On a two-phase overdetermined problem of Serrin type

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1 What are overdetermined problems?

Before stating our results on two-phase overdetermined problems, we would like to first give a brief introduction concerning elliptic overdetermined problems in general and how they are related to symmetry.

Let us start by introducing maybe the most famous example of overdetermined problem in the realm of elliptic PDE's. Let Ω be a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with sufficiently smooth, say $\mathcal{C}^2$, boundary $\partial\Omega$ and consider the following boundary value problem

$$-\Delta u = 1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

It is well known that such a problem admits a unique solution of class $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ independently of the shape of the domain Ω. In particular, if Ω is a ball, then, we obtain that the unique solution of (1.1) must be radial and hence, its outward normal derivative $\partial_n u$ verifies

$$\partial_n u = d \quad \text{on } \partial\Omega, \quad \text{for some } d \in \mathbb{R}. \tag{1.2}$$

It is natural to ask, whether the ball is the only domain Ω for which the solution of (1.1) satisfies (1.2) as well. We say that (1.2) is a so-called *overdetermined condition* (or simply *overdetermination*) and a problem such as (1.1)-(1.2) is called an *overdetermined problem.* It is clear that, unlike usual boundary value problems, overdetermined problems might not be solvable in general. One of the basic questions in the study of overdetermined problems lies in asking for which particular domains a specific overdetermined problem (i.e. any pair of the form "boundary value problem"+"overdetermination") admits a solution. We will say that such domains are the *solutions* of the overdetermined problem in question.

As far as the overdetermined problem (1.1)-(1.2) is concerned, Serrin proved the following result.

Theorem 1.1 (Radial symmetry, [Se]). *The overdetermined problem* (1.1)-(1.2) *admits a solution $u \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ if and only if Ω is a ball of radius $-Nd$.*

The original proof of Serrin employs the use of the so-called *method of moving planes*, a refinement of *Aleksandrov's reflection principle* (see [Al]).

We remark Theorem 1.1 makes no assumptions on the eventual holes that the domain Ω might have. In particular, Theorem 1.1 says something more than the mere radial symmetry of Ω: if the same constant d is chosen for all connected components of $\partial\Omega$ (which is precisely the point of the overdetermined condition (1.2)), then, no annular domains, not even radially symmetrical ones, are allowed as a solution.

This raises the question, whether more complicated solutions can be obtained if we impose an overdetermined condition on only part of the boundary, say, one of its connected components. The answer is affirmative and is given by the so-called *Bernoulli overdetermined problem.* In what follows we will briefly describe the problem setting for the Bernoulli overdetermined problem.

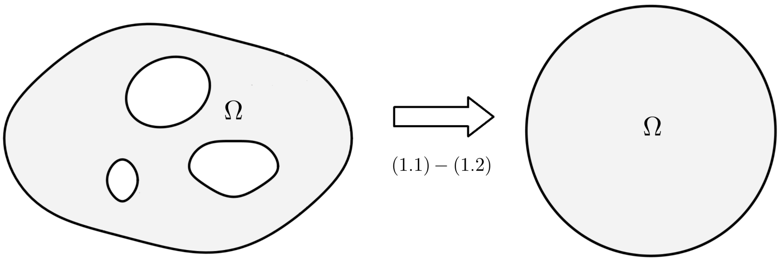

Figure 1: Serrin's symmetry result

Let K and Ω be sufficiently smooth, simply connected, bounded domains of $\mathbb{R}^N$ $(N \geq 2)$ that satisfy $\overline{K} \subset \Omega$. For a given constant $d \in \mathbb{R}$, the classical Bernoulli overdetermined problem reads:

$$-\Delta u = 0 \text{ in } \Omega \setminus \overline{K}, \quad u = 1 \text{ on } \partial K, \quad u = 0 \text{ on } \partial\Omega, \quad \partial_n u = d \text{ on } \partial\Omega. \tag{1.3}$$

Such an overdetermined problem arises naturally as optimality condition when searching for a domain Ω of a given volume, that minimizes the relative capacity $\mathrm{cap}_\Omega(K)$. The existence of solutions to the overdetermined problem (1.3) is therefore studied by means of the (somewhat) easier task of finding minimizers of the capacity functional $\Omega \mapsto \mathrm{cap}_\Omega(K)$ for a given K, under volume constraint. In particular, the shape of the solution Ω depends on both the shape of the hole K and the constant d.

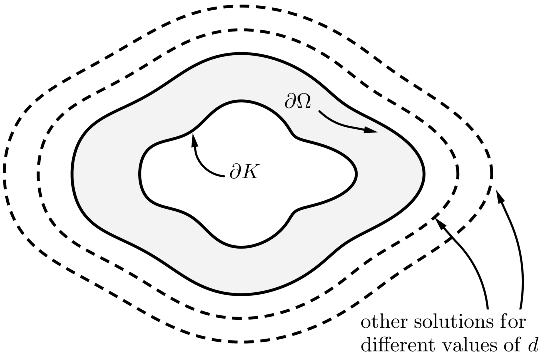

Figure 2: A family of non symmetric solutions of the Bernoulli overdetermined problem

We refer to [HP, pp. 249–262], where some geometrical properties of the solution Ω are examined, such as convexity, the sign of the mean curvature of the boundary and star-shapedness.

2 A two-phase overdetermined problem

As shown in the end of the previous section, we can say that the Bernoulli overdetermined problem admits non symmetric solutions because, despite the overdetermination on part of its boundary, it has a degree of freedom left (namely the shape of the hole K). There is a way of ensuring a degree of freedom (and thus non symmetric solutions) even though the overdetermination is assumed on the whole boundary, namely replacing the Laplacian by a two-phase operator in divergence form.

Let D and Ω be two sufficiently smooth bounded domains of $\mathbb{R}^N$ $(N \geq 2)$ that satisfy $\overline{D} \subset \Omega$.

Moreover, let $\sigma = \sigma(x)$ be the piecewise constant function given by

$$\sigma(x) = \begin{cases} \sigma_c & \text{in } D, \\ 1 & \text{in } \Omega \setminus D, \end{cases}$$

where σ_c is a positive constant such that $\sigma_c \neq 1$.

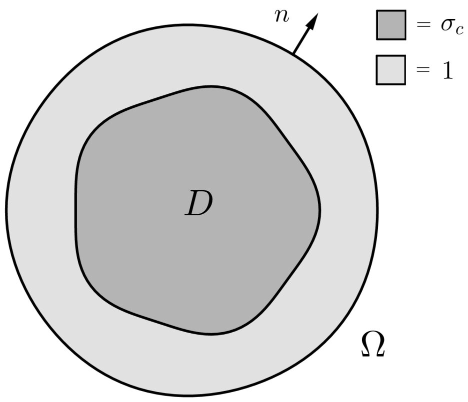

Figure 3: Problem setting for a two-phase overdetermined problem

We consider the following two-phase counterpart of (1.1):

$$-\operatorname{div}(\sigma \nabla u) = 1 \quad \text{in } \Omega, \tag{2.4}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.5}$$

where the solution u is defined weakly, as the unique function $u \in H_0^1(\Omega)$ that satisfies

$$\int_\Omega \sigma \nabla u \cdot \nabla \psi = \int_\Omega \psi \quad \text{for all } \psi \in H_0^1(\Omega).$$

Moreover, we consider the following overdetermined condition on the boundary:

$$\partial_n u = d \quad \text{on } \partial\Omega, \quad \text{for some } d \in \mathbb{R}. \tag{2.6}$$

We notice that, if (D, Ω) is a solution of the overdetermined problem (2.4)-(2.5)-(2.6), then integration by parts shows that the constant d is linked to the geometry of Ω by the formula

$$d = d(\Omega) = -\frac{|\Omega|}{|\partial\Omega|}. \tag{2.7}$$

If D and Ω are concentric balls, then the unique solution of (2.4)-(2.5) is radial and so it satisfies (2.6) for some d. In what follows we will refer to any pair of concentric balls as a *trivial solution* of the overdetermined problem (2.4)-(2.5)-(2.6).

Since the overdetermined problem (2.4)-(2.5)-(2.6) has one degree of freedom, it is natural to expect it to admit nontrivial (i.e. non symmetric) solutions just like the Bernoulli overdetermined problem does. The search for nontrivial solutions can be divided in the so-called inner and outer problems.

Problem 1 (Inner problem)**.** *For a given domain* Ω *and a real number* $0 < V_0 < |\Omega|$*, find a domain* $D \subset \overline{D} \subset \Omega$ *with volume* $|D| = V_0$*, such that the pair* (D, Ω) *is a solution of the overdetermined problem* (2.4)-(2.5)-(2.6)*.*

Problem 2 (Outer problem). *For a given domain D and a real number $V_0 > |D|$, find a domain $\Omega \supset \overline{D}$ with volume $|\Omega| = V_0$, such that the pair (D, Ω) is a solution of the overdetermined problem* (2.4)-(2.5)-(2.6).

In what follows we will state the main results concerning the existence of nontrivial solutions for the overdetermined problem (2.4)-(2.5)-(2.6), framed for both inner and outer problem. For a given $R \in (0,1)$ let $D_0 \subset \Omega_0$ denote two concentric balls with radii R and 1 respectively whose common center can be thought to be the origin. Notice that any trivial solution coincides with (D_0, Ω_0) modulo a suitable translation and dilation.

The following theorems show the existence of nontrivial solutions as suitable perturbations of the trivial one (D_0, Ω_0).

Theorem I (Local unique solvability for the inner problem, [CMS]). *For every domain Ω of class $\mathcal{C}^{2+\alpha}$ sufficiently close to Ω_0, there exists a domain D of class $\mathcal{C}^{2+\alpha}$ sufficiently close to D_0 such that the outer problem of the overdetermined problem* (2.4)-(2.5)-(2.6) *admits a solution for the pair (D, Ω).*

Theorem II (Local unique solvability for the outer problem, [CY]). *Let us define*

$$\begin{aligned} s(k) &= \frac{k(N+k-1) - (N+k-2)(k-1)R^{2-N-2k}}{k(N+k-1) + k(k-1)R^{2-N-2k}} \quad \textit{for } k \in \mathbb{N}, \\ \Sigma &= \{ s \in (0, \infty) \ : \ s = s(k) \ \textit{ for some } k \in \mathbb{N} \}. \end{aligned} \tag{2.8}$$

If $\sigma_c \notin \Sigma$, then for every domain D of class $\mathcal{C}^{2+\alpha}$ sufficiently close to D_0, there exists a domain Ω of class $\mathcal{C}^{2+\alpha}$ sufficiently close to Ω_0 such that the outer problem of the overdetermined problem (2.4)-(2.5)-(2.6) *admits a solution for the pair (D, Ω).*

To our knowledge, [CMS] and later [CY] are the only works concerning nontrivial solutions for the overdetermined problem like (2.4)-(2.5)-(2.6), although the study of symmetry results analogous to Theorem 1.1 in a two-phase setting (for both elliptic and parabolic boundary value problems) is a bit older (see for example [Sa1] and [Sa2]).

3 Two main tools

The proof of Theorems I and II is based on a perturbation argument in which the degree of freedom coming from the two-phase setting is fully exploited. In what follows, we will introduce the two main tools used: the implicit function theorem and shape derivatives.

3.1 Implicit function theorem

We first recall the definition of Fréchet derivative for a general functional between Banach spaces. Let V and W be Banach spaces (whose norms will be indistinctly denoted by $\|\cdot\|$) and let $U \subset V$ be an open subset of V. A function $F : U \to W$ is said to be Fréchet differentiable at $x_0 \in U$ if there exists a bounded linear operator $A : V \to W$ such that

$$\lim_{x \to 0} \frac{\left\| F(x_0 + x) - F(x_0) - Ax \right\|}{\|x\|} = 0.$$

It is easy to show that, when such an operator A exists, then it is also unique. This bounded linear operator will be denoted by $F'(x_0)$ and referred to as the Fréchet derivative of F at x_0 (the term "differential" is also commonly used in this case).

In the particular case when the domain of F is a product of two Banach spaces $V = \mathcal{X} \times \mathcal{Y}$, then we can define the *partial Fréchet derivatives* of F at $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$ as follows:

$$\partial_x F(x_0, y_0) : \mathcal{X} \to W, \quad x \mapsto F'(x_0, y_0)(x, 0),$$
$$\partial_y F(x_0, y_0) : \mathcal{Y} \to W, \quad y \mapsto F'(x_0, y_0)(0, y).$$

The following version of the implicit function theorem will play a fundamental role in proving Theorems (I) and (II).

Theorem 3.1 (Implicit function theorem, [Ni])**.** *Suppose that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are three Banach spaces, $\mathcal{U}$ is an open subset of $\mathcal{X} \times \mathcal{Y}$, $(x_0, y_0) \in \mathcal{U}$, and $\Psi : \mathcal{U} \to \mathcal{Z}$ is a Fréchet differentiable mapping such that $\Psi(x_0, y_0) = 0$. Assume that the partial derivative $\partial_y \Psi(x_0, y_0)$ with respect to the variable y at (x_0, y_0) is a bounded invertible linear transformation from $\mathcal{Y}$ to $\mathcal{Z}$. Then there exists a neighborhood $\mathcal{U}_0$ of x_0 in $\mathcal{X}$ and a unique continuous function $g : \mathcal{U}_0 \to \mathcal{Y}$ such that $g(x_0) = y_0$, $(x, g(x)) \in \mathcal{U}$ and $\Psi(x, g(x)) = 0$ for all $x \in \mathcal{U}$. Moreover, the function g is Fréchet differentiable in $\mathcal{U}_0$ and its Fréchet differential g' can be written as*

$$g'(x) = -\partial_y \Psi(x, g(x))^{-1}\, \partial_x \Psi(x, g(x)) \qquad \textit{for } x \in \mathcal{U}_0.$$

3.2 Shape derivatives

Our aim is to apply the functional machinery developed in the previous subsection in order to find a solution to the two-phase overdetermined problem (2.4)-(2.5)-(2.6). The first difficulty lies in the fact that the set of "shapes", where the solution belongs, is not endowed with a linear structure in any natural way. In what follows we will overcome this obstacle by introducing the so-called *shape derivative*.

We first need some basic notation. Let $\omega \subset \mathbb{R}^N$ be a smooth domain at which we will compute the derivative of a real valued shape functional J (we will therefore require $J(\widetilde{\omega})$ to be defined at least for all domains $\widetilde{\omega}$ "sufficiently close" to the reference domain ω). Let φ be an element of a Banach space V of vector fields from $\mathbb{R}^N$ to itself. For $\|\varphi\|$ small enough, the perturbation of the identity $\mathrm{Id} + \varphi$ is a diffeomorphism of $\mathbb{R}^N$ into itself. Let then $\omega_\varphi = (\mathrm{Id} + \varphi)(\omega)$ denote the deformed domain and set $\mathcal{J}(\varphi) = J(\omega_\varphi)$, whenever the right hand side is well defined. Notice that, by hypothesis, the functional $\mathcal{J}$ is well defined in a neighborhood of $0 \in V$. The shape derivative of J at ω with respect to the direction φ is then defined as a Fréchet derivative:

$$J'(\omega)(\varphi) := \mathcal{J}'(0)(\varphi). \tag{3.9}$$

Of course, the definition above can be extended to functionals that take several domains as input as well.

The concept of shape derivative can be applied to shape functionals that take values in a general Banach space too. A fairly common example is given by a smoothly varying family of smooth functions $u_\varphi : \omega_\varphi \to \mathbb{R}$. In many practical applications u_φ is the solution to some boundary value problem defined on the perturbed domain ω_φ and the shape derivative of u_φ is then defined as the Fréchet derivative of the function $\varphi \mapsto u_\varphi$, as before.

We refer to [HP, Chapter 5] for a self-contained introduction on the topic of shape derivatives and their computation.

4 Sketch of the proofs of Theorems I and II

The proofs of Theorems I and II rely on a perturbation argument based on the implicit function theorem (Theorem 3.1). We will construct a functional Ψ in such a way that its zeros are in a one-to-one correspondence with the solutions of the overdetermined problem (2.4)-(2.5)-(2.6) (near the trivial solution (D_0, Ω_0)).

4.1 The functional Ψ

In what follows we will introduce the preliminary notations in order to define Ψ. For $\alpha \in (0,1)$, let $\varphi \in \mathcal{C}^{2+\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ be a sufficiently small perturbation field such that the map $\mathrm{Id}+\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is a diffeomorphism that satisfies

$$\varphi = fn \quad \text{on } \partial D_0 \quad \text{and} \quad \varphi = gn \quad \text{on } \partial\Omega_0,$$

where f and g are given functions of class $\mathcal{C}^{2+\alpha}$ and n indistinctly denotes the outward unit normal to both D_0 and Ω_0. Next we define the perturbed domains

$$D_f = (\mathrm{Id} + \varphi)(D_0) \quad \text{and} \quad \Omega_g = (\mathrm{Id} + \varphi)(\Omega_0).$$

We will also require f and g to be sufficiently small, so that the inclusion $\overline{D}_f \subset \Omega_g$ holds and so the boundary value problem (2.4)-(2.5) is well defined for the pair (D_f, Ω_g). In order to apply the implicit function theorem (Theorem 3.1), we consider the following Banach spaces (equipped with the standard norms):

$$\mathcal{F} = \left\{ f \in C^{2+\alpha}(\partial D_0) \ : \ \textstyle\int_{\partial D_0} f \ = 0 \right\}, \quad \mathcal{G} = \left\{ g \in C^{2+\alpha}(\partial \Omega_0) \ : \ \textstyle\int_{\partial \Omega_0} g \ = 0 \right\},$$
$$\mathcal{H} = \left\{ h \in C^{1+\alpha}(\partial \Omega_0) \ : \ \textstyle\int_{\partial \Omega_0} h \ = 0 \right\}.$$

Finally, we define the map $\Psi : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ by

$$\Psi(f,g) = \partial_{n_g} v_{f,g} - \frac{1}{|\partial\Omega_0|} \int_{\partial\Omega_0} \partial_{n_g} v_{f,g} \quad \text{for } (f,g) \in \mathcal{X} \times \mathcal{Y}. \tag{4.10}$$

Here $v_{f,g}$ denotes the solution of the boundary value problem (2.4)-(2.5) corresponding to the deformed configuration (D_f, Ω_g), similarly n_g denotes the outer normal of Ω_g. Moreover, by a slight abuse of notation, the notation $\partial_{n_g} v_{f,g}$ is used to represent the function of value

$$\nabla v_{f,g}\left(x + g(x)\, n(x)\right) \cdot n_g(x + g(x)\, n(x)) \quad \text{at any } x \in \partial\Omega_0. \tag{4.11}$$

Remark 4.1. *It is clear that $\Psi(f,g) = 0$ if and only if the quantity defined in* (4.11) *is constant on $\partial\Omega_0$, that means if and only if the pair (D_f, Ω_g) is a solution to the overdetermined problem* (2.4)-(2.5)-(2.6). *In particular, we know that $\Psi(0,0) = 0$.*

4.2 Applying the implicit function theorem

The Fréchet differentiability of the map Ψ in a neighborhood of $(0,0) \in \mathcal{F} \times \mathcal{G}$ can be proved in a standard way by combining the ideas of [HP, Theorem 5.3.2, pp. 183-184] with the Schauder's theory for elliptic operators with piecewise constant coefficients.

Again, by standard computations based on the Hadamard formula ([HP, Theorem 5.2.2, p. 172]) we obtain the following explicit formulas for the two partial Fréchet derivatives of the functional Ψ.

Lemma 4.2 ([CY])**.** *The map $\Psi : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ defined by* (4.10) *is Fréchet differentiable in a neighborhood of* $(0,0)$*. Moreover, for all $(f,g) \in \mathcal{F} \times \mathcal{G}$, its partial Fréchet derivatives are:*

$$\partial_f \Psi(0,0)(f) = \partial_n v'_-, \qquad \partial_g \Psi(0,0)(g) = \partial_n v'_+ + \partial_{nn} v\, g,$$

where the functions $v'_\pm$ are the solutions to the following transmission problems.

$$\begin{cases} \Delta v'_- = 0 & \textit{in } D_0 \cup (\Omega_0 \setminus \overline{D_0}), \\ [\sigma\, \partial_n v'_-] = 0 & \textit{on } \partial D_0, \\ [v'_-] = -[\partial_n v] f & \textit{on } \partial D_0, \\ v'_- = 0 & \textit{on } \partial\Omega_0. \end{cases} \quad (4.12) \qquad \begin{cases} \Delta v'_+ = 0 & \textit{in } D_0 \cup (\Omega_0 \setminus \overline{D_0}), \\ [\sigma\, \partial_n v'_+] = 0 & \textit{on } \partial D_0, \\ [v'_+] = 0 & \textit{on } \partial D_0, \\ v'_+ = -\partial_n v\, g & \textit{on } \partial\Omega_0. \end{cases} \quad (4.13)$$

In the above, v is the solution of (2.4)-(2.5)*, and square brackets are used to denote the jump of a function across the interface ∂D_0.*

A careful analysis of the expressions for $\partial_f \Psi(0,0)$ and $\partial_g \Psi(0,0)$ by means of the spherical harmonic expansions of the functions $v'_\pm$ yields the following theorems.

Theorem 4.3 ([CMS])**.** *The map $\partial_f \Psi(0,0) : \mathcal{F} \to \mathcal{H}$ is a bounded invertible linear transformation.*

Theorem 4.4 ([CY])**.** *Let Σ be the set defined by* (2.8) *and suppose that $\sigma_c \notin \Sigma$. Then, the map $\partial_g \Psi(0,0) : \mathcal{G} \to \mathcal{H}$ is a bounded invertible linear transformation.*

As stated in Remark 4.1, $\Psi(f,g) = 0$ if and only if (D_f, Ω_g) is a solution to the overdetermined problem (2.4)-(2.5)-(2.6). Moreover, we know that $\Psi(0,0) = 0$ and that Ψ is Fréchet differentiable in a neighborhood of $(0,0) \in \mathcal{F} \times \mathcal{G}$. Therefore, in order to apply the implicit function theorem to Ψ, we just need to study the invertibility of the partial Fréchet derivative. Now, in the light of the above, by taking $\mathcal{X} = \mathcal{G}$ and $\mathcal{Y} = \mathcal{F}$ in Theorem 3.1, we see that Theorem 4.3 implies Theorem I. On the other hand, by taking $\mathcal{X} = \mathcal{F}$ and $\mathcal{Y} = \mathcal{G}$ in Theorem 3.1, we see that Theorem 4.4 implies Theorem II.

5 Numerical simulation for the outer problem

In [CY], we also present a numerical algorithm to find an approximate solution of Problem 2. For a given D, we consider the following Kohn–Vogelius functional (we refer to [KV], where a similar functional was first introduced in the study of electrical impedance tomography):

$$\Phi(\Omega) = \int_\Omega \sigma |\nabla v - \nabla w|^2,$$

Here, v is the solution to the Dirichlet boundary value problem (2.4)-(2.5) and w is the solution to the Neumann boundary value problem (2.4)-(2.6), where the constant $d = d(\Omega)$ is the one defined by (2.7) and the solution w is normalized in such a way that $\int_{\partial\Omega} w = 0$. By definition, it is clear that $\Phi(\Omega) = 0$ if and only if the pair (D, Ω) solves the overdetermined problem (2.4)-(2.5)-(2.6). Moreover, if (D, Ω) is not a solution, then $\Phi(D, \Omega) > 0$. This means that, under the hypothesis of Theorem II (i.e. when D and σ_c are such that we have existence of a nontrivial solution), the following constrained minimization problem is equivalent to Problem 2:

$$\min_{|\Omega| = V_0} \Phi(\Omega), \tag{5.14}$$

where the minimization is considered among all sets Ω that contain the closure of D. This observation is at the heart of the iterative algorithm introduced in [CY].

In what follows, we will give a brief description of the algorithm. Starting from a given initial shape Ω_0, we compute the steepest descent with respect to the functional Φ and then iteratively update Ω accordingly. This gives rise to a sequence $(\Omega_k)_{k \geq 0}$ of domains that converges to the desired solution of Problem 2. In order to find the steepest descent of the Kohn–Vogelius functional Φ, one computes the shape derivative of Φ with respect to the perturbation $x \mapsto x + \varphi(x)$ acting on the subdomain $\Omega \setminus \overline{D}$. Using the same notation of (3.9), we can write (after some lengthy calculations)

$$\Phi'(\Omega)(\varphi) = \int_{\partial\Omega} \{-|\nabla w|^2 + 2(1 + d(\Omega)H)w - |\nabla v|^2 + 2d(\Omega)^2\}\varphi \cdot n,$$

where H is the sum of the principal curvatures of $\partial\Omega$ (here we adopt the convention that H is nonnegative if Ω is convex). Therefore, the steepest descent of the functional Φ is given by:

$$x \mapsto x - \left(-|\nabla w|^2 + 2(1 + d(\Omega)H)w - |\nabla v|^2 + 2d(\Omega)^2\right) n, \quad \text{for } x \in \partial\Omega. \tag{5.15}$$

Remark 5.1. *In reality, the actual implementation of the iterative algorithm we discussed above is a bit more complicated. First of all, since* (5.14) *is a constrained minimization problem, we need to introduce a so-called* augmented Lagrangian *(see [NW, Sections 17.3 and 17.4]). By employing the use of the augmented Lagrangian, we turn* (5.14) *into an equivalent (but unconstrained) minimization problem.*

There is yet another subtle point, that has to be addressed when considering the numerical implementation of such an algorithm. The explicit formula (5.15) *for the steepest descent of* Φ *is only defined on the boundary* $\partial\Omega$*. The choice of a well-behaved extension of the descent direction is a crucial part of the numerical implementation. Indeed, a poor choice of the extension is likely to result in a pathologic behavior when updating the mesh according to the descent direction (see [AP] where some standard extension methods are presented).*

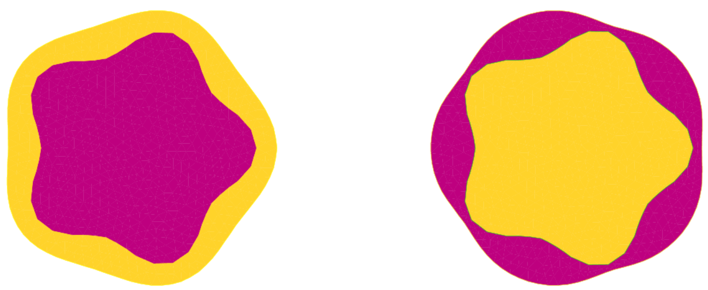

Figure 4: The solution Ω inherits the geometry of D, but in different ways depending on the value of σ_c. The computed solutions for $\sigma_c > 1$ on the left, $\sigma_c < 1$ on the right.

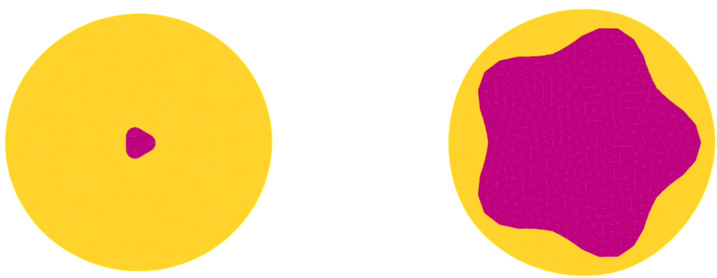

Figure 5: When the influence of D is small, the two-phase overdetermined problem (2.4)-(2.5)-(2.6) can be approximated by the Serrin overdetermined problem (1.1)-(1.2). Here, the computed solutions Ω when D is small (on the left) and when σ_c is close to 1 (on the right). In both cases, the computed Ω is virtually indistinguishable from a ball.

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