Abstract

The purpose of this study is finding a common minimizer of two convex functions on a geodesic space. The iterative schemes for this problem have Halpern iteration, contraction projection method, CQ method and so on. To iterative a common minimizer by Mann type iteration, we use properties of the resolvent in this study. In this paper, we show fundamental properties of a CAT(1) space, our theorem and recent results.

1 Introduction

We know that Mann’s iterative scheme [10] is a very effective method to find a fixed point of a nonexpansive mapping. By using this scheme, a large number of authors have proved various kinds of theorems. Reich [13] proved a weak convergence theorem for Mann type iteration in a Banach space. Takahashi and Tamura [12] proved weak convergence of an iteration generated by two nonexpansive mappings in a Banach space. Motivated by these results, researchers began to investigate the Mann type scheme defined on geodesic spaces. Dhompongsa and Panyanak [3] proved ∆-convergence of an iteration generated by a nonexpansive mapping in a CAT(0) space. Kimura and Nakagawa [7] proved ∆-convergence of an iteration generated by two quasinonexpansive and ∆-demiclosed mappings in a CAT(1) space. We show the theorem in [7].

Theorem 1.1. (Kimura and Nakagawa [7]) Let $X$ be a complete CAT(1) space such that for any $u, v \in X, d(u, v) < \pi/2$. Let $S$ and $T$ be quasinonexpansive and ∆-demiclosed mappings from $X$ into itself with $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences of $[a, b] \subset ]0, 1[$. Define a sequence $\{x_n\} \subset X$ by the following recurrence formula: $x_1 \in X$ and

$$
\begin{align*}
    u_n &= (1 - \beta_n)x_n \oplus \beta_n Sx_n, \\
    v_n &= (1 - \gamma_n)x_n \oplus \gamma_n Tx_n, \\
    x_{n+1} &= (1 - \alpha_n)u_n \oplus \alpha_n v_n
\end{align*}
$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ ∆-converges to a common fixed point of $S$ and $T$.

In this paper, the authors prove Theorem 3.1 based on Theorem 1.1 with the resolvent in a complete CAT(1) space.

The resolvent for a convex function on a Hilbert space is defined as follows: Let $f$ be a proper lower semicontinuous convex function from a Hilbert space $H$ into $[-\infty, \infty]$. The resolvent $J_f$ of $f$ is defined by

$$
J_f x = \arg\min_{y \in H} \{f(y) + \frac{1}{2}\|y - x\|^2\}.
$$
for all \( x \in X \). We know that \( J_f \) is a single-valued mapping from \( H \) to \( H \). On the other hand, the resolvent on a Hadamard space, a complete CAT(0) space, is proposed by Jost [4] and Mayer [11]. Let \( X \) be an Hadamard space. Let \( f \) be a proper lower semicontinuous convex function from a Hadamard space \( X \) into \([\ell, \infty)\]. The resolvent \( R_f \) of \( f \) is defined by

\[
R_f x = \arg\min_{y \in X} \{ f(y) + \frac{1}{2} d(y, x)^2 \}
\]

for all \( x \in X \). We know that \( R_f \) is a single-valued mapping from \( X \) to \( X \). We also know the definition and some properties of the resolvent in a complete CAT(1) space [5, 6]. We show them in preliminaries.

2 Preliminaries

Let \( X \) be a metric space. For \( x, y \in X \), a mapping \( c : [0, l] \to X \) is called a geodesic if \( c \) satisfies

\[
c(0) = x, c(l) = y, \text{ and } d(c(u), c(v)) = |u - v|
\]

for every \( u, v \in [0, l] \). An image \([x, y]\) of \( c \) is called a geodesic segment joining \( x \) and \( y \). If a geodesic exists for every \( x, y \in X \), then we call \( X \) a geodesic space.

Let \( X \) be a geodesic space. For a triangle \( \triangle(x, y, z) \subset X \) such that \( d(x, y) + d(y, z) + d(z, x) < 2\pi \), let a comparison triangle \( \triangle(\bar{x}, \bar{y}, \bar{z}) \) in two-dimensional unit sphere \( S^2 \) be such that each corresponding edge has the same length as that of the original triangle. \( X \) is called a CAT(1) space if for every \( x, y, z \in X \) such that \( d(x, y) + d(y, z) + d(z, x) < 2\pi \), every \( p, q \in \triangle(x, y, z) \) satisfy that

\[
d(p, q) \leq d_{S^2}(\bar{p}, \bar{q}),
\]

where \( d_{S^2} \) is the spherical metric on \( S^2 \).

Let \( X \) be a CAT(1) space. For every \( x, y \in X \) with \( d(x, y) < \pi \) and \( \alpha \in [0, 1] \), if \( z \in [x, y] \) satisfies that \( d(y, z) = \alpha d(x, y) \) and \( d(x, z) = (1 - \alpha)d(x, y) \), then we denote \( z \) by \( z = \alpha x \oplus (1 - \alpha)y \). A subset \( C \subset X \) is called \( \pi \)-convex if \( \alpha x \oplus (1 - \alpha)y \in C \) for every \( x, y \in C \) with \( d(x, y) < \pi \) and \( \alpha \in [0, 1] \).

For every \( x, y, z \in X \) with \( d(x, y) + d(y, z) + d(z, x) < 2\pi \) and \( \alpha \in [0, 1] \), the following inequality holds [8]:

\[
\cos d(x, w) \sin d(y, z) \geq \cos d(x, y) \sin(\alpha d(y, z)) + \cos d(x, z) \sin((1 - \alpha)d(y, z)),
\]

where \( w = \alpha y \oplus (1 - \alpha)z \).

A CAT(1) space \( X \) is said to be admissible if every \( x, y \in X \) satisfy that \( d(x, y) < \pi/2 \).

Let \( X \) be a CAT(1) space and let \( T \) be a mapping from \( X \) to \( X \) such that the set \( F(T) = \{ z \in X : z = Tz \} \) of fixed points of \( T \) is not empty. If \( d(Tx, p) \leq d(x, p) \) for every \( x \in X \) and \( p \in F(T) \), then we call \( T \) a quasi-nonexpansive mapping.

\( T \) is said to be strongly quasi-nonexpansive if \( T \) is a quasi-nonexpansive mapping, and \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \) whenever \( \{x_n\} \subset X \) satisfies \( \sup_{n \in \mathbb{N}} d(x_n, p) < \pi/2 \) and \( \lim_{n \to \infty}(\cos d(x_n, p)/\cos d(Tx_n, p)) = 1 \) for every \( p \in F(T) \).

We also define a strongly quasi-nonexpansive sequence. A sequence \( \{T_n\} \) of mappings from \( X \) to \( X \) is called a strongly quasi-nonexpansive sequence if each \( T_n \) is a quasi-nonexpansive mapping, and \( \lim_{n \to \infty} d(x_n, T_nx_n) = 0 \) whenever \( \{x_n\} \subset X \) satisfies \( \sup_{n \in \mathbb{N}} d(x_n, p) < \pi/2 \) and \( \lim_{n \to \infty}(\cos d(x_n, p)/\cos d(T_nx_n, p)) = 1 \) for every \( p \in \bigcap_{n=1}^{\infty} F(T_n) \); see [1].

Let \( X \) be a metric space. An element \( z \in X \) is said to be an asymptotic center of \( \{x_n\} \subset X \) if

\[
\lim_{n \to \infty} d(x_n, z) = \inf_{x \in X} \limsup_{n \to \infty} d(x_n, x).
\]
Moreover, \( \{x_n\} \) \( \Delta \)-converges to a \( \Delta \)-limit \( z \) if \( z \) is the unique asymptotic center of any subsequences of \( \{x_n\} \).

Let \( X \) be a CAT(1) space and let \( T \) be a mapping from \( X \) to \( X \) such that \( F(T) \neq \emptyset \). \( T \) is said to be \( \Delta \)-demiclosed if \( z \in F(T) \) whenever \( \{x_n\} \) \( \Delta \)-converges to \( z \) and \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \).

We also define a \( \Delta \)-demiclosed sequence. A sequence \( \{T_n\} \) of mappings from \( X \) to \( X \) is called a \( \Delta \)-demiclosed sequence if \( z \in \bigcap_{n=1}^{\infty} F(T_n) \) whenever \( \{x_n\} \) \( \Delta \)-converges to \( z \) and \( \lim_{n \to \infty} d(x_n, T_n x_n) = 0 \); see [1].

Let \( X \) be a complete CAT(1) space and let \( C \subset X \) be a nonempty closed \( \pi \)-convex subset such that \( d(x, C) = \inf_{y \in C} d(x, y) < \pi/2 \) for every \( x \in X \). Then for every \( x \in X \), there exists a unique point \( x_0 \in C \) satisfying
\[
d(x, x_0) = \inf_{y \in C} d(x, y).
\]

We define the metric projection \( P_C \) from \( X \) onto \( C \) by \( P_C x = x_0 \). We know that the metric projection \( P_C \) is a strongly quasinonexpansive and \( \Delta \)-demiclosed mapping such that \( F(P_C) = C \) [2, 9].

Let \( X \) be an admissible complete CAT(1) space. Let \( f \) be a proper lower semicontinuous convex function from \( X \) into \( ]-\infty, \infty] \). The resolvent \( R_f \) of \( f \) is defined by
\[
R_f x = \arg\min_{y \in X} \{ f(y) + \tan d(y, x) \sin d(y, x) \}
\]
for all \( x \in X \) [5]. We know that \( R_f \) is a single-valued mapping from \( X \) to \( X \). We also know that the resolvent \( R_f \) is strongly quasinonexpansive and \( \Delta \)-demiclosed such that \( F(R_f) = \arg\min_{x \in X} f \) [5, 6].

We introduce some lemmas and theorem used for our results.

**Lemma 2.1.** (Kimura and Satô [8]) Let \( X \) be a CAT(1) space. For every \( x, y, z \in X \) with \( d(x, y) + d(y, z) + d(z, x) < 2\pi \) and \( \alpha \in [0, 1] \), the following inequality holds:
\[
\cos d(x, w) \sin d(y, z) \geq \cos d(x, y) \sin(\alpha d(y, z)) + \cos d(x, z) \sin((1 - \alpha)d(y, z)),
\]
where \( w = \alpha y \oplus (1 - \alpha)z \).

**Lemma 2.2.** (Kimura and Satô [9]) Let \( X \) be a CAT(1) space. For every \( x, y, z \in X \) with \( d(x, y) + d(y, z) + d(z, x) < 2\pi \) and \( \alpha \in [0, 1] \), the following inequality holds:
\[
\cos d(x, w) \geq \alpha \cos d(x, y) + (1 - \alpha) \cos d(x, z),
\]
where \( w = \alpha y \oplus (1 - \alpha)z \).

**Theorem 2.3.** (Espínola and Fernández-León [2]) Let \( X \) be a complete CAT(1) space. Let \( \{x_n\} \) be a sequence in \( X \). If \( r(\{x_n\}) < \pi/2 \), then the following hold:

(a) \( AC(\{x_n\}) \) consists of exactly one point;
(b) \( \{x_n\} \) has a \( \Delta \)-convergent subsequence.

**Lemma 2.4.** Let \( \sigma \) be a real number in \( ]-1, 0[ \) and \( \{b_n\}, \{c_n\} \) real sequences in \( [\sigma, 1] \) and \( \liminf_{n \to \infty} b_n c_n \geq 1 \). Then \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = 1 \).

**Lemma 2.5.** Let \( s \) be a real number in \( ]0, \infty[ \) and \( \{b_n\}, \{c_n\} \) bounded real sequences such that \( b_n \leq 0 \), \( s < c_n \) and \( \lim_{n \to \infty} b_n / c_n = 0 \). Then \( \lim_{n \to \infty} b_n = 0 \).

**Lemma 2.6.** Let \( \{b_n\}, \{c_n\} \) be bounded real sequences such that \( \lim_{n \to \infty} (b_n - c_n) = 0 \). Then \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n \).
3 Main result

To prove this theorem, we employ the technique proposed in [7].

**Theorem 3.1.** Let $X$ be an admissible complete CAT(1) space. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of positive real numbers such that $\inf_n \lambda_n > 0$ and $\inf_n \mu_n > 0$. Let $f$ and $g$ be proper lower semicontinuous convex functions from $X$ into $]-\infty, \infty]$ such that $F = \text{argmin}_X f \cap \text{argmin}_X g \neq \emptyset$. Let $R_{\lambda_n}f$ and $R_{\mu_n}g$ be the resolvents of $\lambda_n f$ and $\mu_n g$, respectively. For a given real number $a \in [0, \frac{1}{2}]$, let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$. Let $\{x_n\}$ be a sequence in $X$ defined by $x_1 \in X$ and

$$
\begin{align*}
    u_n &= \beta_n x_n + (1 - \beta_n) R_{\lambda_n} f x_n, \\
    v_n &= \gamma_n x_n + (1 - \gamma_n) R_{\mu_n} g x_n, \\
    x_{n+1} &= \alpha_n u_n + (1 - \alpha_n) v_n
\end{align*}
$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ $\Delta$-converges to a point of $F$.

**Proof.** Let $z \in F$. By Lemma 2.2, we have

$$
\cos d(u_n, z) \geq \beta_n \cos d(x_n, z) + (1 - \beta_n) \cos d(R_{\lambda_n} f x_n, z) \\
\geq \cos d(x_n, z).
$$

Similarly, we have

$$
\cos d(v_n, z) \geq \cos d(x_n, z).
$$

By these inequalities, we obtain

$$
\cos d(x_{n+1}, z) \geq \alpha_n \cos d(u_n, z) + (1 - \alpha_n) \cos d(v_n, z) \\
\geq \cos d(x_n, z).
$$

Thus, we have $d(x_{n+1}, z) \leq d(x_n, z)$ for all $n \in \mathbb{N}$ and there exists

$$
D = \lim_{n \to \infty} d(x_n, z) \leq d(x_1, z) < \frac{\pi}{2}.
$$

By Lemma 2.1, we have

$$
\cos d(u_n, z) \sin d(x_n, R_{\lambda_n} f x_n) \\
\geq \cos d(x_n, z) \sin \beta_n d(x_n, R_{\lambda_n} f x_n) + \cos d(R_{\lambda_n} f x_n, z) \sin (1 - \beta_n) d(x_n, R_{\lambda_n} f x_n) \\
\geq 2 \cos d(x_n, z) \sin \frac{d(x_n, R_{\lambda_n} f x_n)}{2} \cos \frac{(2\beta_n - 1)d(x_n, R_{\lambda_n} f x_n)}{2}. \quad (1)
$$

Similarly, we get

$$
\cos d(v_n, z) \sin d(x_n, R_{\mu_n} g x_n) \\
\geq \cos d(x_n, z) \sin \gamma_n d(x_n, R_{\mu_n} g x_n) + \cos d(R_{\mu_n} g x_n, z) \sin (1 - \gamma_n) d(x_n, R_{\mu_n} g x_n) \\
\geq 2 \cos d(x_n, z) \sin \frac{d(x_n, R_{\mu_n} g x_n)}{2} \cos \frac{(2\gamma_n - 1)d(x_n, R_{\mu_n} g x_n)}{2}. \quad (2)
$$

Let $d_n = d(x_n, z)$, $f_n = d(x_n, R_{\lambda_n} f x_n)/2$ and $g_n = d(x_n, R_{\mu_n} g x_n)/2$ for all $n \in \mathbb{N}$. If $f_n \neq 0$ and $g_n = 0$, then we have $v_n = x_n$. From (1), (2) and Lemma 2.2, we have

$$
2 \cos d_{n+1} \sin f_n \cos f_n = \cos d_{n+1} \sin 2f_n \\
\geq \alpha_n \cos d(u_n, z) \sin 2f_n + (1 - \alpha_n) \cos d(v_n, z) \sin 2f_n \\
\geq 2\alpha_n \cos d_n \sin f_n \cos (2\beta_n - 1)f_n + (1 - \alpha_n) \cos d_n \sin f_n \cos f_n
$$
Dividing by $2 \sin f_n > 0$, we obtain
\[
\cos d_{n+1} \cos f_n \geq \alpha_n \cos d_n \cos (2\beta_n - 1)f_n + (1 - \alpha_n) \cos d_n \cos f_n. \tag{3}
\]
If $f_n = 0$ and $g_n \neq 0$, then we have $u_n = x_n$. Similarly, we have
\[
\cos d_{n+1} \cos g_n \geq \alpha_n \cos d_n \cos g_n + (1 - \alpha_n) \cos d_n \cos (2\gamma_n - 1)g_n. \tag{4}
\]
If $f_n \neq 0$ and $g_n \neq 0$, then from (1), (2) and Lemma 2.2, we have
\[
\cos d_{n+1} \sin 2f_n \sin 2g_n
\]
\[
\geq \alpha_n \cos d(u_n, z) \sin 2f_n \sin 2g_n + (1 - \alpha_n) \cos d(v_n, z) \sin 2f_n \sin 2g_n
\]
\[
\geq 4 \cos d_n \sin f_n \sin g_n (\alpha_n \cos (2\beta_n - 1)f_n \cos g_n + (1 - \alpha_n) \cos f_n \cos (2\gamma_n - 1)g_n).
\]
Dividing by $4 \sin f_n \sin g_n > 0$, we have
\[
\cos d_{n+1} \cos f_n \cos g_n
\]
\[
\geq \alpha_n \cos d_n \cos (2\beta_n - 1)f_n \cos g_n + (1 - \alpha_n) \cos d_n \cos f_n \cos (2\gamma_n - 1)g_n. \tag{5}
\]
If $f_n = 0$ and $g_n = 0$, then we also have the inequality (5), and the inequality (5) can be reduced to the inequality (3), (4) for each case. From inequality (5), we have
\[
\left( \frac{\epsilon_n \cos f_n}{(1 - \alpha_n) \cos (2\beta_n - 1)f_n} - \frac{\alpha_n}{1 - \alpha_n} \right) \left( \frac{\epsilon_n \cos g_n}{\alpha_n \cos (2\gamma_n - 1)g_n} - \frac{1 - \alpha_n}{\alpha_n} \right) \geq 1,
\]
where $\epsilon_n = \cos d_{n+1}/\cos d_n$ for $n \in \mathbb{N}$. It follows that $\lim_{n \to \infty} \epsilon_n = \cos D/\cos D = 1$. Since \( \{\alpha_n\} \subset [a, 1 - a] \) for all $n \in \mathbb{N}$, we obtain
\[
\liminf_{n \to \infty} \left( \frac{\cos f_n}{(1 - \alpha_n) \cos (2\beta_n - 1)f_n} - \frac{\alpha_n}{1 - \alpha_n} \right) \left( \frac{\cos g_n}{\alpha_n \cos (2\gamma_n - 1)g_n} - \frac{1 - \alpha_n}{\alpha_n} \right) \geq 1, \tag{6}
\]
We show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the following hold:
\[
-\frac{1}{2} \leq \frac{\cos f_n}{(1 - \alpha_n) \cos (2\beta_n - 1)f_n} - \frac{\alpha_n}{1 - \alpha_n} \leq 1 \tag{7}
\]
and
\[
-\frac{1}{2} \leq \frac{\cos g_n}{\alpha_n \cos (2\gamma_n - 1)g_n} - \frac{1 - \alpha_n}{\alpha_n} \leq 1. \tag{8}
\]
We show the second inequality of (7). Since $\{\beta_n\} \subset [a, 1 - a]$ for all $n \in \mathbb{N}$, we obtain
\[
\frac{\cos f_n}{(1 - \alpha_n) \cos (2\beta_n - 1)f_n} - \frac{\alpha_n}{1 - \alpha_n} \leq \frac{1}{1 - \alpha_n} - \frac{\alpha_n}{1 - \alpha_n} = 1.
\]
Similarly, we show the second inequality of (8). Next, we show the first inequality of (7). Let
\[
\sigma_n = \frac{\cos f_n}{(1 - \alpha_n) \cos (2\beta_n - 1)f_n} - \frac{\alpha_n}{1 - \alpha_n} \quad \text{and} \quad \theta_n = \frac{\cos g_n}{\alpha_n \cos (2\gamma_n - 1)g_n} - \frac{1 - \alpha_n}{\alpha_n}.
\]
We assume that the first inequality of (7) does not hold. Then we can get a subsequence $\{\sigma_{n_j}\} \subset \{\sigma_n\}$ such that $\sigma_{n_j} < -1/2$ and $\lim_{j \to \infty} \sigma_{n_j} = \sigma \leq -1/2$. Since $\{\alpha_n\}, \{\gamma_n\} \subset [a, 1 - a]$ and $\{g_n\} \subset [0, \pi/4]$, we get $\{g_n\}$ is bounded. Let $\{\theta_{n_j}\} \subset \{\theta_n\} \subset \{\theta_n\}$ such that $\{\theta_{n_j}\}$ converges to $\theta \in \mathbb{R}$. From the inequality (6), we have
\[
\sigma \theta = \lim_{j \to \infty} \sigma_{n_j} \theta_{n_j} \geq \liminf_{n \to \infty} \sigma_n \theta_n \geq 1.
\]
Therefore, we may assume that $\theta_{n_{ij}} < 0$ for all $j \in \mathbb{N}$. Since $\{f_n\}, \{g_n\} \subset [0, \pi/4]$ and $\{\beta_n\}, \{\gamma_n\} \subset [a, 1 - a]$, we have

$$0 < \frac{\sqrt{2}}{2(1 - a)} \leq \frac{\cos f_n}{(1 - \alpha_n) \cos(2\beta_n - 1)f_n}$$

and

$$0 < \frac{\sqrt{2}}{2(1 - a)} \leq \frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n}.$$  \hspace{1cm} (9)

Let $\rho \in \mathbb{R}$ such that

$$0 < \rho < \min \left\{ \frac{\sqrt{2}}{2(1 - a)}, \frac{2a}{1 - a} \right\}.$$  \hspace{1cm} (10)

From the inequalities (9), (10), we have

$$\rho - \frac{\alpha_{n_{ij}}}{1 - \alpha_{n_{ij}}} \leq \sigma_{n_{ij}} < 0 \text{ and } \rho - \frac{1 - \alpha_{n_{ij}}}{\alpha_{n_{ij}}} \leq \theta_{n_{ij}} < 0.$$  \hspace{1cm} (11)

From the inequalities (11), (12), we have

$$\sigma_{n_{ij}} \theta_{n_{ij}} \leq \left( \rho - \frac{\alpha_{n_{ij}}}{1 - \alpha_{n_{ij}}} \right) \left( \rho - \frac{1 - \alpha_{n_{ij}}}{\alpha_{n_{ij}}} \right)$$

$$= \rho^2 - \left( \frac{\alpha_{n_{ij}}}{1 - \alpha_{n_{ij}}} + \frac{1 - \alpha_{n_{ij}}}{\alpha_{n_{ij}}} \right) \rho + 1$$

$$\leq \rho^2 - \frac{2a}{1 - a} \rho + 1$$

$$= \rho \left( \rho - \frac{2a}{1 - a} \right) + 1.$$  \hspace{1cm} (12)

Then, as $j \to \infty$, we obtain

$$1 \leq \sigma \theta \leq \rho \left( \rho - \frac{2a}{1 - a} \right) + 1 < 1.$$  

This is a contradiction. Similarly, we obtain the left inequality of (8). From the inequalities (7), (8) and Lemma 2.4, we have

$$\lim_{n \to \infty} \left( \frac{\cos f_n}{(1 - \alpha_n) \cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1 - \alpha_n} \right) = \lim_{n \to \infty} \left( \frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n} - \frac{1 - \alpha_n}{\alpha_n} \right) = 1.$$  \hspace{1cm} (13)

Hence, we have

$$\lim_{n \to \infty} \frac{\cos f_n - \cos(2\beta_n - 1)f_n}{(1 - \alpha_n) \cos(2\beta_n - 1)f_n} = 0.$$  \hspace{1cm} (14)

By Lemma 2.5, we get

$$\lim_{n \to \infty} \frac{\cos f_n - \cos(2\beta_n - 1)f_n}{(1 - \alpha_n) \cos(2\beta_n - 1)f_n} = 0.$$  \hspace{1cm} (15)

By Lemma 2.6, we have

$$\lim_{n \to \infty} \cos f_n = \lim_{n \to \infty} \cos(2\beta_n - 1)f_n = \lim_{n \to \infty} \cos |2\beta_n - 1|f_n.$$
Hence, we obtain
\[
\limsup_{n \to \infty} f_n = \limsup_{n \to \infty} (|2\beta_n - 1| f_n) \leq \limsup_{n \to \infty} |2\beta_n - 1| \limsup_{n \to \infty} f_n.
\]
Furthermore, we get
\[
\liminf_{n \to \infty} (1 - |2\beta_n - 1|) \limsup_{n \to \infty} f_n = \left(1 - \limsup_{n \to \infty} |2\beta_n - 1|\right) \limsup_{n \to \infty} f_n \leq 0.
\]
Since \(\{\beta_n\} \subset [a, 1 - a]\) for all \(n \in \mathbb{N}\), we get \(\liminf_{n \to \infty} (1 - |2\beta_n - 1|) > 0\) and thus \(\limsup_{n \to \infty} f_n = 0\). It implies that \(d(x_n, R_{\lambda_n} f x_n) \to 0\). Similarly, we get \(d(x_n, R_{\mu_n} g x_n) \to 0\).

Next, we show \(\{x_n\}\) \(\Delta\)-converges to a point of \(F\). Let \(\{x_{n_k}\}\) be a subsequence of \(\{x_n\}\).
Since \(r(\{x_n\}) \leq D < \pi/2\) and Theorem 2.3(a), there exists a unique asymptotic center \(x_0\) of \(\{x_{n_k}\}\). Since \(r(\{x_{n_k}\}) < \pi/2\) and Theorem 2.3(b), there exists a subsequence \(\{x_{n_{k_i}}\}\) of \(\{x_{n_k}\}\) such that \(\{x_{n_{k_i}}\}\) \(\Delta\)-converges to \(x_0' \in X\). Moreover, since \(d(x_{n_{k_i}}, R_{\lambda_{n_{k_i}}} f x_{n_{k_i}}) \to 0\), \(d(x_{n_{k_i}}, R_{\mu_{n_{k_i}}} g x_{n_{k_i}}) \to 0\) and \(\{R_{\lambda_{n_{k_i}}}, f x_{n_{k_i}}\}, \{R_{\mu_{n_{k_i}}}, g\}\) are \(\Delta\)-demiclosed sequence, we obtain \(x_0' \in F = \text{argmin}_X f \cap \text{argmin}_X g\). If \(x_0 \neq x_0'\), by Theorem 2.3(a) and the uniqueness of the asymptotic centers, we get
\[
\limsup_{k \to \infty} d(x_{n_k}, x_0) < \limsup_{k \to \infty} d(x_{n_k}, x_0') = \lim_{n \to \infty} d(x_n, x_0') = \limsup_{i \to \infty} d(x_{n_{k_i}}, x_0') < \limsup_{i \to \infty} d(x_{n_{k_i}}, x_0) \leq \limsup_{k \to \infty} d(x_{n_k}, x_0).
\]
This is a contradiction. Hence, we have \(x_0 \in F = \text{argmin}_X f \cap \text{argmin}_X g\). Let \(\{u_k\}, \{v_k\}\) be subsequences of \(\{x_n\}\), \(u_0 \in AC(\{u_k\})\) and \(v_0 \in AC(\{v_k\})\). If \(u_0 \neq v_0\), then since \(u_0 \notin AC(\{u_k\})\), \(v_0 \notin AC(\{v_k\})\) and by Theorem 2.3(a), we have
\[
\limsup_{k \to \infty} d(u_k, u_0) < \limsup_{k \to \infty} d(u_k, v_0) = \lim_{n \to \infty} d(x_n, v_0) = \limsup_{k \to \infty} d(v_k, v_0) < \limsup_{k \to \infty} d(v_k, u_0) = \lim_{n \to \infty} d(x_n, u_0) = \limsup_{k \to \infty} d(u_k, u_0).
\]
This is a contradiction. We obtain \(u_0 = v_0\), then we have \(\{x_n\}\) \(\Delta\)-converges to a point of \(F = \text{argmin}_X f \cap \text{argmin}_X g\). This completes the proof.

References


