# Convex property of Wulff shapes and regularity of their convex integrands

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**Abstract**

We study the convexity of the Wulff shape. In this paper, we investigate a convex property of the Wulff shape, both locally and globally.

## 1 Introduction

Let $n \in \mathbb{N}$. We denote by $S^n$ the unit sphere with center at the origin in $\mathbb{R}^{n+1}$. Let $\gamma\colon S^n \longrightarrow \mathbb{R}_{>0}$ be a positive continuous function. For any $\nu \in S^n$, we set

$$\Gamma_{\gamma,\nu} := \{x \in \mathbb{R}^{n+1} | \langle x, \nu\rangle \leq \gamma(\nu)\},$$

where $\langle\ ,\ \rangle$ stands for the standard inner product of $\mathbb{R}^{n+1}$. The definition of the **Wulff shape** for $\gamma$ is as follows :

$$\mathcal{W}_\gamma \overset{\mathrm{def}}{=} \bigcap_{\nu \in S^n} \Gamma_{\gamma,\nu}.$$

We note that the Wulff shape for $\gamma$ is not smooth in general (examples of nonsmooth Wulff shapes are given in Section 2). $\gamma$ is called **convex integrand** if the homogeneous extension $\tilde{\gamma}$ to $\mathbb{R}^{n+1}$ of $\gamma$ is convex function. It is equivalent to the condition $\tilde{\gamma}(x+y) \leq \tilde{\gamma}(x) + \tilde{\gamma}(y)$ for any $x, y \in \mathbb{R}^{n+1}$.

By the fact which J. E. Taylor [3] proved in 1978, it is known that for a given $V > 0$, there exists a unique (up to translations) closed hypersuface $S$ which minimizes an anisotropic surface energy $E_\gamma$ among all closed piecewise smooth hypersurfaces in $\mathbb{R}^{n+1}$ enclosing the same volume $V$, and the minimizer $S$ is homothetic with the boundary of the Wulff shape for $\gamma$. Here, for a smooth hypersurface $S$ in $\mathbb{R}^{n+1}$ (when $S \subset \mathbb{R}^{n+1}$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$, $S$ is called a hypersurface in $\mathbb{R}^{n+1}$) with the outward pointing unit normal vector field $\nu$ on $S$, an anisotropic surface energy $E_\gamma(S)$ of $S$ is as follows :

$$E_\gamma(S) := \int_S \gamma(\nu(p))\ dS,$$

where dS is the $n$-dimensional volume element of $S$. More generally, even if $S$ is a piecewise smooth compact hypersurface without self-intersections, $E_\gamma(S)$ can be defined. In fact, we consider the integral $\int \gamma(\nu(p))\ dS$ over each part in which $S$ is smooth and take the sum of all integrals. Therefore, the Wulff shape for $\gamma$ is the solution of the isoperimetric problem for the functional $E_\gamma$. In the case where $\gamma \equiv 1$, the anisotropic surface energy $E_\gamma(S)$ is the usual $n$-dimensional volume of the hypersurface $S$, and the Wulff shape for $\gamma$ is the unit sphere $S^n$. When $n = 2$, $E_\gamma(S)$ is the surface area of $S$. Thus, the Wulff shape for $\gamma$ is the solution of the classical isoperimetric problem. The solution is known as a mathematical model of a soap bubble. Thus, the Wulff shape is the generalization of the solution of the classical isoperimetric problem and the research for the Wulff shape is practical.

The Wulff shape is known as a mathematical model of a crystal. For example, it is known that the crystal of a salt is a cube and the crystal of an ice is an octahedron which is surrounded by two hexagons and six rectangles normal to them [7]. In general, the Wulff shape is not smooth. In some cases, the Wulff shape is a closed surface containing straight segments, flat faces and edges like a polytope. It is known that if $\gamma$ satisfies the "convexity condition", the Wulff shape for $\gamma$ has no edges, that is, the Wulff shape for $\gamma$ is smooth (Note that we need to assume that $\gamma$ is of $C^2$ to assume the convexity condition for $\gamma$. See [1], [4] for the definition of the "convexity condition".). However, it is not natural to assume the smoothness of the Wulff shape if we regard it as a mathematical model of a crystal. Also, to research more general properties of the Wulff shape, we neither assume high regularity of $\gamma$ nor the convexity condition for $\gamma$ in this paper.

We are interested in when the Wulff shape contains straight segments and flat faces. In 2017, H. Han and T. Nishimura [2] proved that the following (a) and (b) are equivalent :

$(a)$ $\gamma$ is of $C^1$.

$(b)$ The boundary of the Wulff shape for $\gamma$ has no segments.

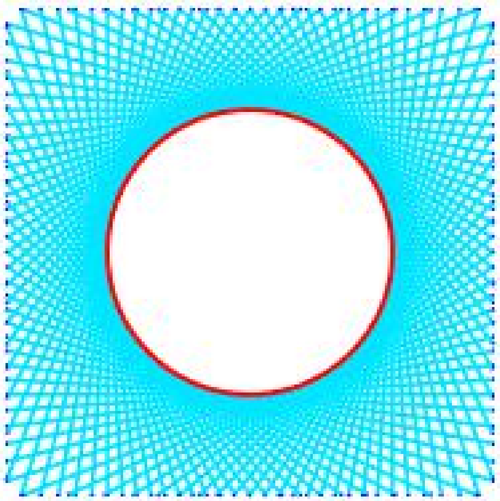
By their theorem, when $\gamma$ is not of $C^1$, the Wulff shape for $\gamma$ contains a straight segment. In this paper, we investigate more detail their relationship.

# 2 Preliminaries

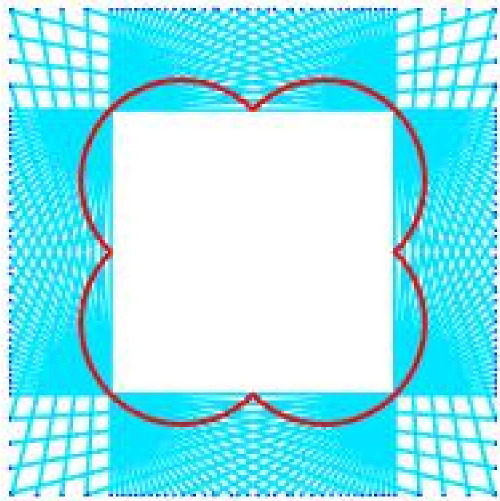
## 2.1 Examples of Wulff shapes

First, we would like to show many examples of Wulff shapes. In the following examples, the white part is the Wulff shape. The red curve is the $\gamma$-plot. Here, $\gamma$-plot is the set $\{ \gamma(\nu)\nu \in \mathbb{R}^{n+1} | \nu \in S^n\}$ The blue lines are the boundaries of the half spaces $\Gamma_{\gamma,\nu}$.

**Example 2.1.** *Let* $\gamma_i : S^1 \longrightarrow \mathbb{R}_{\geq 0},\ i = 1, 2, 3, 4, 5, 6,\ \ \nu = (\nu_1, \nu_2) \in S^1$.



$\gamma_1 \equiv 1$
the Wulff shape is a circle.



$\gamma_2(\nu) = |\nu_1| + |\nu_2|$
the Wulff shape is a square.

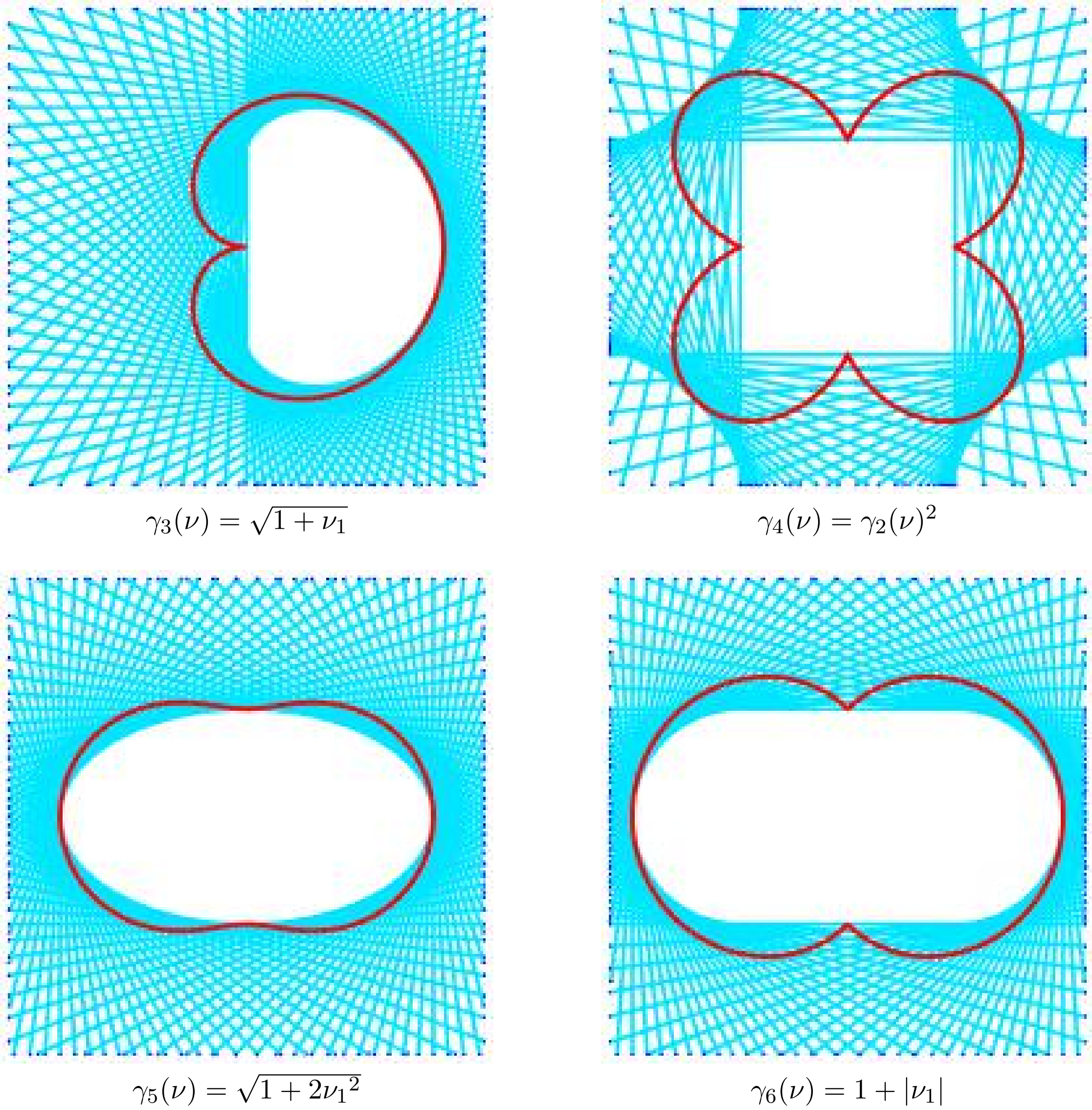



Figure 1: Examples of Wulff shapes

## 2.2 Notations and Definitions

$$\mathcal{K}_0^{n+1} := \{A \subset \mathbb{R}^{n+1} \mid A \text{ is a compact convex set containing the origin.}\}.$$

Let $\gamma\colon S^n \longrightarrow \mathbb{R}_{\geq 0}$ be a non-negative continuous function. Then, by the definition of $\mathcal{W}_\gamma$, we can verify that $\mathcal{W}_\gamma \in \mathcal{K}_0^{n+1}$. Conversely, it is known that for a given $W \in \mathcal{K}_0^{n+1}$ there exists a continuous function $\gamma\colon S^n \longrightarrow \mathbb{R}_{\geq 0}$ such that $W = \mathcal{W}_\gamma$. In fact, for a function $\gamma\colon S^n \longrightarrow \mathbb{R}_{\geq 0}$ defined by

$$\gamma(\nu) := \underset{x\in W}{max}\langle x, \nu\rangle \quad (\forall \nu \in S^n),$$

$\gamma$ is continuous and $W = \mathcal{W}_\gamma$ holds (see [3] for details on the proof). The function $\gamma\colon S^n \longrightarrow \mathbb{R}_{\geq 0}$ constructed as above is called the **convex integrand** for W and denoted by $\gamma_W$. For $W \in \mathcal{K}_0^{n+1}$ we set $C_W^0(S^n, \mathbb{R}_{\geq 0}) := \{\gamma\colon S^n \longrightarrow \mathbb{R}_{\geq 0} \mid \gamma \text{ is continuous and } W = \mathcal{W}_\gamma\}$.

An element of $C_W^0(S^n, \mathbb{R}_{\geq 0})$ is called a **support function** for $W$. As demonstrated above, we obtain $C_W^0(S^n, \mathbb{R}_{\geq 0}) \neq \phi$ (Because $\gamma_W$ is contained in it.). But, $C_W^0(S^n, \mathbb{R}_{\geq 0}) = \{\gamma_W\}$ does not hold in general, that is, a support function for a given $W \in \mathcal{K}_0^{n+1}$ is not unique.

For $W \in \mathcal{K}_0^{n+1}$ and $\nu \in S^n$, we denote by $F(W, \nu)$ the intersection of the boundary of $W$ and the boundary of $\Gamma_{\gamma_W,\nu}$ :

$$F(W, \nu) := \partial W \cap \partial \Gamma_{\gamma_W,\nu},$$

We call the set $F(W, \nu)$ the **$\nu$-way face** of W. By the following remark, $F(W, \nu)$ is a nonempty set.

**Remark 2.1.** *The following $(a)$ and $(b)$ are equivalent.*

$(a)$ $\quad \gamma$ *is the convex integrand for W.*

$(b)$ $\quad \forall \nu \in S^n \ : \ \partial W \cap \partial \Gamma_{\gamma,\nu} \neq \phi.$

**Example 2.2.** *Let $W \in \mathcal{K}_0^2$ be a square whose incircle is $S^1$. Then, for $\nu = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in S^1$ $F(W, \nu)$ is the point $(1, 1)$ that is an edge of $W$. For $\nu = (0, 1) \in S^1$, $F(W, \nu)$ is the segment joining the points $(-1, 1)$ and $(1, 1)$, that is, a side of $W$.*

**Definition 2.1.** *Let $\gamma \colon S^n \longrightarrow \mathbb{R}_{\geq 0}$.*
*Then, the homogeneous extension $\tilde{\gamma} \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}_{\geq 0}$ of $\gamma$ is defined by*

$$\tilde{\gamma}(x) = \begin{cases} \|x\| \gamma(\frac{1}{\|x\|} x), & (x \neq \mathbf{0}), \\ 0, & (x = \mathbf{0}). \end{cases}$$

**Remark 2.2.** *Let $W \in \mathcal{K}_0^{n+1}$, and $\gamma$ be the convex integrand for W. Then, the homogeneous extension $\tilde{\gamma}$ of $\gamma$ is represented as follows :*

$$\tilde{\gamma}(x) = \max_{a \in W} \langle x, a \rangle \quad (\forall x \in \ \mathbb{R}^{n+1}).$$

*Then, $\tilde{\gamma}$ is convex function.*

For $A \subset \mathbb{R}^{n+1}$, we denote by $Int(A)$ the set consisting of interior points of $A$.

**Definition 2.2.** *Let $W \subset \mathbb{R}^{n+1}$. Then, if $\overline{xy} \setminus \{x, y\} \subset Int(W)$ holds for any distinct two points $x, y \in \partial W$, we say that $W$ is* **strictly convex**.

**Example 2.3.** *A sphere is strictly convex. A cube is not strictly convex.*

For $W \in \mathcal{K}_0^{n+1}$, $W$ is strictly convex if and only if the boundary of $W$ has no segments.

## 2.3 Known Results

Now, we list the known results about convex analysis and convex geometry below.

**Fact 2.1** (cf. Schneider 1994 [5])**.** *Let $\nu \in S^n$ and $u \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$. Then the followings hold :*
*(1)* $D_u^+ \tilde{\gamma}_W(\nu) = \max_{x \in F(W,\nu)} \langle x, u \rangle,$

*(2) The set $DD\tilde{\gamma}(\nu)$ is a linear space,*

*(3) The following (a) and (b) are equivalent,*

$(a)$ $\quad \gamma$ *is differentiable at $\nu$,*

$(b)$ $\quad \tilde{\gamma}$ *is differentiable at $\nu$,* *(4) Similarly, for $r \in \mathbb{N}$ the following (a) and (b) are equivalent,*

$(a)$ *$\gamma$ is of $C^r$,*

$(b)$ *$\tilde{\gamma}$ is of $C^r$ on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$,*

*(5) the following $(a)$ and $(b)$ are equivalent,*

$(a)$ *$\tilde{\gamma}$ is differentiable on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$,*

$(b)$ *$\tilde{\gamma}$ is of $C^1$ on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$,*

*(6) The following (a) and (b) are equivalent,*

$(a)$ *$W$ is strictly convex,*

$(b)$ *$F(W,\nu) = \{\text{one point}\}$ $(\forall \nu \in S^n)$,*

*(7) the following (a) and (b) are equivalent,*

$(a)$ *$\gamma_W$ is differentiable at $\nu$,*

$(b)$ *$F(W,\nu) = \{\text{one point}\}$,*

where, $D_u^+\tilde{\gamma}_W(\nu)$ stands for the right-hand derivative of $\tilde{\gamma}_W$ with respect to the direction $u$ at $\nu$ and $DD\tilde{\gamma}(\nu)$ is the set consisting of all directions which $\tilde{\gamma}$ is differentiable at $\nu$ with respect to.

We remark that the result by H. Han and T. Nishimura is the corollary of above fact.

# 3 Main Results

We state the local detail relationship between convex property of Wulff shapes and regularity of their convex integrands.

**Theorem 3.1.** *Let $W \in \mathcal{K}_0^{n+1}$, $\nu \in S^n$. Then the following holds :*

$$DD\gamma_W(\nu) \bigoplus V_{F(W,\nu)} = \langle \nu \rangle^{\perp},$$

*where $V_{F(W,\nu)}$ is the vector space parallel to $F(W,\nu)$ in $R^{n+1}$and $DD\gamma_W(\nu) = \{\frac{dc}{dt}|_{t=0} \in \mathbb{R}^n \mid \gamma \circ c$ is differentiable at $t = 0$ for an arc $c$ on $S^n$ such that $c(0) = \nu\}$.*

**Corollary 3.1.** *The following holds :*

$$\dim F(W,\nu) = n - \dim DD\gamma_W(\nu),$$

*where $\dim F(W,\nu)$ is the dimension of the vector space $V_{F(W,\nu)}$.*

# 4 Proof of Main Results

In this section, since it immediately follows Corollary 3.1 from Theorem 3.1, we only prove Theorem 3.1. Let $\nu \in S^n$, $W \in \mathcal{K}_0^{n+1}$. We need the following lemma :

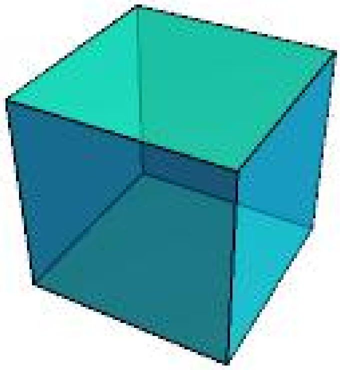
**Lemma 4.1.** *The following holds :*

$$DD\gamma_W(\nu) = DD\tilde{\gamma}_W(\nu) \cap \langle \nu \rangle^{\perp}.$$

Since it is easy to show this lemma, we skip the proof. By this lemma, it is sufficient to show that $DD\tilde{\gamma}(\nu) = V_{F(W,\nu)}{}^{\perp}$. $u \in DD\tilde{\gamma}_W(\nu)$ if and only if $D_u^+\tilde{\gamma}_W(\nu) = -D_{-u}^+\tilde{\gamma}_W(\nu)$. This equation is equivalent to the equation $\underset{x \in F(W,\nu)}{max} \langle x, u \rangle = \underset{x \in F(W,\nu)}{min} \langle x, u \rangle$. This equation holds if and only if $F(W,\nu)$ is contained a plane perpendicular to the direction $u$, it is equivalent to $u \in V_{F(W,\nu)}{}^{\perp}$. Therefore we have finished the proof of this theorem.

# 5 Examples of Main Results

**Example 5.1.**



$\gamma(\nu) = |\nu_1| + |\nu_2| + |\nu_3|$
$(\nu = (\nu_1, \nu_2, \nu_3) \in S^2)$
$W_\gamma$ *is a cube.*

*For* $\nu = (0, 0, 1) \in S^2$,
$F(W, \nu)$ *is the flat face of cube.*
$\dim F(W, \nu) = 2.$

*For* $\nu = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in S^2$,
$F(W, \nu)$ *is the segment of cube.*
$\dim F(W, \nu) = 1.$

*For* $\nu = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \in S^2$,
$F(W, \nu)$ *is the vertex of cube.*
$\dim F(W, \nu) = 0.$

## References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