CAT(1) 空間上での 2 つのリゾルベントによる近似列

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Abstract

The purpose of this study is finding a common minimizer of two convex functions on a geodesic space. The iterative schemes for this problem have Halpern iteration, contraction projection method, CQ method and so on. To iterative a common minimizer by Mann type iteration, we use properties of the resolvent in this study. In this paper, we show fundamental properties of a CAT(1) space, our theorem and recent results.

1 Introduction

We know that Mann's iterative scheme [10] is a very effective method to find a fixed point of a nonexpansive mapping. By using this scheme, a large number of authors have proved various kinds of theorems. Reich [13] proved a weak convergence theorem for Mann type iteration in a Banach space. Takahashi and Tamura [12] proved weak convergence of an iteration generated by two nonexpansive mappings in a Banach space. Motivated by these results, researchers began to investigate the Mann type scheme defined on geodesic spaces. Dhompongsa and Panyanak [3] proved Δ-convergence of an iteration generated by a nonexpansive mapping in a CAT(0) space. Kimura and Nakagawa [7] proved Δ-convergence of an iteration generated by two quasinonexpansive and Δ-demiclosed mappings in a CAT(1) space. We show the theorem in [7].

Theorem 1.1. (Kimura and Nakagawa [7]) *Let X be a complete CAT(1) space such that for any $u, v \in X, d(u, v) < \pi/2$. Let S and T be quasinonexpansive and Δ-demiclosed mappings from X into itself with $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences of $[a, b] \subset]0, 1[$. Define a sequence $\{x_n\} \subset X$ by the following recurrence formula: $x_1 \in X$ and*

$$\begin{cases} u_n = (1 - \beta_n)x_n \oplus \beta_n S x_n, \\ v_n = (1 - \gamma_n)x_n \oplus \gamma_n T x_n, \\ x_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n v_n \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a common fixed point of S and T.

In this paper, the authors prove Theorem 3.1 based on Theorem 1.1 with the resolvent in a complete CAT(1) space.

The resolvent for a convex function on a Hilbert space is defined as follows: Let f be a proper lower semicontinuous convex function from a Hilbert space H into $]-\infty, \infty]$. The resolvent J_f of f is defined by

$$J_f x = \operatorname*{argmin}_{y \in H} \{f(y) + \frac{1}{2} \|y - x\|^2\}$$

for all $x \in X$. We know that J_f is a single-valued mapping from H to H. On the other hand, the resolvent on a Hadamard space, a complete CAT(0) space, is proposed by Jost [4] and Mayer [11]. Let X be an Hadamard space. Let f be a proper lower semicontinuous convex function from a Hadamard space X into $]-\infty, \infty]$. The resolvent R_f of f is defined by

$$R_f x = \operatorname*{argmin}_{y \in X}\{f(y) + \frac{1}{2}d(y,x)^2\}$$

for all $x \in X$. We know that R_f is a single-valued mapping from X to X. We also know the definition and some properties of the resolvent in a complete CAT(1) space [5, 6]. We show them in preliminaries.

2 Preliminaries

Let X be a metric space. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic if c satisfies

$$c(0) = x, c(l) = y, \text{ and } d(c(u), c(v)) = |u - v|$$

for every $u, v \in [0, l]$. An image $[x, y]$ of c is called a geodesic segment joining x and y. If a geodesic exists for every $x, y \in X$, then we call X a geodesic space.

Let X be a geodesic space. For a triangle $\triangle(x, y, z) \subset X$ such that $d(x, y)+d(y, z)+d(z, x) < 2\pi$, let a comparison triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$ in two-dimensional unit sphere $\mathbb{S}^2$ be such that each corresponding edge has the same length as that of the original triangle. X is called a CAT(1) space if for every $x, y, z \in X$ such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$, every $p, q \in \triangle(x, y, z)$ and their corresponding points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z})$ satisfy that

$$d(p, q) \leq d_{\mathbb{S}^2}(\bar{p}, \bar{q}),$$

where $d_{\mathbb{S}^2}$ is the spherical metric on $\mathbb{S}^2$.

Let X be a CAT(1) space. For every $x, y \in X$ with $d(x, y) < \pi$ and $\alpha \in [0, 1]$, if $z \in [x, y]$ satisfies that $d(y, z) = \alpha d(x, y)$ and $d(x, z) = (1-\alpha)d(x, y)$, then we denote z by $z = \alpha x \oplus (1-\alpha)y$. A subset $C \subset X$ is called π-convex if $\alpha x \oplus (1 - \alpha)y \in C$ for every $x, y \in C$ with $d(x, y) < \pi$ and $\alpha \in [0, 1]$.

For every $x, y, z \in X$ with $d(x, y)+d(y, z)+d(z, x) < 2\pi$ and $\alpha \in [0, 1]$, the following inequality holds [8]:

$$\cos d(x, w) \sin d(y, z) \geq \cos d(x, y) \sin(\alpha d(y, z)) + \cos d(x, z) \sin((1 - \alpha)d(y, z)),$$

where $w = \alpha y \oplus (1 - \alpha)z$.

A CAT(1) space X is said to be admissible if every $x, y \in X$ satisfy that $d(x, y) < \pi/2$.

Let X be a CAT(1) space and let T be a mapping from X to X such that the set $F(T) = \{z \in X : z = Tz\}$ of fixed points of T is not empty. If $d(Tx, p) \leq d(x, p)$ for every $x \in X$ and $p \in F(T)$, then we call T a quasinonexpansive mapping.

T is said to be strongly quasinonexpansive if T is a quasinonexpansive mapping, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ whenever $\{x_n\} \subset X$ satisfies $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, p)/ \cos d(Tx_n, p)) = 1$ for every $p \in F(T)$.

We also define a strongly quasinonexpansive sequence. A sequence $\{T_n\}$ of mappings from X to X is called a strongly quasinonexpansive sequence if each T_n is a quasinonexpansive mapping, and $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$ whenever $\{x_n\} \subset X$ satisfies $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, p)/ \cos d(T_n x_n, p)) = 1$ for every $p \in \bigcap_{n=1}^{\infty} F(T_n)$; see [1].

Let X be a metric space. An element $z \in X$ is said to be an asymptotic center of $\{x_n\} \subset X$ if

$$\limsup_{n\to\infty} d(x_n, z) = \inf_{x\in X} \limsup_{n\to\infty} d(x_n, x).$$

Moreover, $\{x_n\}$ Δ-converges to a Δ-limit z if z is the unique asymptotic center of any subsequences of $\{x_n\}$.

Let X be a CAT(1) space and let T be a mapping from X to X such that $F(T) \neq \emptyset$. T is said to be Δ-demiclosed if $z \in F(T)$ whenever $\{x_n\}$ Δ-converges to z and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

We also define a Δ-demiclosed sequence. A sequence $\{T_n\}$ of mappings from X to X is called a Δ-demiclosed sequence if $z \in \bigcap_{n=1}^{\infty} F(T_n)$ whenever $\{x_n\}$ Δ-converges to z and $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$; see [1].

Let X be a complete CAT(1) space and let $C \subset X$ be a nonempty closed π-convex subset such that $d(x, C) = \inf_{y\in C} d(x, y) < \pi/2$ for every $x \in X$. Then for every $x \in X$, there exists a unique point $x_0 \in C$ satisfying

$$d(x, x_0) = \inf_{y\in C} d(x, y).$$

We define the metric projection P_C from X onto C by $P_C x = x_0$. We know that the metric projection P_C is a strongly quasinonexpansive and Δ-demiclosed mapping such that $F(P_C) = C$ [2, 9].

Let X be an admissible complete CAT(1) space. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. The resolvent R_f of f is defined by

$$R_f x = \operatorname*{argmin}_{y\in X}\{f(y) + \tan d(y, x) \sin d(y, x)\}$$

for all $x \in X$ [5]. We know that R_f is a single-valued mapping from X to X. We also know that the resolvent R_f is strongly quasinonexpansive and Δ-demiclosed such that $F(R_f) = \operatorname{argmin}_{x\in X} f$ [5, 6].

We introduce some lemmas and theorem used for our results.

Lemma 2.1. (Kimura and Satô [8]) *Let X be a CAT(1) space. For every $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi$ and $\alpha \in [0, 1]$, the following inequality holds:*

$$\cos d(x, w) \sin d(y, z) \geq \cos d(x, y) \sin(\alpha d(y, z)) + \cos d(x, z) \sin((1 - \alpha)d(y, z)),$$

where $w = \alpha y \oplus (1 - \alpha)z$.

Lemma 2.2. (Kimura and Satô [9]) *Let X be a CAT(1) space. For every $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi$ and $\alpha \in [0, 1]$, the following inequality holds:*

$$\cos d(x, w) \geq \alpha \cos d(x, y) + (1 - \alpha) \cos d(x, z),$$

where $w = \alpha y \oplus (1 - \alpha)z$.

Theorem 2.3. (Espínola and Fernández-León [2]) *Let X be a complete CAT(1) space. Let $\{x_n\}$ be a sequence in X. If $r(\{x_n\}) < \pi/2$, then the following hold:*

(a) *$AC(\{x_n\})$ consists of exactly one point;*
(b) *$\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 2.4. *Let σ be a real number in $]-1, 0[$ and $\{b_n\}, \{c_n\}$ real sequences in $[\sigma, 1]$ and $\liminf_{n\to\infty} b_n c_n \geq 1$. Then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 1$.*

Lemma 2.5. *Let s be a real number in $]0, \infty[$ and $\{b_n\}, \{c_n\}$ bounded real sequences such that $b_n \leq 0, s < c_n$ and $\lim_{n\to\infty} b_n/c_n = 0$. Then $\lim_{n\to\infty} b_n = 0$.*

Lemma 2.6. *Let $\{b_n\}, \{c_n\}$ be bounded real sequences such that $\lim_{n\to\infty}(b_n - c_n) = 0$. Then $\liminf_{n\to\infty} b_n = \liminf_{n\to\infty} c_n$.*

3 Main result

To prove this theorem, we employ the technique proposed in [7].

Theorem 3.1. *Let X be an admissible complete CAT(1) space. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of positive real numbers such that $\inf_n \lambda_n > 0$ and $\inf_n \mu_n > 0$. Let f and g be proper lower semicontinuous convex functions from X into $]-\infty, \infty]$ such that $F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g \neq \emptyset$. Let $R_{\lambda_n f}$ and $R_{\mu_n g}$ be the resolvents of $\lambda_n f$ and $\mu_n g$, respectively. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$. Let $\{x_n\}$ be a sequence in X defined by $x_1 \in X$ and*

$$\begin{cases} u_n = \beta_n x_n \oplus (1-\beta_n) R_{\lambda_n f} x_n, \\ v_n = \gamma_n x_n \oplus (1-\gamma_n) R_{\mu_n g} x_n, \\ x_{n+1} = \alpha_n u_n \oplus (1-\alpha_n) v_n \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point of F.

Proof. Let $z \in F$. By Lemma 2.2, we have

$$\begin{aligned} \cos d(u_n, z) &\geq \beta_n \cos d(x_n, z) + (1-\beta_n) \cos d(R_{\lambda_n f} x_n, z) \\ &\geq \cos d(x_n, z). \end{aligned}$$

Similarly, we have

$$\cos d(v_n, z) \geq \cos d(x_n, z).$$

By these inequalities, we obtain

$$\begin{aligned} \cos d(x_{n+1}, z) &\geq \alpha_n \cos d(u_n, z) + (1-\alpha_n) \cos d(v_n, z) \\ &\geq \cos d(x_n, z). \end{aligned}$$

Thus, we have $d(x_{n+1}, z) \leq d(x_n, z)$ for all $n \in \mathbb{N}$ and there exists

$$D = \lim_{n\to\infty} d(x_n, z) \leq d(x_1, z) < \frac{\pi}{2}.$$

By Lemma 2.1, we have

$$\begin{aligned} &\cos d(u_n, z) \sin d(x_n, R_{\lambda_n f} x_n) \\ &\quad \geq \cos d(x_n, z) \sin \beta_n d(x_n R_{\lambda_n f} x_n) + \cos d(R_{\lambda_n f} x_n, z) \sin(1-\beta_n) d(x_n, R_{\lambda_n f} x_n) \\ &\quad \geq 2 \cos d(x_n, z) \sin \frac{d(x_n, R_{\lambda_n f} x_n)}{2} \cos \frac{(2\beta_n - 1) d(x_n, R_{\lambda_n f} x_n)}{2}. \end{aligned} \tag{1}$$

Similarly, we get

$$\begin{aligned} &\cos d(v_n, z) \sin d(x_n, R_{\mu_n g} x_n) \\ &\quad \geq \cos d(x_n, z) \sin \gamma_n d(x_n R_{\lambda_n f} x_n) + \cos d(R_{\mu_n g} x_n, z) \sin(1-\gamma_n) d(x_n, R_{\mu_n g} x_n) \\ &\quad \geq 2 \cos d(x_n, z) \sin \frac{d(x_n, R_{\mu_n g} x_n)}{2} \cos \frac{(2\gamma_n - 1) d(x_n, R_{\mu_n g} x_n)}{2}. \end{aligned} \tag{2}$$

Let $d_n = d(x_n, z)$, $f_n = d(x_n, R_{\lambda_n f} x_n)/2$ and $g_n = d(x_n, R_{\mu_n g} x_n)/2$ for all $n \in \mathbb{N}$.

If $f_n \neq 0$ and $g_n = 0$, then we have $v_n = x_n$. From (1), (2) and Lemma 2.2, we have

$$\begin{aligned} 2 \cos d_{n+1} \sin f_n \cos f_n &= \cos d_{n+1} \sin 2f_n \\ &\geq \alpha_n \cos d(u_n, z) \sin 2f_n + (1-\alpha_n) \cos d(v_n, z) \sin 2f_n \\ &\geq 2\alpha_n \cos d_n \sin f_n \cos(2\beta_n - 1) f_n + 2(1-\alpha_n) \cos d_n \sin f_n \cos f_n \end{aligned}$$

Dividing by $2\sin f_n > 0$, we obtain

$$\cos d_{n+1}\cos f_n \geq \alpha_n \cos d_n \cos(2\beta_n - 1)f_n + (1-\alpha_n)\cos d_n \cos f_n. \tag{3}$$

If $f_n = 0$ and $g_n \neq 0$, then we have $u_n = x_n$. Similarly, we have

$$\cos d_{n+1}\cos g_n \geq \alpha_n \cos d_n \cos g_n + (1-\alpha_n)\cos d_n \cos(2\gamma_n - 1)g_n. \tag{4}$$

If $f_n \neq 0$ and $g_n \neq 0$, then from $(1), (2)$ and Lemma 2.2, we have

$$\begin{aligned}
&\cos d_{n+1}\sin 2f_n \sin 2g_n \\
&\qquad \geq \alpha_n \cos d(u_n, z)\sin 2f_n \sin 2g_n + (1-\alpha_n)\cos d(v_n, z)\sin 2f_n \sin 2g_n \\
&\qquad \geq 4\cos d_n \sin f_n \sin g_n(\alpha_n \cos(2\beta_n - 1)f_n \cos g_n + (1-\alpha_n)\cos f_n \cos(2\gamma_n - 1)g_n).
\end{aligned}$$

Dividing by $4\sin f_n \sin g_n > 0$, we have

$$\begin{aligned}
&\cos d_{n+1}\cos f_n \cos g_n \\
&\qquad \geq \alpha_n \cos d_n \cos(2\beta_n - 1)f_n \cos g_n + (1-\alpha_n)\cos d_n \cos f_n \cos(2\gamma_n - 1)g_n).
\end{aligned} \tag{5}$$

If $f_n = 0$ and $g_n = 0$, then we also have the inequality (5), and the inequality (5) can be reduced to the inequality $(3), (4)$ for each case. From inequality (5), we have

$$\left(\frac{\epsilon_n \cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n}\right)\left(\frac{\epsilon_n \cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n}\right) \geq 1,$$

where $\epsilon_n = \cos d_{n+1}/\cos d_n$ for $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty}\epsilon_n = \cos D/\cos D = 1$. Since $\{\alpha_n\} \subset [a, 1-a]$ for all $n \in \mathbb{N}$, we obtain

$$\liminf_{n\to\infty}\left(\frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n}\right)\left(\frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n}\right) \geq 1, \tag{6}$$

We show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,the following hold:

$$-\frac{1}{2} \leq \frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n} \leq 1 \tag{7}$$

and

$$-\frac{1}{2} \leq \frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n} \leq 1. \tag{8}$$

We show the second inequality of (7). Since $\{\beta_n\} \subset [a, 1-a]$ for all $n \in \mathbb{N}$, we obtain

$$\frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n} \leq \frac{1}{1-\alpha_n} - \frac{\alpha_n}{1-\alpha_n} = 1.$$

Similarly, we show the second inequality of (8). Next, we show the first inequality of (7). Let

$$\sigma_n = \frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n} \quad \text{and} \quad \theta_n = \frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n}.$$

We assume that the first inequality of (7) does not hold. Then we can get a subsequence $\{\sigma_{n_i}\} \subset \{\sigma_n\}$ such that $\sigma_{n_i} < -1/2$ and $\lim_{i\to\infty}\sigma_{n_i} = \sigma \leq -1/2$. Since $\{\alpha_n\}, \{\gamma_n\} \subset [a, 1-a]$ and $\{g_n\} \subset [0, \pi/4[$, we get $\{g_n\}$ is bounded. Let $\{\theta_{n_{i_j}}\} \subset \{\theta_{n_i}\} \subset \{\theta_n\}$ such that $\{\theta_{n_{i_j}}\}$ converges to $\theta \in \mathbb{R}$. From the inequality (6), we have

$$\sigma\theta = \lim_{j\to\infty}\sigma_{n_{i_j}}\theta_{n_{i_j}} \geq \liminf_{n\to\infty}\sigma_n\theta_n \geq 1.$$

Therefore, we may assume that $\theta_{n_{i_j}} < 0$ for all $j \in \mathbb{N}$. Since $\{f_n\}, \{g_n\} \subset [0, \pi/4[$ and $\{\beta_n\}, \{\gamma_n\} \subset [a, 1-a]$, we have

$$0 < \frac{\sqrt{2}}{2(1-a)} \le \frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1) f_n} \tag{9}$$

and

$$0 < \frac{\sqrt{2}}{2(1-a)} \le \frac{\cos g_n}{\alpha_n \cos(2\gamma_n - 1) g_n}. \tag{10}$$

Let $\rho \in \mathbb{R}$ such that

$$0 < \rho < \min\left\{\frac{\sqrt{2}}{2(1-a)}, \frac{2a}{1-a}\right\}. \tag{11}$$

From the inequalities $(9), (10)$, we have

$$\rho - \frac{\alpha_{n_{i_j}}}{1-\alpha_{n_{i_j}}} \le \sigma_{n_{i_j}} < 0 \quad \text{and} \quad \rho - \frac{1-\alpha_{n_{i_j}}}{\alpha_{n_{i_j}}} \le \theta_{n_{i_j}} < 0. \tag{12}$$

From the inequalities $(11), (12)$, we have

$$\begin{aligned}
\sigma_{n_{i_j}}\theta_{n_{i_j}} &\le \left(\rho - \frac{\alpha_{n_{i_j}}}{1-\alpha_{n_{i_j}}}\right)\left(\rho - \frac{1-\alpha_{n_{i_j}}}{\alpha_{n_{i_j}}}\right) \\
&= \rho^2 - \left(\frac{\alpha_{n_{i_j}}}{1-\alpha_{n_{i_j}}} + \frac{1-\alpha_{n_{i_j}}}{\alpha_{n_{i_j}}}\right)\rho + 1 \\
&\le \rho^2 - \frac{2a}{1-a}\rho + 1 \\
&= \rho\left(\rho - \frac{2a}{1-a}\right) + 1.
\end{aligned}$$

Then, as $j \to \infty$, we obtain

$$1 \le \sigma\theta \le \rho\left(\rho - \frac{2a}{1-a}\right) + 1 < 1.$$

This is a contradiction. Similarly, we obtain the left inequality of (8). From the inequalities $(7), (8)$ and Lemma 2.4, we have

$$\lim_{n\to\infty}\left(\frac{\cos f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} - \frac{\alpha_n}{1-\alpha_n}\right) = \lim_{n\to\infty}\left(\frac{\cos g_n}{\alpha_n\cos(2\gamma_n - 1)g_n} - \frac{1-\alpha_n}{\alpha_n}\right) = 1.$$

Hence, we have

$$\lim_{n\to\infty}\frac{\cos f_n - \cos(2\beta_n - 1)f_n}{(1-\alpha_n)\cos(2\beta_n - 1)f_n} = 0.$$

By Lemma 2.5, we get

$$\lim_{n\to\infty}(\cos f_n - \cos(2\beta_n - 1)f_n) = 0.$$

By Lemma 2.6, we have

$$\liminf_{n\to\infty}\cos f_n = \liminf_{n\to\infty}\cos(2\beta_n - 1)f_n = \liminf_{n\to\infty}\cos|2\beta_n - 1|\, f_n.$$

Hence, we obtain

$$\limsup_{n\to\infty} f_n = \limsup_{n\to\infty} \left(|2\beta_n - 1|\, f_n\right) \le \limsup_{n\to\infty} |2\beta_n - 1| \limsup_{n\to\infty} f_n.$$

Furthermore, we get

$$\liminf_{n\to\infty}(1 - |2\beta_n - 1|) \limsup_{n\to\infty} f_n = \left(1 - \limsup_{n\to\infty} |2\beta_n - 1|\right) \limsup_{n\to\infty} f_n \le 0.$$

Since $\{\beta_n\} \subset [a, 1-a]$ for all $n \in \mathbb{N}$, we get $\liminf_{n\to\infty}(1 - |2\beta_n - 1|) > 0$ and thus $\limsup_{n\to\infty} f_n = 0$. It implies that $d(x_n, R_{\lambda_n f} x_n) \to 0$. Similarly, we get $d(x_n, R_{\mu_n g} x_n) \to 0$.

Next, we show $\{x_n\}$ Δ-converges to a point of F. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Since $r(\{x_n\}) \le D < \pi/2$ and Theorem 2.3(a), there exists a unique asymptotic center x_0 of $\{x_{n_k}\}$. Since $r(\{x_{n_k}\}) < \pi/2$ and Theorem 2.3(b), there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_l}}\}$ Δ-converges to $x'_0 \in X$. Moreover, since $d(x_{n_{k_l}}, R_{\lambda_{n_{k_l}} f} x_{n_{k_l}}) \to 0$, $d(x_{n_{k_l}}, R_{\mu_{n_{k_l}} g} x_{n_{k_l}}) \to 0$ and $\{R_{\lambda_{n_{k_l}} f}\}, \{R_{\mu_{n_{k_l}} g}\}$ are Δ-demiclosed sequence, we obtain $x'_0 \in F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g$. If $x_0 \neq x'_0$, by Theorem 2.3(a) and the uniqueness of the asymptotic centers, we get

$$\begin{aligned}
\limsup_{k\to\infty} d(x_{n_k}, x_0) &< \limsup_{k\to\infty} d(x_{n_k}, x'_0) \\
&= \lim_{n\to\infty} d(x_n, x'_0) \\
&= \limsup_{l\to\infty} d(x_{n_{k_l}}, x'_0) \\
&< \limsup_{l\to\infty} d(x_{n_{k_l}}, x_0) \\
&\le \limsup_{k\to\infty} d(x_{n_k}, x_0).
\end{aligned}$$

This is a contradiction. Hence, we have $x_0 \in F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g$. Let $\{u_k\}, \{v_k\}$ be subsequences of $\{x_n\}$, $u_0 \in AC(\{u_k\})$ and $v_0 \in AC(\{v_k\})$. If $u_0 \neq v_0$, then since $u_0 \notin AC(\{v_k\}), v_0 \notin AC(\{u_k\})$ and by Theorem 2.3(a), we have

$$\begin{aligned}
\limsup_{k\to\infty} d(u_k, u_0) &< \limsup_{k\to\infty} d(u_k, v_0) \\
&= \lim_{n\to\infty} d(x_n, v_0) \\
&= \limsup_{k\to\infty} d(v_k, v_0) \\
&< \limsup_{k\to\infty} d(v_k, u_0) \\
&= \lim_{n\to\infty} d(x_n, u_0) \\
&= \limsup_{k\to\infty} d(u_k, u_0).
\end{aligned}$$

This is a contradiction. We obtain $u_0 = v_0$, then we have $\{x_n\}$ Δ-converges to a point of $F = \operatorname{argmin}_X f \cap \operatorname{argmin}_X g$. This completes the proof. □

References

[1] K. Aoyama and Y. Kimura, *Strong convergence theorems for strongly nonexpansive sequences*, Appl. Math. Comput. **217** (2011), 7537–7545.

[2] R. Espínola and A. Fernández-León, *CAT(k)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **353** (2009), 410–427.

[3] S. Dhompongsa and B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl. **56**(10) (2008), 2572–2579.

[4] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. **70** (1995), 659–673.

[5] Y. Kimura and F. Kohsaka, *Spherical nonspreadingness of resolvents of convex functions in geodesic spaces*, J. Fixed Point Theory Appl. **18** (2016), 93–115.

[6] Y. Kimura and F. Kohsaka, *Two modified proximal point algorithms in geodesic spaces with curvature bounded above*, Rend. Circ. Mat. Palermo (2), to appear.

[7] Y. Kimura and K. Nakagawa, *Another type of Mann iterative scheme for two mappings in a complete geodesic space*, J. Inequal. Appl. **2014** (2014), 9pages.

[8] Y. Kimura and K. Satô, *Convergence of subsets of a complete geodesic space with curvature bounded above*, Nonlinear Anal. **75** (2012), 5079-5085.

[9] Y. Kimura and K. Satô, *Halpern iteration for strongly quasinonexpansive mappings on a geodesic space with curvature bounded above by one*, Fixed Point Theory Appl. **2013** (2013), 14pages.

[10] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[11] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–253.

[12] W. Takahashi and T Tamura, *Convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67**(1) (1998), 45–56.

[13] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.