Singularities of non-$\mathbb{Q}$-Gorenstein varieties

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1 Introduction

On smooth complex variety, the classical canonical sheaf makes sense, but we can consider the analogous sheaf in normal variety not necessary smooth by the following way;

Definition 1.1. Let X be a normal complex variety and $\pi : Y \to X$ be a resolution of singularities. We define the *canonical sheaf* ω_X on X by

$$\omega_X := \pi_* \omega_Y.$$

This sheaf is very difficult to deal in the general setting, but since resolutions are isomorphism in codimension one points on X, so this sheaf has good condition in codimension one. In particular, we can define the divisor corresponding to the canonical sheaf.

Definition 1.2. Let X be a normal complex variety. We define the *canonical divisor* K_X on X by

$$K_X := \sum_{E:\ \text{prime divisor}} \mathrm{ord}_E(\omega_X)$$

The canonical sheaf on smooth variety is invertible, and it means the K_X is a Cartier divisor. However, in general, K_X is not Cartier divisor if X is not smooth. Therefore, we want the following mild condition;

Definition 1.3. Let X be a normal complex variety.

- We say that K_X is $\mathbb{Q}$-*Cartier* if there exists positive integer m such that mK_X is a Cartier divisor.
- We say that X is $\mathbb{Q}$-*Gorenstein* if K_X is $\mathbb{Q}$-Cartier.

Many people studied geometric property of non $\mathbb{Q}$-Gorenstein variety, but non $\mathbb{Q}$-Gorenstein variety exists.

Example 1.4. Let X be a smooth complex projective variety and $\mathcal{L}$ be a ample line bundle on X. Then we define the *affine cone* C associated $\mathcal{L}$ by

$$C := \mathrm{Spec}(\bigoplus_m \mathrm{H}^0(X, \mathcal{L}^m)).$$

Here, C is $\mathbb{Q}$-Cartier if and only if there exists positive integers m_1 and m_2 such that $\mathcal{L}^{m_1} \simeq \mathcal{O}(m_2 K_X)$. It means that affine cones are usually not $\mathbb{Q}$-Gorenstein.

In section 2, we recall the definition of singularities and related results in $\mathbb{Q}$-Gorenstein setting. Next, in section 3, we explain the analogous singularities to those of $\mathbb{Q}$-Gorenstein varieties and results.

2 Singularities of $\mathbb{Q}$-Gorenstein varieties

In this section, we recall the definition of singularities of $\mathbb{Q}$-Gorenstein varieties. As following way, pullback of the canonical divisor naturally makes sense in $\mathbb{Q}$-Gorenstein setting.

Definition 2.1. Let X be a normal $\mathbb{Q}$-Gorenstein varieties and $\pi : Y \to X$ be a birational morphism with Y normal. We define the *pullback* of K_X by

$$\pi^* K_X = \frac{1}{m} f^*(mK_X),$$

where m is an positive integer such that mK_X is a Cartier divisor and $f^*(mK_X)$ is the pullback of Cartier divisor.

Next, we define the log discrepancy, it is very important for birational geometry.

Definition 2.2. Let X be a normal $\mathbb{Q}$-Gorenstein varieties and $\pi : Y \to X$ be a birational morphism. For prime divisor E on Y, we define the *log discrepancy* of X with respect to E by

$$a(E, X) := \mathrm{ord}_E(K_Y - f^* K_X) + 1$$

X is *log canonical* if for any birational morphism $\pi : Y \to X$ with Y normal and prime divisor E on Y, $a(E, X) \geq 1$ holds.

First we introduce the result related to MMP.

Theorem 2.3 ([OX12])**.** *Let X be a normal $\mathbb{Q}$-Gorenstein variety. Then there exists the birational morphism $\pi : Y \to X$ with Y normal such that Y satisfies the following conditions;*

- $K_Y + E^\pi$ *is π-ample, and*
- (Y, E^π) *is log canonical,*

where E^π is the sum of the prime exceptional divisors.

Next, we introduce the result related to polarized endomorphisms.

Definition 2.4. Let X be a normal projective variety and $f : X \to X$ be a surjective endomorphism. f is *polarized endomorphism* if there exists ample line bundle $\mathcal{L}$ and positive integer m such that $\mathcal{L}^m \simeq f^* \mathcal{L}$ holds.

Theorem 2.5 ([BH14]). *Let X be a normal $\mathbb{Q}$-Gorenstein projective variety admitting a non-invertible polarized endomorphism. Then X is log canonical.*

3 Singularities of non-$\mathbb{Q}$-Gorenstein varieties

In this section, we consider the following questions;

Question 3.1.

(1) How to define log canonical in non-$\mathbb{Q}$-Gorenstein setting ?
(2) Does non-$\mathbb{Q}$-Gorenstein variety have good model as in Theorem 2.3 ?
(3) Are there relationships between log canonicalness and existence of polarized endomorphism ?

The answer of Question (1) is the following;

Definition 3.2 (cf. [dFH09], [BdFF],[Has18],[Y]). Let X be a $\mathbb{Q}$-Gorenstein variety, $\pi : Y \to X$ be a birational morphism with Y normal and E be a prime divisor on Y. Then we define the following notations.

- $a^+(E, X) = \mathrm{ord}_E(K_Y) + \lim_{m\to\infty} \frac{1}{m}(\mathrm{ord}_E(\mathcal{O}_X(mK_X))) + 1$
- $a^-(E, X) = \mathrm{ord}_E(K_Y) - \lim_{m\to\infty} \frac{1}{m}(\mathrm{ord}_E(\mathcal{O}_X(-mK_X))) + 1$

Furthermore, X is pseudo-log canonical (resp. valuative log canonical) if $a^-(E, X) \geq 0$ (resp. $a^+(E, X) \geq 0$) for any $\pi : Y \to X$ be a birational morphism with Y normal and E be a prime divisor on Y.

The definitions of a^+ and a^- are very similar, but these are not same in general.
The answer of (2) is the following;

Theorem 3.3 ([Has18]). *Let X be a normal pseudo-log canonical variety. Then there exists the birational morphism $\pi : Y \to X$ with Y normal such that*

- *Y is $\mathbb{Q}$-Gorenstein log canonical and*
- *π is isomorphism in codimension one points of Y,*

where second condition is equivalent to the condition that Y has no exceptional prime divisor.

The answer of Question 3 is the following;

Theorem 3.4 ([Y]). *Let X be a normal projective variety admitting a non-invertible polarized endomorphism. Then X is valuative log canonical.*

参考文献

[BdFF] S. Boucksom, T. de Fernex, C. Favre, The volume of an isolated singularity, Duke Math. J. **161** (2012), no. 8, 1455–1520.

[BH14] A. Broustet and A. Höring, Singularities of varieties admitting an endomorphism. Math. Ann. **360** (2014), no. 1-2, 439–456.

[dFH09] T. de Fernex and C. Hacon, Singularities on normal varieties, Compos. Math. **145** (2009), no.2, 393–414.

[Has18] K. Hashizume, A class of singularity of arbitrary pairs and log canonicalizations, arXiv:1804.06326v3, preprint (2018).

[OX12] Y. Odaka, C. Xu, Log-canonical models of singular pairs and its applications, Math. Res. Lett, 19(2):325–334, 2012.

[Y] S. Yoshikawa, Singularities of non-$\mathbb{Q}$-Gorenstein varieties admitting a polarized endomorphism, preprint (2018), arXiv:1811.01795v3.