Complex interpolation of B_w^u-function spaces

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1 Introduction

In this note, we inverstigate the first and second complex interplation of $\dot{B}_w^u$-spaces. Let us recall the definition of these spaces (see [4]). Let $1 \le u, p \le \infty$ and $w : (0, \infty) \to (0, \infty)$ be a nonincreasing function. For $r > 0$ denote by $B(r)$ the open ball in $\mathbb{R}^n$ centered at the origin. The space $\dot{B}_w^u(L^p)$ and $B_w^u(L^p)$ are defined to be the sets of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{\dot{B}_w^u(L^p)} \equiv \begin{cases} \left(\int_0^\infty (w(r)\|f\|_{L^p(B(r))})^u \frac{dr}{r}\right)^{\frac{1}{u}}, & \text{for } u < \infty \\ \sup\limits_{r>0} w(r)\|f\|_{L^p(B(r))}, & \text{for } u = \infty. \end{cases}$$

and

$$\|f\|_{B_w^u(L^p)} \equiv \begin{cases} \left(\int_1^\infty (w(r)\|f\|_{L^p(B(r))})^u \frac{dr}{r}\right)^{\frac{1}{u}}, & \text{for } u < \infty \\ \sup\limits_{r\ge 1} w(r)\|f\|_{L^p(B(r))}, & \text{for } u = \infty. \end{cases}$$

are finite. In order to guarantee that $\chi_{B(R)} \in \dot{B}_w^u(L^p)$, we assume that

$$\int_0^R \frac{w(r)^u |B(r)|^{u/p}}{r} \, dr < \infty \quad \text{and} \quad \int_R^\infty \frac{w(r)^u}{r} \, dr < \infty, \tag{1}$$

for every $R > 0$. Note that if $w(r) = 1$ and $u = \infty$, then $\dot{B}_w^u(L^p) = L^p$.

We now recall the definition of the complex interpolation method, introduced by A. P. Calderón (see [1, 2]). Let $\overline{S} := \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1\}$ and S be its

*joint work with Shohei Nakamura (Tokyo Metropolitan University), Yoshihiro Sawano (Tokyo Metropolitan University), and Takuya Sobukawa (Waseda University) interior. Recall that a pair (X_0, X_1) is said to be a compatible couple of Banach spaces if there exists a Hausdorff topological vector space Z such that X_0 and X_1 are subspaces of Z.

Definition 1.1 (Calderón's first complex interpolation functor)**.** Let (X_0, X_1) be a compatible couple of Banach spaces. The space $\mathcal{F}(X_0, X_1)$ is defined to be the set of all continuous functions $F : \overline{S} \to X_0 + X_1$ such that

1. $\sup\limits_{z \in \overline{S}} \|F(z)\|_{X_0+X_1} < \infty$,
2. F is holomorphic on S,
3. the functions $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0, 1$.

The norm on $\mathcal{F}(X_0, X_1)$ is defined by

$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\left\{\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|F(1 + it)\|_{X_1}\right\}.$$

Definition 1.2 (Calderón's first complex interpolation spaces)**.** Let $\theta \in (0, 1)$ and (X_0, X_1) be a compatible couple of Banach spaces. The complex interpolation space $[X_0, X_1]_\theta$ with respect to (X_0, X_1) is defined by

$$[X_0, X_1]_\theta := \{f \in X_0 + X_1 : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}$$

The norm on $[X_0, X_1]_\theta$ is defined by

$$\|f\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}.$$

The fact that $[X_0, X_1]_\theta$ is a Banach space can be seen in [2] and [1, Theorem 4.1.2]. When X_0 and X_1 are Lebesgue spaces, Calderón gave the following description of $[X_0, X_1]_\theta$.

Theorem 1.3. [2] *Let $\theta \in (0, 1)$, $1 \le p_0 \le \infty$, and $1 \le p_1 \le \infty$. Then*

$$[L^{p_0}, L^{p_1}]_\theta = L^p$$

where p is defined by

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

Note that the Riesz-Thorin complex interpolation theorem can be seen as a corollary of Theorem 1.3 and the following Calderón's result.

Theorem 1.4. [2] *Let* (X_0, X_1) *and* (Y_0, Y_1) *be two compatible couples of Banach spaces and let* $\theta \in (0,1)$. *Suppose that* T *is a bounded linear operator from* X_k *to* Y_k *for* $k = 0, 1$. *Then,* T *is bounded from* $[X_0, X_1]_\theta$ *to* $[Y_0, Y_1]_\theta$.

We now move on to the second complex interpolation method. First let us recall the definition of Banach space-valued Lipschitz continuous functions. Given a Banach spaces X. The space $\mathrm{Lip}(\mathbb{R}, X)$ is defined as the set of all functions $f : \mathbb{R} \to X$ for which

$$\|f\|_{\mathrm{Lip}(\mathbb{R},X)} := \sup_{-\infty<s<t<\infty} \frac{\|f(t) - f(s)\|_X}{|t-s|}$$

is finite.

Definition 1.5. [1, 2](Calderón's second complex interpolation functor) Let (X_0, X_1) be a compatible couple of Banach spaces. Denote by $\mathcal{G}(X_0, X_1)$ the set of all continuous functions $G : \bar{S} \to X_0 + X_1$ such that:

1. G is holomorphic on S,
2. $\sup\limits_{z\in\bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$,
3. the functions
$$t \in \mathbb{R} \mapsto G(j + it) - G(j) \in X_j$$
are Lipschitz continuous on $\mathbb{R}$ for $j = 0, 1$.

The space $\mathcal{G}(X_0, X_1)$ is equipped with the norm

$$\|G\|_{\mathcal{G}(X_0,X_1)} := \max\left\{ \|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)},\ \|G(1 + i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)} \right\}. \tag{2}$$

Definition 1.6. [1, 2](Calderón's second complex interpolation space) Let $\theta \in (0,1)$. The second complex interpolation space $[X_0, X_1]^\theta$ with respect to (X_0, X_1) is defined to be the set of all $f \in X_0 + X_1$ such that $f = G'(\theta)$ for some $G \in \mathcal{G}(X_0, X_1)$. The norm on $[X_0, X_1]^\theta$ is defined by

$$\|f\|_{[X_0,X_1]^\theta} := \inf\{\|G\|_{\mathcal{G}(X_0,X_1)} : f = G'(\theta) \text{ for some } G \in \mathcal{G}(X_0, X_1)\}.$$

2 Main results

We now state our main results. Suppose that we have 3 functions $w_0, w_1, w : (0,\infty) \to (0,\infty)$ and 7 parameters $0 < \theta < 1 \le u_0, p_0, u_1, p_1, u, p \le \infty$ satisfying

$$p_0 \ne p_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}, \quad w = {w_0}^{1-\theta}{w_1}^{\theta}. \tag{3}$$

For the case u_0 and u_1 are finite, in addition to (3), we assume that

$$\frac{u_0}{p_0} = \frac{u_1}{p_1} \quad \text{and} \quad \omega_0^{u_0} = \omega_1^{u_1}. \tag{4}$$

We first describe the first complex interpolation $[\dot{B}^{u_0}_{w_0}(L^{p_0}), \dot{B}^{u_1}_{w_1}(L^{p_1})]_\theta$ and the second complex interpolation $[\dot{B}^{u_0}_{w_0}(L^{p_0}), \dot{B}^{u_1}_{w_1}(L^{p_1})]^\theta$ for the case u_0 and u_1 are finite.

Theorem 2.1. [3] *Suppose that* $u_0, u_1 < \infty$*. Assume that* (3) *and* (4) *hold. Then*

1. $[\dot{B}^{u_0}_{w_0}(L^{p_0}), \dot{B}^{u_1}_{w_1}(L^{p_1})]_\theta = [\dot{B}^{u_0}_{w_0}(L^{p_0}), \dot{B}^{u_1}_{w_1}(L^{p_1})]^\theta = \dot{B}^{u}_{w}(L^{p})$.
2. $[B^{u_0}_{w_0}(L^{p_0}), B^{u_1}_{w_1}(L^{p_1})]_\theta = [B^{u_0}_{w_0}(L^{p_0}), B^{u_1}_{w_1}(L^{p_1})]^\theta = B^{u}_{w}(L^{p})$.

For the case $u_0 = u_1 = u = \infty$, we have the following result.

Theorem 2.2. [3] *Let* $\theta \in (0,1)$*,* $1 \le p_0, p_1 < \infty$*, and* $w_0, w_1 : (0,\infty) \to (0,\infty)$*. Suppose that* $w_0(r)^{p_0} = w_1(r)^{p_1}$*. Define* p *and* w *by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad w := w_0^{1-\theta} w_1^{\theta}.$$

Then

$$[\dot{B}^{\infty}_{w_0}(L^{p_0}), \dot{B}^{\infty}_{w_1}(L^{p_1})]_\theta = \left\{ f \in \dot{B}^{\infty}_{w}(L^{p}) : \lim_{j\to\infty} \|f - \chi_{\{\frac{1}{j} \le |f| \le j\}} f\|_{\dot{B}^{\infty}_{w}(L^{p})} = 0 \right\} \tag{5}$$

and

$$[\dot{B}^{\infty}_{w_0}(L^{p_0}), \dot{B}^{\infty}_{w_1}(L^{p_1})]^\theta = \dot{B}^{\infty}_{w}(L^{p}) \tag{6}$$

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