The factorization method for the acoustic inverse scattering problems

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We consider the inverse scattering problem of time-harmonic acoustic plane waves by multiple impenetrable obstacles. For the purpose, we derive the factorization method, which is a sampling method for solving certain kinds of inverse problems where the shape and location of a domain have to be reconstructed. It has first been introduced by Kirsch [2] for inverse acoustic scattering problems. For details of the factorization method, we refer to [3]. Recently, it has been applied to reconstruction of a scatterer with different physical properties, see e.g., [5], [6].

Let $k > 0$ be the wave number and for $\theta \in \mathbb{S}^2$ we set

$$u^i(x) := \mathrm{e}^{ik\theta\cdot x}, \ x \in \mathbb{R}^3, \tag{1}$$

where i in the left hand side stands for *incident plane wave*. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with C^2 boundary and let its exterior $\mathbb{R}^3 \setminus \overline{\Omega}$ be connected. We assume that Ω consists of two bounded open sets, i.e., $\Omega = \Omega_1 \cup \Omega_2$ such that $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$. Note that each Ω_j may consist of finitely many connected components whose closures are mutually disjoint. Consider the following exterior mixed boundary value problem:

$$(\Delta + k^2)u^s = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega} \tag{2}$$

$$u^s = -u^i \text{ on } \partial\Omega_1 \tag{3}$$

$$\frac{\partial u^s}{\partial \nu_{\Omega_2}} = -\frac{\partial u^i}{\partial \nu_{\Omega_2}} \text{ on } \partial\Omega_2 \tag{4}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{5}$$

where $r = |x|$, and (5) is the *Sommerfeld radiation condition*. Here and throughout this paper, $\nu_{\Omega_2}(x)$ denotes the unit normal vector at $x \in \partial\Omega_2$. We refer to Theorem 7.15 in [7] for the existence and uniqueness of the problem (2)–(5). It is well knwon that u^s has the following asymptotic behavior:

$$u^s(x,\theta) = \frac{\mathrm{e}^{ik|x|}}{4\pi|x|}u^\infty(\hat{x},\theta) + O\left(\frac{1}{|x|^2}\right), \ |x| \to \infty, \quad \hat{x} := \frac{x}{|x|}. \tag{6}$$

The function u^{∞} is called the far field pattern of u^s. With the far field pattern u^{∞}, we define the far field operator $F^{Mix}_{\Omega_1,\Omega_2} : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$F^{Mix}_{\Omega_1,\Omega_2} g(\hat{x}) := \int_{\mathbb{S}^2} u^{\infty}(\hat{x},\theta) g(\theta) ds(\theta),\ \hat{x} \in \mathbb{S}^2. \tag{7}$$

The inverse scattering problem we consider is to reconstruct the unknown obstacle Ω from the far field pattern $u^{\infty}(\hat{x},\theta)$ for all $\hat{x},\theta \in \mathbb{S}^2$. In other words, given the operator $F^{Mix}_{\Omega_1,\Omega_2}$, reconstruct Ω.

Recently, we have obtained the following result by modifying the original factorization method:

Theorem 1 ([1])**.** *Let a bounded domain $B = B_1 \cup B_2$ be known a priori. Assume that $\overline{B_1} \subset \Omega_1$, $\overline{\Omega_2} \subset B_2$, $\overline{\Omega_1} \cap \overline{B_2} = \emptyset$. (See* Figure 1*). Then for $z \in \mathbb{R}^3 \setminus \overline{B_2}$*

$$z \in \Omega_1 \Longleftrightarrow \sum_{n=1}^{\infty} \frac{|(\phi_z, \varphi_n)_{L^2(S^2)}|^2}{\lambda_n} < \infty \tag{8}$$

where (λ_n, φ_n) is a complete eigensystem of the self adjoint and positive operator $F_{\#}$ given by

$$F_{\#} := \left| \mathrm{Re}\left[\mathrm{e}^{i\pi} (F^{Mix}_{\Omega_1,\Omega_2} + F^{Imp}_{B,i\lambda_0}) \right] \right| + \mathrm{Im}(F^{Mix}_{\Omega_1,\Omega_2} + F^{Imp}_{B,i\lambda_0}) \tag{9}$$

Here, $F^{Imp}_{B,i\lambda_0}$ is the far field operator for the pure impedance boundary condition on B with an impedance function $i\lambda_0$ $\left(i.e., \frac{\partial u^s}{\partial \nu_B} + i\lambda_0 u^s = -\frac{\partial u^i}{\partial \nu_B} - i\lambda_0 u^i \text{ on } \partial B\right)$, where λ_0 is arbitrary positive number.

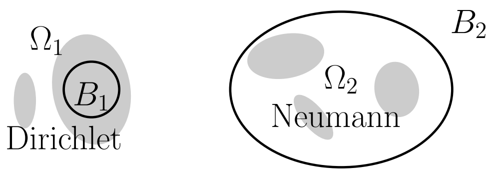

Figure 1:

Theorem 1 empolys the following two ideas: The first, coming from [6], is to make use of an a priori known outer estimation B_2 for a part of unkonw obstacles; the second, coming from [4], inner estimation B_1.

Compare our work with previous works. In Theorem 2.5 of [6], reconstruction of unkonwn obstacles, like Theorem 1, has been shown by imposing an eigenvalue of $-\Delta$ in the unknown obstacle Ω (but we do not need an inner estimation B_1). On the other hand, our work does not require eigenvales instead of using an inner estimation B_1. Therefore, by our work we can expand the application of the factorization method for some inverse acoustic scattering problems.

References

[1] T. Furuya, in preparation.

[2] A. Kirsch, Characterization of the shape of a scattering obstacle using the spectral data of the far field operator, Inverse Problems **14**, (1998), 1489–1512.

[3] A. Kirsch and N. Grinberg, *The factorization method for inverse problems*, Oxford University Press, (2008).

[4] A. Kirsch and X. Liu, *A modification of the factorization method for the classical acoustic inverse scattering problems*, Inverse Problems **30**, (2014), 1–14.

[5] A. Kirsch and X. Liu, *Direct and inverse acoustic scattering by a mixed-type scatterer*, Inverse Problems **29** (2013), 1–19.

[6] X. Liu, *The factorization method for scatterers with different physical properties*, Discrete Contin. Dyn. Syst. Ser. S **8**, (2015), 563–577.

[7] W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000