CHARACTERISTIC QUASI-POLYNOMIALS AND A CONNECTION TO EHRHART THEORY

TAN NHAT TRAN

This note contains a recent joint work with Masahiko Yoshinaga

ABSTRACT. Kamiya-Takemura-Terao proved that the cardinality of the complement of a central hyperplane arrangement with integral coefficients modulo positive integers q is a quasi-polynomial in q. They called it the characteristic quasi-polynomial as its first constituent coincides with the characteristic polynomial of the hyperplane arrangement, and left the task of understanding the other constituents to be an interesting problem. In this note, we present two interpretations for the constituents through subspace and toric arrangements. We then briefly describe the recent work of Yoshinaga on how the characteristic and Ehrhart quasi-polynomials are relevant to study the arrangements arising from root systems.

1. INTRODUCTION

When a finite list $\mathcal{A}$ of integer vectors in $\mathbb{Z}^\ell$ is given, we may naturally associate to it a *hyperplane arrangement* $\mathcal{A}(G)$ in the vector space G^ℓ with G is $\mathbb{R}$ or $\mathbb{C}$, and a *toric arrangement* $\mathcal{A}(G)$ in the torus G^ℓ with G is $\mathbb{S}^1$ or $\mathbb{C}^\times$. The study of a hyperplane/toric arrangement typically goes along with the study of its *characteristic polynomial* as the polynomial carries combinatorial and topological information of the arrangement (e.g., [OS80], [OT92], [Loo93], [DCP05]). Many attempts have been made in order to compute and to make a broader understanding the characteristic polynomials (e.g., [Ath96], [BS98], [KTT08], [ERS09], [Law11], [Moc12], [DM13], [BM14]). One of the first and classical ways is to define a polynomial in more than one variable which specializes to the characteristic polynomial. It is well-known that the characteristic polynomial of any hyperplane arrangement gets generalized to the *Tutte polynomial* [Tut54], going back to Whitney (e.g., [Sta07, Theorem 2.4]). More recently, an arithmetical generalization of the ordinary Tutte polynomial, the *arithmetic* Tutte polynomial was introduced [Moc12] to extend the analysis on the toric case. For another interesting way of generalizing the polynomials, we would encode the information of the characteristic polynomials of these arrangements in a single quasi-polynomial, the *characteristic quasi-polynomial* of $\mathcal{A}$ [KTT08]. This quasi-polynomial was defined to evaluate the cardinality of the complement of the *q-reduced arrangement* $\mathcal{A}(\mathbb{Z}_q)$ in $\mathbb{Z}_q^\ell$. Then the characteristic polynomial of the hyperplane arrangement coincides with the first constituent of the characteristic quasi-polynomial, which is known as a central result of the "finite field method" (e.g., [CR70], [Ath96], [BE97], [BS98], [KTT08]).

In a more general setting, all of the concepts mentioned above can be redefined through a finite list $\mathcal{A}$ of elements in a finitely generated abelian group Γ. The notions of hyperplane, toric and q-reduced arrangements are unified by the notion of *G-plexifications* $\mathcal{A}(G)$ [LTY17], which are defined by means of group homomorphisms from Γ to certain abelian groups G. The G-plexifications when $G = F \times (\mathbb{S}^1)^p \times \mathbb{R}^q$, with F is a finite abelian group are (non-trivial) examples of arrangements of *abelian Lie groups*. In [LTY17], their topologies provided that $q > 0$ have been described by the *G-characteristic polynomials*. Studying problems related to this specific class of abelian Lie group arrangements is a source of our interest and suffices the purpose of generalizing the previous concepts. The first result of the paper is obtained through the motivation of giving a combinatorial framework that describes the intersection patterns of $\mathcal{A}(G)$. Inspired by the pioneered work in [Moc12], we will associate to $\mathcal{A}(G)$ intersection posets and prove that its *partial* and *total* characteristic polynomials are also expressible in terms of the G-characteristic polynomials (Section 3).

Key words and phrases. G-Tutte polynomial, chromatic quasi-polynomial, characteristic quasi-polynomial, constituent, abelian Lie group arrangement, Worpitzky partition, Ehrhart quasi-polynomial.

The cardinality of the complement of $\mathcal{A}(\mathbb{Z}_q)$ appeared as an analogue of the chromatic polynomial defined on a graph, and is called the *chromatic quasi-polynomial* [BM14]. We are also interested in a question that given a constituent of a chromatic polynomial how we can describe it in connection with arrangement characteristic polynomials? Less is known, except for the first and the last. Some attempts were made to describe certain classes of the constituents appeared in [Tra18], [DFM17, §10.3.3]. The second result of the paper is two complete interpretations for the constituents through subspace and toric viewpoints. The subspace interpretation is obtained from the combinatorics of $\mathcal{A}(\mathbb{R}^{\dim(G)} \times \mathbb{Z}_k)$, while the toric interpretation is obtained from the arithmetics of $\mathcal{A}(\mathbb{S}^1)$ (or $\mathcal{A}(\mathbb{C}^\times)$) by appropriately extracting its intersection poset (Section 4).

Much of the motivation for the study of the hyperplane and toric arrangements comes from the arrangements that are defined by irreducible root systems. Apart from the theoretical aspects, the "finite field method" and its toric analogue proved to have efficient applications to compute the characteristic (quasi-)polynomials of several arrangements arising from these vector configurations (e.g., [Ath96], [BS98], [KTT07], [ACH15], [Yos18a]). More concrete computational results have also been derived to assist the observation of interesting coincidences, which we choose to mention some important examples in our study. The surprising connection between independent calculations on the Ehrhart quasi-polynomials [Sut98] and the characteristic quasi-polynomials [KTT07] produced a main flavor to the analysis on the deformations of root system arrangements in [Yos18b] (Section 5). Combining the computation on the arithmetic Tutte polynomials of classical root systems [ACH15] with the previously mentioned calculations provided the authors in [LTY17] with the key observation of the identification between the last constituent of the characteristic quasi-polynomial and the corresponding toric arrangement.

The remainder of the paper is organized as follows. In Section 2, we recall definitions and basic facts of the generalized toric arrangements, G-plexifications with the associated G-Tutte polynomials, and chromatic quasi-polynomials. In Section 3, we define for $\mathcal{A}(G)$ with $G = F \times (\mathbb{S}^1)^p \times \mathbb{R}^q$ the total and partial intersection posets, and express the corresponding characteristic polynomials in terms of G-characteristic polynomials (Theorem 3.7, Corollary 3.8). In Section 4, we obtain the subspace interpretation (Corollary 4.1) for every constituent of a chromatic quasi-polynomial immediately from the preceding section. The toric interpretation (Theorem 4.6) is derived from an important arithmetical result on the intersection poset (Lemma 4.4). In Section 5, we describe the Worpitzky partition (Theorem 5.1) introduced by Yoshinaga and show the relation between the characteristic and Ehrhart quasi-polynomials (Theorem 5.2).

This report is written based on the preprint [TY18], to which the interested reader is suggested to refer for many details and for all the proofs, which are omitted here.

2. Preliminaries

Let us first fix some definitions and notations throughout the paper. Let Γ be a finitely generated abelian group, and let $\mathcal{A} \subseteq \Gamma$ be a finite list (multiset) of elements in Γ. For each sublist $\mathcal{S} \subseteq \Gamma$, we denote by $r_{\mathcal{S}}$ the *rank* (as an abelian group) of the subgroup $\langle \mathcal{S} \rangle \le \Gamma$ generated by $\mathcal{S}$. Given a group K, denote by K_{tor} the torsion subgroup of K. Denote $\mathcal{S}^{\mathrm{tor}} := \mathcal{S} \cap \Gamma_{\mathrm{tor}}$.

2.1. **Generalized toric arrangements.** Let $(\mathcal{P}, \le_{\mathcal{P}})$ be a finite poset. The Möbius function $\mu_{\mathcal{P}}$ of $\mathcal{P}$ is the function $\mu_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \longrightarrow \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(a,b) := \begin{cases} 0 & \text{if } a \not\le_{\mathcal{P}} b, \\ 1 & \text{if } a =_{\mathcal{P}} b, \\ -\sum_{a \le c < b} \mu_{\mathcal{P}}(a,c) & \text{if } a <_{\mathcal{P}} b. \end{cases}$$

A poset $\mathcal{P}$ is said to be *ranked* if for every $a \in \mathcal{P}$, all maximal chains among those with a as greatest element have the same length, denoted this common number by $\mathrm{rk}_{\mathcal{P}}(a)$.

Now we briefly recall what has been known on combinatorics of generalized toric arrangements following [Moc12, §5]. Set $T := \mathrm{Hom}(\Gamma, G)$ with G is either $\mathbb{S}^1$ or $\mathbb{C}^\times$. Each $\alpha \in \mathcal{A}$ determines the subvariety of T as follows $H_\alpha := \{\varphi \in T \mid \varphi(\alpha) = 1\}$. The collection $\mathcal{T}(\mathcal{A}) := \{H_\alpha \mid \alpha \in \mathcal{A}\}$ is called the *generalized toric arrangement* defined by $\mathcal{A}$ on T. In particular, when Γ is free, T is a torus and $\mathcal{T}(\mathcal{A})$ is called the *toric arrangement*. To describe the combinatorics of $\mathcal{T}(\mathcal{A})$, we associate to it an *intersection poset* $L_{\mathcal{T}(\mathcal{A})}$, which is the set of all the connected components of all the intersections of the subvarieties H_α. The poset $L_{\mathcal{T}(\mathcal{A})}$ is ranked by the dimension of its elements (*layers*). The combinatorics is encoded in the *characteristic polynomial* defined by

$$\chi^{\mathrm{toric}}_{\mathcal{A}}(t) := \sum_{\mathcal{C} \in L_{\mathcal{T}(\mathcal{A})}} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})},$$

where $T^{\mathcal{C}}$ is the connected component of T that contains $\mathcal{C}$. To compute $\chi^{\mathrm{toric}}_{\mathcal{A}}(t)$ (in the same way as the Whitney's theorem showing how the characteristic polynomial of a hyperplane arrangement is computed by the Tutte polynomial), Moci introduced the *arithmetic Tutte polynomial*

$$T^{\mathrm{arith}}_{\mathcal{A}}(x,y) := \sum_{\mathcal{S} \subseteq \mathcal{A}} \#(\Gamma/\langle \mathcal{S} \rangle)_{\mathrm{tor}} (x-1)^{r_{\mathcal{A}} - r_{\mathcal{S}}} (y-1)^{\#\mathcal{S} - r_{\mathcal{S}}}.$$

Theorem 2.1 ([Moc12]). *If Γ is free and $0_\Gamma \notin \mathcal{A}$ (or even if Γ is arbitrary with $\mathcal{A}^{\mathrm{tor}} = \emptyset$), then*

$$\chi^{\mathrm{toric}}_{\mathcal{A}}(t) = (-1)^{r_{\mathcal{A}}} \cdot t^{r_\Gamma - r_{\mathcal{A}}} \cdot T^{\mathrm{arith}}_{\mathcal{A}}(1-t, 0).$$

2.2. **G-plexifications.** Let G be an arbitrary abelian group. We recall the notions of G-plexifications and G-Tutte polynomials of $\mathcal{A}$ following [LTY17, §3]. We regard $T = \mathrm{Hom}(\Gamma, G)$ as our total group. For each $\alpha \in \mathcal{A}$, we define the *G-hyperplane* associated to α as follows:

$$H_{\alpha,G} := \{\varphi \in T \mid \varphi(\alpha) = 0\} \leq T.$$

Then the *G-plexification* $\mathcal{A}(G)$ of $\mathcal{A}$ is the collection of the subgroups $H_{\alpha,G}$

$$\mathcal{A}(G) := \{H_{\alpha,G} \mid \alpha \in \mathcal{A}\}.$$

The *G-complement* $\mathcal{M}(\mathcal{A}; \Gamma, G)$ of $\mathcal{A}(G)$ is defined by

$$\mathcal{M}(\mathcal{A}; \Gamma, G) := T \smallsetminus \bigcup_{\alpha \in \mathcal{A}} H_{\alpha,G}.$$

In what follows, we assume further that G is torsion-wise finite i.e., $G[d] := \{x \in G \mid d \cdot x = 0\}$ is finite for all $d \in \mathbb{Z}_{>0}$. The *G-multiplicity* $m(\mathcal{S}; G)$ for each $\mathcal{S} \subseteq \mathcal{A}$ is defined by

$$m(\mathcal{S}; G) := \# \mathrm{Hom}\left((\Gamma/\langle \mathcal{S} \rangle)_{\mathrm{tor}}, G\right).$$

Definition 2.2.

(1) The *G-Tutte polynomial* $T^G_{\mathcal{A}}(x,y)$ of $\mathcal{A}$ is defined by

$$T^G_{\mathcal{A}}(x,y) := \sum_{\mathcal{S} \subseteq \mathcal{A}} m(\mathcal{S}; G)(x-1)^{r_{\mathcal{A}} - r_{\mathcal{S}}} (y-1)^{\#\mathcal{S} - r_{\mathcal{S}}}.$$

(2) The *G-characteristic polynomial* $\chi^G_{\mathcal{A}}(t)$ of $\mathcal{A}$ is defined by

$$\chi^G_{\mathcal{A}}(t) := (-1)^{r_{\mathcal{A}}} \cdot t^{r_\Gamma - r_{\mathcal{A}}} \cdot T^G_{\mathcal{A}}(1-t, 0).$$

Proposition 2.3. *The leading coefficient of $\chi^G_{\mathcal{A}}(t)$ equals $\#\mathcal{M}(\mathcal{A}^{\mathrm{tor}}; \Gamma_{\mathrm{tor}}, G)$.*

Various specializations of the G-plexifications and G-Tutte polynomials have appeared in the literature which we refer the reader to [LTY17] for more details. In particular, the hyperplane arrangements and generalized toric arrangements are G-plexifications by viewing $G = \mathbb{R}$ (or $G = \mathbb{C}$) and $G = \mathbb{S}^1$ (or $G = \mathbb{C}^\times$), respectively.

2.3. **Chromatic quasi-polynomials.** For each $\mathcal{S} \subseteq \mathcal{A}$, by the Structure Theorem, we may write $\Gamma/\langle\mathcal{S}\rangle \simeq \bigoplus_{i=1}^{n_\mathcal{S}} \mathbb{Z}_{d_{\mathcal{S},i}} \oplus \mathbb{Z}^{r_\Gamma - r_\mathcal{S}}$ where $n_\mathcal{S} \geq 0$ and $1 < d_{\mathcal{S},i} | d_{\mathcal{S},i+1}$. The *LCM-period* $\rho_\mathcal{A}$ of $\mathcal{A}$ is defined by

$$\rho_\mathcal{A} := \operatorname{lcm}(d_{\mathcal{S},n_\mathcal{S}} \mid \mathcal{S} \subseteq \mathcal{A}).$$

It is proved in [BM14] that $\#\mathcal{M}(\mathcal{A};\Gamma,\mathbb{Z}_q)$ is a quasi-polynomial in $q \in \mathbb{Z}_{>0}$ for which $\rho_\mathcal{A}$ is a period. The quasi-polynomial is called the *chromatic quasi-polynomial* of $\mathcal{A}$, and denoted by $\chi^{\text{quasi}}_\mathcal{A}(q)$. More precisely, there exist polynomials $f^k_\mathcal{A}(t) \in \mathbb{Z}[t]$ $(1 \leq k \leq \rho_\mathcal{A})$, called the *$k$-constituents*, such that

$$\#\mathcal{M}(\mathcal{A};\Gamma,\mathbb{Z}_q) = f^k_\mathcal{A}(q) \ \text{ if } q \equiv k \bmod \rho_\mathcal{A}.$$

The chromatic quasi-polynomial is precisely the $\mathbb{Z}_q$-characteristic polynomial in variable q, i.e., $\chi^{\text{quasi}}_\mathcal{A}(q) = \chi^{\mathbb{Z}_q}_\mathcal{A}(q)$ (e.g., [LTY17, Theorem 5.4]), and also the Chen-Wang's quasi-polynomial [Tra18]. In particular, when $\Gamma = \mathbb{Z}^\ell$, the $\mathbb{Z}_q$-plexification is the q-reduced arrangement defined on $\mathcal{A}$ with $\chi^{\text{quasi}}_\mathcal{A}(q)$ is the characteristic quasi-polynomial in the sense of [KTT08]. A partial description of the constituents in connection with arrangement theory is known.

Theorem 2.4 ([Tra18])**.**

$$f^1_\mathcal{A}(t) = \begin{cases} 0 & \text{if } \mathcal{A}^{\text{tor}} \neq \emptyset, \\ \chi_{\mathcal{A}(\mathbb{R})}(t) & \text{if } \mathcal{A}^{\text{tor}} = \emptyset. \end{cases}$$

Theorem 2.5 ([LTY17])**.** *Assume that Γ is free and $0_\Gamma \notin \mathcal{A}$. Then*

$$f^{\rho_\mathcal{A}}_\mathcal{A}(t) = \chi^{\text{toric}}_\mathcal{A}(t).$$

3. The combinatorics

Unless otherwise stated, throughout this section, we assume that G is an abelian Lie group with finitely many connected components i.e. $G \simeq (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ with $g := \dim(G) = p + q \geq 0$ and F is a finite abelian group. For each $\mathcal{S} \subseteq \mathcal{A}$, by [LTY17, Proposition 3.6], we have

$$\begin{aligned} H_{\mathcal{S},G} &:= \bigcap_{\alpha\in\mathcal{S}} H_{\alpha,G} \\ &\simeq \operatorname{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\text{tor}}, G) \times F^{r_\Gamma - r_\mathcal{S}} \times \left((\mathbb{S}^1)^p \times \mathbb{R}^q\right)^{r_\Gamma - r_\mathcal{S}}. \end{aligned} \tag{3.1}$$

We agree that $T := H_{\emptyset,G}$. Each connected component of $H_{\mathcal{S},G}$ is isomorphic to $\left((\mathbb{S}^1)^p \times \mathbb{R}^q\right)^{r_\Gamma - r_\mathcal{S}}$. If either $r_\Gamma = 0$ or $g = 0$, it can be identified with a point. The set of the connected components of $H_{\mathcal{S},G}$ is denoted by $\operatorname{cc}(H_{\mathcal{S},G})$. The following lemma is somewhat more general than [Moc12, Lemma 5.4].

Lemma 3.1. $\#\operatorname{cc}(H_{\mathcal{S},G}) = m(\mathcal{S};G) \cdot (\#F)^{r_\Gamma - r_\mathcal{S}}$.

Most of the main concepts in this section are defined by inspiration of [Moc12, §5] and [LTY17, §7].

Definition 3.2.

(1) The *total* intersection poset of $\mathcal{A}(G)$ is defined by

$$L = L^{\text{tot}}_{\mathcal{A}(G)} := \{\text{connected components of nonempty } H_{\mathcal{S},G} \mid \mathcal{S} \subseteq \mathcal{A}\},$$

whose elements, called *layers*, are ordered by reverse inclusion ($\mathcal{D} \leq_L \mathcal{C}$ if $\mathcal{D} \supseteq \mathcal{C}$).

(2) The *total* characteristic polynomial of $\mathcal{A}(G)$ is defined by

$$\chi^{\text{tot}}_{\mathcal{A}(G)}(t) := \sum_{\mathcal{C}\in L} \mu(T^\mathcal{C}, \mathcal{C}) t^{\dim(\mathcal{C})}.$$

Here $T^\mathcal{C}$ is the connected component of T that contains $\mathcal{C}$, and $\mu := \mu_L$.

The set of minimal elements of L is exactly $\mathrm{cc}(T)$. The connected components of $H_{\mathcal{A},G}$ are maximal elements of L but the converse is not necessarily true. For each $\mathcal{C} \in L$, set

$$\mathcal{R}(\mathcal{C}) := \{\mathcal{S} \subseteq \mathcal{A} \mid \mathcal{C} \in \mathrm{cc}(H_{\mathcal{S},G})\}.$$

One observes that $\dim(\mathcal{C}) = \dim(H_{\mathcal{S},G}) = g(r_\Gamma - r_\mathcal{S})$ for every $\mathcal{S} \in \mathcal{R}(\mathcal{C})$. The *localization* of $\mathcal{A}$ with respect to $\mathcal{C}$ is defined by

$$\mathcal{A}_\mathcal{C} := \{\alpha \in \mathcal{A} \mid \mathcal{C} \subseteq H_{\alpha,G}\}.$$

Stated differently, $\mathcal{A}_\mathcal{C}$ is the unique maximal element of $\mathcal{R}(\mathcal{C})$ in the sense that $\mathcal{S} \subseteq \mathcal{A}_\mathcal{C}$ for every $\mathcal{S} \in \mathcal{R}(\mathcal{C})$. We also can write

$$\mathcal{R}(\mathcal{C}) = \{\mathcal{S} \subseteq \mathcal{A}_\mathcal{C} \mid r_\mathcal{S} = r_{\mathcal{A}_\mathcal{C}}\}. \tag{3.2}$$

Thus L is a ranked poset with a rank function is given by $\mathrm{rk}_L(\mathcal{C}) := r_{\mathcal{A}_\mathcal{C}} = \mathrm{codim}(\mathcal{C})/g$ for $\mathcal{C} \in L$.

We are interested in a particular subset of $\mathrm{cc}(T)$,

$$\begin{aligned}\mathrm{scc}(T) &:= \{T_i \in \mathrm{cc}(T) \mid (\mathcal{A}_{T_i})^{\mathrm{tor}} = \emptyset\} \\ &= \mathrm{cc}(T) \smallsetminus \bigcup_{\alpha \in \mathcal{A}^{\mathrm{tor}}} \mathrm{cc}(H_{\alpha,G}).\end{aligned}$$

By using the Inclusion-Exclusion principle,

$$\#\mathrm{scc}(T) = \#\mathcal{M}(\mathcal{A}^{\mathrm{tor}}; \Gamma_{\mathrm{tor}}, G) \cdot (\#F)^{r_\Gamma}. \tag{3.3}$$

Definition 3.3.

(1) The *partial* intersection poset of $\mathcal{A}(G)$ is defined by

$$L^{\mathrm{par}} := \{\mathcal{C} \in L \mid T^\mathcal{C} \in \mathrm{scc}(T)\},$$

with the Möbius function of L^{par} is the restriction of μ i.e., $\mu_{L^{\mathrm{par}}} = \mu|_{L^{\mathrm{par}} \times L^{\mathrm{par}}}$.

(2) The *partial* characteristic polynomial of $\mathcal{A}(G)$ is defined by

$$\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) := \sum_{\mathcal{C} \in L^{\mathrm{par}}} \mu(T^\mathcal{C}, \mathcal{C}) t^{\dim(\mathcal{C})}.$$

In other words, L^{par} is the dual order ideal (e.g., [Sta11, §3.1]) of L generated by $\mathrm{scc}(T)$. It follows from the definition above that $\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) = 0$ if $\mathrm{scc}(T) = \emptyset$.

Remark 3.4. Removing from $\mathcal{A}(G)$ the hyperplanes $H_{\alpha,G}$ with $\alpha \in \mathcal{A}^{\mathrm{tor}}$ does not affect the structure of the poset i.e.,

$$L_{\mathcal{A}(G)} = L_{(\mathcal{A} \smallsetminus \mathcal{A}^{\mathrm{tor}})(G)} = L^{\mathrm{par}}_{(\mathcal{A} \smallsetminus \mathcal{A}^{\mathrm{tor}})(G)}.$$

As a consequence,

$$\chi^{\mathrm{tot}}_{\mathcal{A}(G)}(t) = \chi^{\mathrm{tot}}_{(\mathcal{A} \smallsetminus \mathcal{A}^{\mathrm{tor}})(G)}(t) = \chi^{\mathrm{par}}_{(\mathcal{A} \smallsetminus \mathcal{A}^{\mathrm{tor}})(G)}(t).$$

In particular, $\chi^{\mathrm{tot}}_{\mathcal{A}(G)}(t) = \chi^{\mathrm{par}}_{\mathcal{A}(G)}(t)$ if $\mathcal{A}^{\mathrm{tor}} = \emptyset$.

In the lemma below, we generalize the result in [Moc12, Lemma 5.5] as we include the possibility $\mathcal{A}^{\mathrm{tor}} \neq \emptyset$.

Lemma 3.5. *If $\mathcal{C} \in L$, then*

$$\sum_{\mathcal{S} \in \mathcal{R}(\mathcal{C})} (-1)^{\#\mathcal{S}} = \begin{cases} \mu(T^\mathcal{C}, \mathcal{C}) & \text{if } \mathcal{C} \in L^{\mathrm{par}}, \\ 0 & \text{if } \mathcal{C} \notin L^{\mathrm{par}}. \end{cases}$$

Corollary 3.6. *The Möbius function of L strictly alternates in sign. That is, for all $\mathcal{C} \in L$,*

$$(-1)^{\mathrm{rk}_L(\mathcal{C})} \mu(T^\mathcal{C}, \mathcal{C}) > 0.$$

Theorem 3.7. *Let $G \simeq (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ with $g = p + q \in \mathbb{Z}_{>0}$. Then*

$$\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) = \chi^G_\mathcal{A}(\#F \cdot t^g).$$

Corollary 3.8. *Let $G \simeq (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ with $g = p+q \in \mathbb{Z}_{>0}$. Then*

$$\chi^{\text{tot}}_{\mathcal{A}(G)}(t) = \chi^{G}_{\mathcal{A}\smallsetminus\mathcal{A}^{\text{tor}}}(\#F \cdot t^g).$$

Remark 3.9. Although either Theorem 3.7 or Corollary 3.8 may not be valid when $g = 0$, there is no loss of information in these formulations. Namely, $\chi^{\text{par}}_{\mathcal{A}(G)}(t) = \#\text{scc}(T)$, and by equality (3.3) and Proposition 2.3, this equals the "leading part" of $\chi^G_{\mathcal{A}}(\#F)$ (the value of the leading term of $\chi^G_{\mathcal{A}}(t)$ evaluated at $\#F$). Similarly, $\chi^{\text{tot}}_{\mathcal{A}(G)}(t) = \#\text{cc}(T)$, which is equal to the leading part of $\chi^G_{\mathcal{A}\smallsetminus\mathcal{A}^{\text{tor}}}(\#F)$.

Remark 3.10. Note that when $G = \mathbb{S}^1$ (or $G = \mathbb{C}^\times$ if the dimension is defined over $\mathbb{C}$) and $\mathcal{A}^{\text{tor}} = \emptyset$, $\chi^{\text{par}}_{\mathcal{A}(G)}(t) = \chi^{\text{tot}}_{\mathcal{A}(G)}(t) = \chi^{\text{toric}}_{\mathcal{A}}(t)$. The result of Moci (Theorem 2.1) is a special case of Corollary 3.8.

4. The constituents

4.1. **Via subspace viewpoint.** Our first result in this section is the interpretation for chromatic polynomials and their constituents through the real subspace arrangement viewpoint. Combining Theorem 3.7 with the property of the chromatic polynomials (e.g., [Tra18, Proposition 3.6]), we obtain

Corollary 4.1. *Let $G = \mathbb{R}^g \times \mathbb{Z}_k$ with $g > 0$ and $1 \le k \le \rho_{\mathcal{A}}$. Then*

$$\chi^{\text{par}}_{\mathcal{A}(G)}(t) = f^k_{\mathcal{A}}(k \cdot t^g).$$

Let us explain Corollary 4.1 in more detail. For nontriviality, we assume that $\text{scc}(T) \neq \emptyset$ (e.g., when $\mathcal{A}^{\text{tor}} = \emptyset$), and $r_\Gamma > 0$. Each connected component of $T = \text{Hom}(\Gamma, \mathbb{R}^g \times \mathbb{Z}_k)$ is isomorphic to $\mathbb{R}^{gr_\Gamma}$. For each $T_i \in \text{scc}(T)$, the poset $L_i = \{\mathcal{C} \in L \mid \mathcal{C} \subseteq T_i\}$ is isomorphic to the total (or equivalently, partial) intersection poset of a $\mathbb{R}^g$-plexification $\mathcal{G}_i$ in $\mathbb{R}^{gr_\Gamma}$ (or *g-plexification* in the sense of [Bjö94, §5.2])), with each $\mathcal{G}_i$ is possibly empty and defined over the integers. Thus after a rescaling of variable, each constituent records the summation of the total characteristic polynomials of the $\mathcal{G}_i$'s i.e.,

$$f^k_{\mathcal{A}}(kt^q) = \sum_{T_i \in \text{scc}(T)} \chi^{\text{tot}}_{\mathcal{G}_i}(t). \tag{4.1}$$

Remark 4.2. In particular, when $g = 1$, each $\mathcal{G}_i$ becomes an integral hyperplane arrangement $\mathcal{H}_i$ [Tra18, Proposition 4.5]. The conclusion related to the first constituent ($k = 1$) in Corollary 4.1 is the same as that stated in Theorem 2.4. In particular, if $\Gamma = \mathbb{Z}^\ell$, each hyperplane $H_{\alpha,\mathbb{R}\times\mathbb{Z}_k}$ in T can be identified with $H_{\alpha,\mathbb{R}} \times H_{\alpha,\mathbb{Z}_k}$ in $\mathbb{R}^\ell \times \mathbb{Z}^\ell_k$. Each arrangement $\mathcal{H}_i$ turns out to be a subarrangement of $\mathcal{A}(\mathbb{R})$, and in which components of T that the components of $H_{\alpha,\mathbb{R}\times\mathbb{Z}_k}$ locate depends on the arithmetics of the list $\mathcal{A}$.

4.2. **Via toric viewpoint.** We may expect that if there exists a "nicer" expression to describe every constituent without making any rescaling of variable. It turns out that such expression can be obtained from the toric arrangement by appropriately extracting its poset of layers. Now let us turn to the second interpretation via toric arrangement viewpoint. In the remainder of this section, we assume that G is either $\mathbb{S}^1$ or $\mathbb{C}^\times$. We retain the notation of the total group $T = \text{Hom}(\Gamma, G)$, with the identity is denoted by $\mathbf{1}$. For each $k \in \mathbb{Z}$, consider the homomorphism

$$E_k : T \longrightarrow T \quad \text{via} \quad \varphi \mapsto \varphi^k := \varphi \cdots \varphi.$$

Definition 4.3.

(1) For each $k \in \mathbb{Z}$, the *k-total* intersection poset of $\mathcal{A}(G)$ is defined by

$$L[k] = \{\mathcal{C} \in L \mid \mathbf{1} \in E_k(\mathcal{C})\}.$$

(2) The *k-total* characteristic polynomial of $\mathcal{A}(G)$ is defined by

$$\chi^{k\text{-tot}}_{\mathcal{A}(G)}(t) := \sum_{\mathcal{C} \in L[k]} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})}.$$

The cover relation in L is preserved in $L[k]$ i.e., if $\mathcal{C}$ covers $\mathcal{D}$ in L and $\mathcal{C} \in L[k]$ then $\mathcal{D} \in L[k]$, which implies that $L[k]$ is an *order ideal* (e.g., [Sta11, §3.1]). For each $\mathcal{S} \subseteq \mathcal{A}$, note that $H_{\mathcal{S},G}$ is a subtorus of T whose each connected component is isomorphic to the torus $G^{r_\Gamma - r_\mathcal{S}}$. Let $\mathcal{C}^{\mathbf{1}}_{\mathcal{S}} \in \mathrm{cc}(H_{\mathcal{S},G})$ be the *identity component* of $H_{\mathcal{S},G}$, that is, the connected component that contains $\mathbf{1}$. Thus $\mathrm{cc}(H_{\mathcal{S},G})$ can be identified with the quotient group $H_{\mathcal{S},G}/\mathcal{C}^{\mathbf{1}}_{\mathcal{S}}$. In the lemma below, we generalize [Moc12, Lemma 5.4] in an arithmetical manner.

Lemma 4.4. *Fix $k \in \mathbb{Z}_{>0}$. For each $\mathcal{S} \subseteq \mathcal{A}$, we have*

$$\# \left(\mathrm{cc}(H_{\mathcal{S},G}) \cap L[k]\right) = m(\mathcal{S}; \mathbb{Z}_k).$$

Definition 4.5.

(1) For each $k \in \mathbb{Z}$, the *k-partial* intersection poset of $\mathcal{A}(G)$ is defined by

$$L^{\mathrm{par}}[k] := \{\mathcal{C} \in L^{\mathrm{par}} \mid \mathbf{1} \in E_k(\mathcal{C})\}.$$

(2) The *k-partial* characteristic polynomial of $\mathcal{A}(G)$ is defined by

$$\chi^{k\text{-par}}_{\mathcal{A}(G)}(t) := \sum_{\mathcal{C} \in L^{\mathrm{par}}[k]} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})}.$$

Theorem 4.6. *If $q \in \mathbb{Z}_{>0}$, then*

$$\chi^{q\text{-par}}_{\mathcal{A}(G)}(q) = \chi^{\mathrm{quasi}}_{\mathcal{A}}(q).$$

Corollary 4.7. *If $1 \le k \le \rho_\mathcal{A}$, then*

$$\chi^{k\text{-par}}_{\mathcal{A}(G)}(t) = f^k_{\mathcal{A}}(t).$$

Corollary 4.8. *If $q \in \mathbb{Z}_{>0}$ and $1 \le k \le \rho_\mathcal{A}$, then*

$$\begin{aligned} \chi^{q\text{-tot}}_{\mathcal{A}(G)}(q) &= \chi^{\mathrm{quasi}}_{\mathcal{A} \smallsetminus \mathcal{A}^{\mathrm{tor}}}(q), \\ \chi^{k\text{-tot}}_{\mathcal{A}(G)}(t) &= f^k_{\mathcal{A} \smallsetminus \mathcal{A}^{\mathrm{tor}}}(t). \end{aligned}$$

5. A CONNECTION TO EHRHART THEORY

5.1. **Root systems.** Let $V := \mathbb{R}^\ell$ with $\ell > 0$ be the Euclidean space with the standard inner product $(\cdot,\cdot)$. Let Φ be an irreducible (crystallographic) root system in V with the exponents $e_1, \dots, e_\ell$, the Coxeter number h and the Weyl group W. For any integer $k \in \mathbb{Z}$ and $\alpha \in \Phi$, the *affine hyperplane* $\widetilde{H}_{\alpha,k}$ is defined by

$$\widetilde{H}_{\alpha,k} := \{x \in V \mid (\alpha, x) = k\}.$$

Fix a positive system $\Phi^+ \subseteq \Phi$ and the associated set of simple roots (base) $\Delta := \{\alpha_1, \dots, \alpha_\ell\} \subseteq \Phi^+$. For $\beta = \sum_{i=1}^{\ell} n_i \alpha_i \in \Phi^+$, the *height* of β is defined by $\mathrm{ht}(\beta) := \sum_{i=1}^{\ell} n_i$. The highest root, denoted by $\tilde{\alpha} \in \Phi^+$, can be expressed uniquely as a linear combination $\tilde{\alpha} = \sum_{i=1}^{\ell} c_i \alpha_i$ ($c_i \in \mathbb{Z}_{>0}$). We also set $\alpha_0 := -\tilde{\alpha}$ and $c_0 := 1$. Then we have the linear relation

$$c_0\alpha_0 + c_1\alpha_1 + \cdots + c_\ell\alpha_\ell = 0.$$

Define the partial order $\succeq$ on $\Phi^+ \sqcup \{0\}$ such that $\beta_1 \succeq \beta_2$ (resp., $\beta_1 \succ \beta_2$) if and only if $\beta_1 - \beta_2 = \sum_{i=1}^{\ell} d_i \alpha_i$ with all $d_i \in \mathbb{Z}_{\ge 0}$ (resp., with all $d_i \in \mathbb{Z}_{\ge 0}$ and at least one $d_j \in \mathbb{Z}_{>0}$). For every $w \in W$, the *ascent* $\mathrm{asc}(w)$ and the *descent* $\mathrm{dsc}(w)$ of w are the functions $\mathrm{asc}, \mathrm{dsc} : W \to \mathbb{Z}$ defined as follows:

$$\mathrm{asc}(w) := \sum_{\substack{0 \le i \le \ell \\ w(\alpha_i) \succ 0}} c_i,$$

$$\mathrm{dsc}(w) := \sum_{\substack{0 \le i \le \ell \\ w(\alpha_i) \prec 0}} c_i.$$

Then we have the relation

$$\mathrm{asc}(w) + \mathrm{dsc}(w) = \mathrm{h}. \tag{5.1}$$

The *coweight lattice* $Z(\Phi)$ and the *coroot lattice* $\check{Q}(\Phi)$ are defined as follows:

$$Z(\Phi) := \{x \in V \mid (\alpha_i, x) \in \mathbb{Z}, \alpha_i \in \Delta\},$$
$$\check{Q}(\Phi) := \sum_{\alpha \in \Phi} \mathbb{Z} \cdot \frac{2\alpha}{(\alpha, \alpha)}.$$

The coroot lattice $\check{Q}(\Phi)$ is a subgroup of the coweight lattice $Z(\Phi)$ of finite index. The index $\#\frac{Z(\Phi)}{\check{Q}(\Phi)} =: f$ is called the *index of connection*. Let $\{\varpi_1^\vee, \ldots, \varpi_\ell^\vee\} \subseteq Z(\Phi)$ be the dual basis of the base Δ, that is, $(\alpha_i, \varpi_j^\vee) = \delta_{ij}$ (the Kronecker delta) for all $1 \le i \le \ell$. Then $Z(\Phi)$ is a free abelian group generated by $\{\varpi_1^\vee, \ldots, \varpi_\ell^\vee\}$. We also have $c_i = (\varpi_i^\vee, \tilde{\alpha})$.

A connected component of $V \smallsetminus \bigcup_{\alpha \in \Phi^+} \widetilde{H}_{\alpha,\mathbb{Z}}$ is called an *alcove*. Let us define the *fundamental alcove* A° by

$$A^\circ = \left\{ x \in V \,\middle|\, \begin{array}{c} (\alpha_i, x) > 0\,(1 \le i \le \ell), \\ (\alpha_0, x) > -1 \end{array} \right\}.$$

The closure $\overline{A^\circ} = \{x \in V \mid (\alpha_i, x) \ge 0\,(1 \le i \le \ell), (\alpha_0, x) \ge -1\}$ is a simplex, which is the convex hull of $0, \varpi_1^\vee/c_1, \ldots, \varpi_\ell^\vee/c_\ell \in V$. The supporting hyperplanes of the facets of $\overline{A^\circ}$ are $\widetilde{H}_{\alpha_1,0}, \ldots, \widetilde{H}_{\alpha_\ell,0}, \widetilde{H}_{\alpha_0,-1}$. We also note that $\overline{A^\circ}$ is a fundamental domain of the affine Weyl group $W_{\text{aff}} = W \ltimes \check{Q}(\Phi)$.

The *fundamental parallelepiped* $P^\diamondsuit$ is defined by

$$\begin{aligned} P^\diamondsuit &:= \sum_{i=1}^{\ell} (0,1]\varpi_i^\vee \\ &= \{x \in V \mid 0 < (\alpha_i, x) \le 1\,(1 \le i \le \ell)\}. \end{aligned}$$

Then $P^\diamondsuit$ is the fundamental domain of the coweight lattice $Z(\Phi)$. Denote by Ξ a finite set having the cardinality $\#\Xi = \#W/f$. Following [Yos18b], let us write $\Sigma := \{A_\xi^\circ \mid \xi \in \Xi\}$ for the set of alcoves contained in $P^\diamondsuit$, where each A_ξ° can be written uniquely as

$$A_\xi^\circ = \left\{ x \in V \,\middle|\, \begin{array}{c} (\alpha, x) > k_\alpha\,(\alpha \in I), \\ (\beta, x) < k_\beta\,(\beta \in J) \end{array} \right\},$$

for some subsets $I, J \subseteq \Phi^+$ with $\#(I \sqcup J) = \ell + 1$, and $k_\alpha, k_\beta \in \mathbb{Z}$ $(\alpha \in I, \beta \in J)$. The facets of A_ξ° are supported by the hyperplanes $\widetilde{H}_{\alpha,k_\alpha}$ $(\alpha \in I)$ and $\widetilde{H}_{\beta,k_\beta}$ $(\beta \in J)$.

Let A' be an arbitrary alcove. We can write $A' = w(A^\circ) + \gamma$ for some $w \in W$ and $\gamma \in \check{Q}(\Phi)$. The value $\operatorname{asc}(w)$ is uniquely determined among all $w \in W$ having this property [Yos18b, Lemma 4.3(2)]. Then we can extend asc as a function $\Sigma \to \mathbb{Z}$ by setting:

$$\operatorname{asc}(A') := \operatorname{asc}(w).$$

For each $A_\xi^\circ \in \Sigma$, we also define the *partial closure* $A_\xi^\diamondsuit$ of A_ξ° as follows:

$$A_\xi^\diamondsuit := \left\{ x \in V \,\middle|\, \begin{array}{c} (\alpha, x) > k_\alpha\,(\alpha \in I), \\ (\beta, x) \le k_\beta\,(\beta \in J) \end{array} \right\}. \tag{5.2}$$

Theorem 5.1 (Worpitzky partition)**.**

$$P^\diamondsuit = \bigsqcup_{\xi \in \Xi} A_\xi^\diamondsuit.$$

As a consequence, for any $q \in \mathbb{Z}_{>0}$,

$$qP^\diamondsuit \cap Z(\Phi) = \bigsqcup_{\xi \in \Xi} qA_\xi^\diamondsuit \cap Z(\Phi).$$

The partition of $P^\diamondsuit$ above is probably well-known among experts, for instance [Hum90, Exercise 4.3]. Yoshinaga [Yos18b, Proposition 2.5] named the latter *Worpitzky partition* as it recovers the classical *Worpitzky identity* when Φ is of type A_ℓ.

5.2. **Connection with lattice point counting.** Let Ψ be a subset of Φ^+. We assume that an $\ell \times \#\Psi$ integral matrix $S_\Psi = [S_{ij}]$ satisfies

$$\Psi = \left\{ \sum_{i=1}^{\ell} S_{ij}\alpha_i \,\middle|\, 1 \le j \le \#\Psi \right\}.$$

In other words, S_Ψ is a coefficient matrix of Ψ with respect to the base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Set $S := S_{\Phi^+}$. In our language as in the previous sections, we can also consider S_Ψ as a finite list of vectors in $\Gamma = \mathbb{Z}^\ell$ whose elements are nothing but the columns of S_Ψ.
Notation: We will denote the list in $\mathbb{Z}^\ell$ corresponding to S_Ψ by $\mathcal{A}_\Psi$, and denote $\mathcal{A} := \mathcal{A}_{\Phi^+}$. For example, if $\Phi = B_2$, then the matrix $S = S_{\Phi^+}$ is given by

$$S = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix},$$

and $\mathcal{A} = \{(1,0),(0,1),(1,1),(1,2)\} \subseteq \mathbb{Z}^2$.
Denote $\mathbb{Z}_q^\times := \mathbb{Z}_q \smallsetminus \{\overline{0}\}$. The $\mathbb{Z}_q$*-complement* $\mathcal{M}(\mathcal{A}_\Psi; \mathbb{Z}^\ell, \mathbb{Z}_q)$ can be formulated as

$$\mathcal{M}(\mathcal{A}_\Psi; \mathbb{Z}^\ell, \mathbb{Z}_q) = \{\mathbf{z} \in \mathbb{Z}_q^\ell \mid \mathbf{z} \cdot S_\Psi \in (\mathbb{Z}_q^\times)^{\#\Psi}\}.$$

Associating to each subset $\Psi \subseteq \Phi^+$, we define the real *arrangement* $\mathcal{H}_\Psi := \{H_\alpha \mid \alpha \in \Psi\}$, where H_α is the real hyperplane orthogonal to α. It is not hard to see that $H_\alpha \simeq H_{\alpha,\mathbb{R}} = \{x \in \mathbb{R}^\ell \mid x \cdot S_{\{\alpha\}} = 0\}$ as vector spaces, so we can view H_α as the $\mathbb{R}$-hyperplane of the element $\mathcal{A}_{\{\alpha\}}$ of the list $\mathcal{A}$. Hence $\mathcal{H}_\Psi$ is the $\mathbb{R}$-plexification of the list $\mathcal{A}_\Psi$. Note also that $\mathcal{H} := \mathcal{H}_{\Phi^+}$ is called the *Weyl arrangement* of Φ^+, and $\mathcal{H}_\Psi$ is a subarrangement of it.

Let Γ be a lattice. For any polytope $\mathcal{P}$ with vertices in the rational vector space generated by Γ, the *Ehrhart quasi-polynomial* $\mathrm{L}_{\mathcal{P}}(q)$ of $\mathcal{P}$ with respect to Γ is defined by $\mathrm{L}_{\mathcal{P}}(q) := \#(q\mathcal{P} \cap \Gamma)$ for $q \in \mathbb{Z}_{>0}$. The Ehrhart quasi-polynomials we are considering are defined with respect to the coweight lattice $\Gamma = Z(\Phi)$.

The characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathcal{A}}(q)$ of $\mathcal{A}$ has an interesting relation to the lattice point counting quasi-polynomial function $\mathrm{L}_{\overline{A^\circ}}(q)$ (e.g., [KTT10], [Yos18b, Proposition 3.7]).

Theorem 5.2.

$$\chi^{\mathrm{quasi}}_{\mathcal{A}}(q) = \frac{\#W}{f}\mathrm{L}_{A^\circ}(q) = \frac{\#W}{f}\mathrm{L}_{\overline{A^\circ}}(q - \mathrm{h}).$$

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TAN NHAT TRAN, DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, KITA 10, NISHI 8, KITA-KU, SAPPORO 060-0810, JAPAN.

E-mail address: trannhattan@math.sci.hokudai.ac.jp