リー群による軌道分解と不変超関数

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概要

リー群 G による多様体 X の軌道分解や X 上の G 不変な超関数の空間は表現論的に X を理解する一つの手がかりになると考えられる。また G と X の複素化をそれぞれ $G_{\mathbb{C}}$ と $X_{\mathbb{C}}$ としたとき $G_{\mathbb{C}}$ による $X_{\mathbb{C}}$ の軌道分解は $\mathcal{D}$ 加群の理論より X 上の G 不変な超関数の空間に関する情報を与える。これを踏まえ今回の講演では X 上の G 軌道の個数は有限だが $X_{\mathbb{C}}$ 上の $G_{\mathbb{C}}$ 軌道が無限となる系列が存在することに幾何学的手法と表現論的手法の二通りの証明を与える。

1 Introduction

Let G be a real reductive Lie group (for example, general linear group $GL(n,\mathbb{R})$) and H a closed subgroup of G. Then T. Kobayashi and T. Oshima established the criterion of finite-multiplicity property for the regular representation on G/H.

Fact 1.1 ([7, Thm. A])**.** Suppose that G and H are defined algebraically over $\mathbb{R}$. Then the following two conditions on the pair (G, H) are equivalent:

(i) $\dim \mathrm{Hom}_G(\pi, C^\infty(G/H,\tau)) < \infty$ for any $(\pi,\tau) \in \hat{G}_{\mathrm{smooth}} \times \hat{H}_{\mathrm{alg}}$,
(ii) G/H is real spherical.

Here $\hat{G}_{\mathrm{smooth}}$ denotes the set of equivalence classes of irreducible smooth admissible Fréchet representations of G with moderate growth (see [12] for the definition of a smooth admissible Fréchet representation with moderate growth), and $\hat{H}_{\mathrm{alg}}$ that of algebraic irreducible finite-dimensional representations of H. Given $\tau \in \hat{H}_{\mathrm{alg}}$, we write $C^\infty(G/H,\tau)$ for the Fréchet space of smooth sections of the G-homogeneous vector bundle over G/H associated to τ, namely, the induced representation of G induced from a representation τ of H. The terminology *real sphericity* was introduced by Kobayashi [5] in his study of a broader framework for global analysis on homogeneous spaces than the usual (e.g., reductive symmetric spaces).

Definition 1.2. A homogeneous space G/H is *real spherical* if a minimal parabolic subgroup P of G has an open orbit on G/H.

Remark 1.3. (1) In the case $G = GL(n,\mathbb{R})$, a minimal parabolic subgroup P of G is given by the non-singular upper triangular matrices.
(2) G/H is real spherical if and only if the number $\#(H\backslash G/P)$ of H-orbits on G/P is infinite

[1]. This is a consequence of the rank one reduction of T. Matsuki [9] and the classification of real spherical varieties in the case of real rank one by B. Kimelfeld [3].

Let $G_{\mathbb{C}}$ be the complexification of G, that is, $G_{\mathbb{C}}$ is a complex Lie group which contains G as a closed Lie subgroup and its Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is equal to $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathfrak{g}$ is a Lie algebra of G (for example, $G_{\mathbb{C}} = GL(n, \mathbb{C})$ in the case $G = GL(n, \mathbb{R})$).

By finding an upper and lower estimate of the dimensions of $\mathrm{Hom}_G(\pi, C^\infty(G/H, \tau))$, Kobayashi and Oshima also established the criterion of the uniform boundedness of the multiplicities for induced representations.

Fact 1.4 ([7, Thm. B])**.** Suppose that G and H are defined algebraically over $\mathbb{R}$. Then the following two conditions on the pair (G, H) are equivalent:

(i) $\displaystyle\sup_{\tau \in \hat{H}_{\mathrm{alg}}} \sup_{\pi \in \hat{G}_{\mathrm{smooth}}} \frac{1}{\dim \tau} \dim \mathrm{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$,

(ii) $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.

Here we say that a homogeneous space $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit on $G_{\mathbb{C}}/H_{\mathbb{C}}$.

Remark 1.5. (1) In the case that $G_{\mathbb{C}}$ is the general linear group $GL(n, \mathbb{C})$ over $\mathbb{C}$, a Borel subgroup B of $G_{\mathbb{C}}$ is given by the non-singular upper triangular matrices.

(2) $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if and only if $\#(H_{\mathbb{C}} \backslash G_{\mathbb{C}}/B) < \infty$ holds [2, 11]. This also follows from the argument in [9].

Moreover, we proved in [10] the following bounded multiplicity property.

Fact 1.6 ([10])**.** Suppose that $\#(H_{\mathbb{C}} \backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$. Then we have

$$\sup_{\tau \in \hat{H}_{\mathrm{f}}} \sup_{\eta \in \hat{P}_{\mathrm{f}}} \frac{1}{\dim \eta \cdot \dim \tau} \dim \mathrm{Hom}_G(C^\infty(G/P, \eta), C^\infty(G/H, \tau)) < \infty.$$

Here $\hat{P}_{\mathrm{f}}$ denotes the set of equivalence classes of irreducible finite dimensional representations of P.

Remark 1.7. (1) In general, $\#(H_{\mathbb{C}} \backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$ is weaker condition on the pair (G, H) than $\#(H_{\mathbb{C}} \backslash G_{\mathbb{C}}/B) < \infty$ when $P_{\mathbb{C}}$ is not a Borel subgroup B of $G_{\mathbb{C}}$.

(2) In [10], it is proved that Fact 1.6 is also true if we replace a minimal parabolic subgroup P by a general parabolic subgroup Q of G.

(3) In Fact 1.6, individual terms are finite by Fact 1.1. Moreover the supremum of these terms is also finite if $P_{\mathbb{C}}$ is a Borel subgroup B of $G_{\mathbb{C}}$ by the proof of Fact 1.4 in [7] (The finiteness of supremum is also true if we replace $\hat{H}_{\mathrm{alg}}$ by $\hat{H}_{\mathrm{f}}$ in Fact 1.4).

2 Main Theorem

As we have seen in Section 1, the orbit decomposition of H on G/P and its complexification have information of harmonic analysis on G/H. In particular, the finiteness of the number of H-orbits on G/P, or $H_{\mathbb{C}}$-orbits on $G_{\mathbb{C}}/B$ characterizes the finite/bounded multiplicity property of the regular representation on G/H. Moreover $\#(H_{\mathbb{C}}\backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$ also implies bounded multiplicity property. Therefore we want to know what happens in the intermediate case, namely, the case that $\#(H\backslash G/P) < \infty$ holds although $\#(H_{\mathbb{C}}\backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$ does not hold. While there are many pairs (G, H) satisfying $\#(H\backslash G/P) < \infty$ and $\#(H_{\mathbb{C}}\backslash G_{\mathbb{C}}/B) = \infty$ (for example, see [6]), it is not easy to construct examples that $\#(H\backslash G/P) < \infty$ holds although $\#(H_{\mathbb{C}}\backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$ does not hold. As a first step in the study of the intermediate case, we give examples that the number of H-orbits on G/P is finite but the number of $H_{\mathbb{C}}$-orbits on $G_{\mathbb{C}}/P_{\mathbb{C}}$ is infinite.

Theorem 2.1. For a real reductive Lie group G without compact factors, the following three conditions are equivalent:

(i) $\mathfrak{l}$ is abelian,
(ii) $\#(N_{\mathbb{C}}\backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$,
(iii) $G_{\mathbb{C}}/N_{\mathbb{C}}$ is spherical,

where P is a minimal parabolic subgroup of G and $P = LN$ is its Levi decomposition.

Remark 2.2. (1) For any real reductive Lie group, we have $\#(N\backslash G/P) < \infty$ by the Bruhat decomposition (for example, see [4, Thm. 7.40]). Therefore Theorem 2.1 implies that the number of N-orbits on G/P is finite but the number of $N_{\mathbb{C}}$-orbits on $G_{\mathbb{C}}/P_{\mathbb{C}}$ is infinite if $\mathfrak{l}$ is not abelian.

(2) In Fact 1.4 and Theorem 2.1, we consider the orbit decompositions of $H_{\mathbb{C}}$ on $G_{\mathbb{C}}/B$ and $G_{\mathbb{C}}/P_{\mathbb{C}}$, respectively. Note that $\mathfrak{l}$ is abelian if and only if $P_{\mathbb{C}} = B$ holds.

(3) Let G be the special indefinite unitary group $SU(1, n)$. Then it is pointed out that $\#(N_{\mathbb{C}}\backslash G_{\mathbb{C}}/P_{\mathbb{C}}) = \infty$ holds if $n \geq 3$ by Matsuki [9, Remark 7].

It is obvious that (i) and (iii) are equivalent by the Bruhat decomposition and that (iii) implies (ii) by (2) of Remark 1.5. Therefore, it is sufficient to prove that the implication (ii) $\Rightarrow$ (i) is true for Theorem 2.1. We give two proofs of this claim. One is a representation theoretic proof and the other is a geometric proof.

Sketch of a geometric proof. By the generalized Bruhat decomposition, we have

$$G_{\mathbb{C}} = \coprod_{w \in W/W_L} BwP_{\mathbb{C}}$$

where W and W_L are Weyl groups of $G_{\mathbb{C}}$ and $L_{\mathbb{C}}$. Because $N_{\mathbb{C}} \subset B$, $N_{\mathbb{C}}$ acts on $BwP_{\mathbb{C}}/P_{\mathbb{C}}$ for any $w \in W/W_L$. If $\mathfrak{l}$ is not abelian and G has no compact factors, we can show that there exists $w_0 \in W/W_L$ such that the dimension of every $N_{\mathbb{C}}$-orbit on $Bw_0P_{\mathbb{C}}/P_{\mathbb{C}}$ is smaller than $\dim Bw_0P_{\mathbb{C}}/P_{\mathbb{C}}$. This implies $\#(N_{\mathbb{C}}\backslash Bw_0P_{\mathbb{C}}/P_{\mathbb{C}}) = \infty$, in particular, $\#(N_{\mathbb{C}}\backslash G_{\mathbb{C}}P_{\mathbb{C}}) = \infty$ holds. □

Sketch of a representation theoretic proof. By Fact 1.6, it is sufficient to prove the following proposition.

Proposition 2.3. *Suppose that G has no compact factors and $\mathfrak{l}$ is not abelian. Then there exists a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ of one-dimensional representations of P satisfying*

$$\lim_{k\to\infty} \dim \mathrm{Hom}_G(C^\infty(G/P,\eta_k), C^\infty(G/N)) = \infty.$$

We prove this proposition by constructing concrete intertwining operators, which are differential operators in the sense of [8, Def. 2.1]. Let $R : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(C^\infty(G))$ be a differential of the right regular representation on $C^\infty(G)$ of G. In other words, for $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, define $R(X)f \in C^\infty(G)$ by

$$(R(X)f)(x) := \left.\frac{d}{dt} f(xe^{tX})\right|_{t=0}$$

where $x \in G$. Then, R extends to : $U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(C^\infty(G))$ where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. If G has no compact factors and $\mathfrak{l}$ is not abelian, we can take an element $Y \in \mathfrak{g}_{\mathbb{C}}$ satisfying the following property.

Let V_k be a subrepresentation of L on $U(\mathfrak{g})$ is generated by Y^k for any $k \in \mathbb{N}$. Then there exists a one-dimensional representation η_k of P such that $R(V_k) \subset \mathrm{Hom}_G(C^\infty(G/P,\eta_k), C^\infty(G/N))$ and $\lim_{k\to\infty} \dim R(V_k) = \infty$ hold.

Here we abbreviate $R(V_k) \subset \mathrm{Hom}_G(C^\infty(G/P,\eta_k), C^\infty(G/N))$ because the image of the restriction of $R(V_k)$ on $C^\infty(G/P,\eta_k) \subset C^\infty(G)$ is contained in $C^\infty(G/N)$ although $R(V_k)$ is a priori contained in $\mathrm{End}_{\mathbb{C}}(C^\infty(G))$. This implies Proposition 2.3. □

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