the set of all polynomial functions.

Definition 1.1. Let B be a ball on $\mathbb{R}^n$.

(1) $||f||_{\mathrm{BMO}} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|dx$,

(2) $\mathrm{BMO}(\mathbb{R}^n) = \{\, f \in \mathcal{S}'(\mathbb{R}^n) \mid ||f||_{\mathrm{BMO}} < \infty \,\}/\mathcal{P}$,

where $f_B = \frac{1}{|B|}\int_B f(x)dx$.

Definition 1.2. Let B be a ball on $\mathbb{R}^n$ and $0 \le \alpha < n$.

(1) $||f||_{\mathcal{L}_\alpha} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^{1+\frac{\alpha}{n}}}\int_B |f(x) - f_B|dx$,

(2) $\mathcal{L}_\alpha(\mathbb{R}^n) = \{\, f \in \mathcal{S}'(\mathbb{R}^n) \mid ||f||_{\mathcal{L}_\alpha} < \infty \,\}/\mathcal{P}$,

where $f_B = \frac{1}{|B|}\int_B f(x)dx$.

A $C^{2[n/2]+2}$-function ϕ is said to be a bump function if it satisfies following conditions:

$$(1)\, \mathrm{supp}\, \phi \subset B(0,10), \qquad (2) |\partial_x^\alpha \phi(x)| \le 1 \quad (|\alpha| \le 2[n/2]+2).$$

We define the operator $\tau_{x_0} f(x) = f(x - x_0)$ for $x, x_0 \in \mathbb{R}^n$.

Definition 1.3. Let $0 \le \alpha < n$. For a linear operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, T is said to be α-weakly bounded ($T \in WB_\alpha$) if there exists a constant C such that:

$$|\langle T\tau_{x_0}(f_R), \tau_{y_0}(g_R)\rangle| \le CR^{-n-\alpha},$$

for all bump functions f, g and $x_0, y_0 \in \mathbb{R}^n$.

The following is the classical T1 theorem.

Theorem 1.4 ([3]). *Let $T \in \mathrm{CZO}(\delta)$. Then, $T1, T^*1 \in BMO$ and $T \in WB_0$ if and only if T is bounded on $L^2(\mathbb{R}^n)$.*

Where $T1$ is the image T for the function 1.

2 Main theorem and techenical lemmas

The following is our main theorem.

Theorem 2.1. *Let $1 < p, q < \infty$, $1/p = 1/q + \alpha/n$ and $T \in CZO_\alpha(\delta)$. Then, we have*

(1) *If $T \in WB_\alpha$ and $T1, T^*1 \in \mathcal{L}_\alpha(\mathbb{R}^n)$, then T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

(2) *Conversely, if T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $T \in WB_\alpha$ and T is also bounded from $L^\infty(\mathbb{R}^n)$ to $\mathcal{L}_\alpha(\mathbb{R}^n)$ (i.e. $T1, T^*1 \in \mathcal{L}_\alpha(\mathbb{R}^n)$).*

To prove this theorem, we claim the following technical lemmas.

Lemma 2.2. *Let $0 \le \alpha < n/2$ and ϕ be a positive function on $[0, \infty)$ such that $\phi(|x|) \in L^1(\mathbb{R}^n)$. We assume that $|\Phi(x)| \le \phi(|x|)$. Furthermore, let $\Psi \in L^1(\mathbb{R}^n)$ be a function satisfies $\int \Psi = 0$ and the following conditions:*

$$(1)\ |\Psi(x)| \lesssim (1+|x|)^{-n-\delta}, \quad (2)\ \sup_{\xi \in \mathbb{R}^n} \int_0^\infty |t\hat{\Psi}(\xi)|^2 \frac{dt}{t} < \infty,$$

for some constant $\delta > 0$. Then, we have

$$\left(\int_{\mathbb{R}^{n+1}_+} |\Phi_t(f)(x)|^2 |\Psi_t(g)|^2 dx \frac{dt}{t} \right)^{1/2} \lesssim ||f||_p ||g||_{\mathcal{L}_\alpha},$$

where $1/p = 1/2 + \alpha/n$.

We say a function $a \in L^\infty(\mathbb{R}^n)$ is (∞, α)-*atom* associated with ball $B \in \mathbb{R}^n$ if it satisfies $\operatorname{supp} a \subset B$, $\int a = 0$ and $||a||_\infty \le |B|^{-1+\alpha/n}$. Let $H^1_\alpha(\mathbb{R}^n)$ be the set of all functions $f \in \mathcal{S}(\mathbb{R}^n)$ such that there exists a sequence of numbers $\{\lambda_j\}_j \subset \mathbb{C}$ and sequence of (∞, α)-*atoms* $\{a_j\}_j$ associated with ball $\{B_j\}_j \subset \mathbb{R}^n$ and it satisfies $f = \sum_j \lambda_j a_j$.

Lemma 2.3. *Let $0 \leq \alpha < n$. Then the dual of $H^1_\alpha(\mathbb{R}^n)$ is $\mathcal{L}_\alpha(\mathbb{R}^n)$. More precisely, we have the following assertions:*

(1) *If $b \in \mathcal{L}_\alpha(\mathbb{R}^n)$, then the mapping*

$$\ell = \ell_b : f \in L^\infty_{\text{comp}}(\mathbb{R}^n) \mapsto \int f(x)b(x)dx \in \mathbb{C}$$

can be extended to a bounded linear functional on H^1_α. Also, we have

$$||\ell|| \lesssim ||b||_{\mathcal{L}_\alpha}.$$

(2) *Conversely, if ℓ is a continuous liner functional on H^1_α, then there exists $b \in \mathcal{L}_\alpha(\mathbb{R}^n)$ such that $\ell(f) = \int f(x)b(x)dx$ for all L^∞_{comp} and that*

$$||b||_{\mathcal{L}_\alpha} \lesssim ||\ell||.$$

Lemma 2.4. *Let T be a linear operator satisfies (1.1) and the kernel $K(x,y)$ satisfies*

$$|K(x,y) - K(x,x_0)| \lesssim \frac{|y-x_0|^\alpha}{|x-y|^n},$$

for any $x, y, x_0 \in \mathbb{R}^n$ with $|y-x_0| < |x-y|/2$. Then, if T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $1 < p, q < \infty$ with $1/p = 1/q + \alpha/n$, T is also bounded from $L^{p'}(\mathbb{R}^n)$ to $L^{q'}(\mathbb{R}^n)$ for any $1 < p', q' < \infty$ with $1/p' = 1/q' + \alpha/n$.

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