Bianchi-Bäcklund transformation for spacelike constant mean curvature surfaces in Minkowski 3-space*

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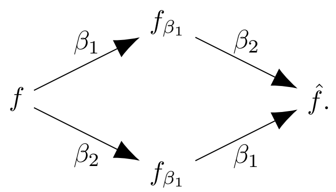
1 Background

Given a surface with constant negative Gaussian curvature (CNC) in the Euclidean 3-space $\mathbb{R}^3$, one may use the Bäcklund transformation to obtain a new CNC surface using tangential line congruence depending on a spectral parameter. It is using this transformation that Bianchi proved the Bianchi permutability theorem in [1], which says that given a seed CNC surface f, and two Bäcklund transforms f_{β_1} and f_{β_2} using spectral parameters β_1 and β_2 respectively, there exists a fourth surface $\hat{f}$ such that

$$\hat{f} = (f_{\beta_1})_{\beta_2} = (f_{\beta_2})_{\beta_1},$$

or schematically,



Using the permutability, in [2], Bianchi considered twice successive transformations using complexified tangential line congruence to obtain a constant positive Gaussian curvature surface (CPC) from a given CPC surface, called the Bianchi-Bäcklund transformation. From this, one may construct a new constant mean curvature (CMC) surface from a given CMC surface, using the fact that a CMC surface is a parallel surface of a CPC surface.

In this presentation, we give a method of constructing a new spacelike CMC surface from a given one in Minkowski 3-space $\mathbb{R}^{2,1}$ by developing an analogue of the classical Bianchi-Backlund transformation, and show that Bianchi permutability also holds for this transformation.

2 Bianchi-Bäcklund transformation for CMC surfaces in $\mathbb{R}^3$

We first briefly review the Bianchi-Bäcklund transformation in $\mathbb{R}^3$. (For further details, see [2], [8], or [5], for example.) Let $\Sigma \subset \mathbb{R}^2$ be a simply-connected domain with coordinates $(u, v) \in \Sigma$, and let $f : \Sigma \to \mathbb{R}^3$ be a conformally immersed surface with curvature line coodinates (u, v). Since $f(u, v)$ is conformal, for some function $\omega : \Sigma \to \mathbb{R}$,

$$\mathrm{d}s^2 = e^{2\omega}(\mathrm{d}u^2 + \mathrm{d}v^2).$$

We choose a unit normal vector field $e_3 : \Sigma \to \mathbb{S}^2$. We further assume that the mean curvature $H = \frac{1}{2}$ and the Hopf differential factor $Q = -\frac{1}{4}$. Then integrability condition, or the Gauss equation, becomes

$$\Delta\omega + \sinh\omega\cosh\omega = 0, \tag{1}$$

the well-known sinh-Gordon equation.

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By considering complexified tangential line congruence of a given surface with metric function ω satisfying (1), one may construct another solution $\vartheta : \Sigma \to \mathbb{C}$ to the same equation by solving

$$\begin{cases} (\vartheta - \omega)_z = \frac{1}{2} e^{\beta} \sinh(\vartheta + \omega) \\ (\vartheta + \omega)_{\bar{z}} = -\frac{1}{2} e^{-\beta} \sinh(\vartheta - \omega) \end{cases} \tag{2}$$

where β is some constant in $\mathbb{R}$, and $z = u + iv$.

To obtain another real solution to (1), we perform two iterations of this as follows: We first obtain ϑ and ϑ^* from a given solution ω using the constants β and β^*, respectively, via (2), where

$$\beta^* := \pi i - \beta. \tag{3}$$

Then by performing another iteration, starting with ϑ and ϑ^*, and using β^* and β, respectively, we obtain a new solution ω^N via the Bianchi permutability formula

$$\tanh\left(\frac{\omega^N - \omega}{2}\right) = \coth\left(\frac{\beta - \beta^*}{2}\right) \tanh\left(\frac{\vartheta - \vartheta^*}{2}\right).$$

The choice of β^* in (3) forces the new solution ω^N to be a real function defined on Σ. The new solution ω^N is called the Bianchi-Bäcklund transformation of ω.

3 Complexified tangential line congruence for CGC $K = 1$ surface in $\mathbb{R}^{2,1}$

Now we switch our attention to $\mathbb{R}^{2,1}$. Let $\Sigma \subset \mathbb{R}^2$ be a simply-connected domain with coordinates $(u, v) \in \Sigma$, and let $f : \Sigma \to \mathbb{R}^{2,1}$ be an immersion with conformal curvature line coordinates (u, v). Since $f(u, v)$ is conformal, for some function $\omega : \Sigma \to \mathbb{R}$,

$$\mathrm{d}s^2 = e^{2\omega}(\mathrm{d}u^2 + \mathrm{d}v^2).$$

We choose a timelike unit normal vector field $n = e_3 : \Sigma \to \mathbb{H}^2$, and let e_1 and e_2 be the unit tangent vectors in the direction of f_u and f_v, respectively. We further assume that the mean curvature $H = \frac{1}{2}$ and the Hopf differential factor $Q = -\frac{1}{4}$. Then the Gauss-Weingarten equations become

$$\begin{cases} f_{uu} = \omega_u f_u - \omega_v f_v - e^{\omega} \sinh \omega \, n \\ f_{vv} = -\omega_u f_u + \omega_v f_v - e^{\omega} \cosh \omega \, n \\ f_{uv} = \omega_v f_u + \omega_u f_v \\ n_u = -e^{-\omega} \sinh \omega f_u \\ n_v = -e^{-\omega} \cosh \omega f_v. \end{cases}$$

Therefore, the integrability condition, or the Gauss equation, becomes

$$\Delta\omega - \sinh \omega \cosh \omega = 0. \tag{4}$$

Now, we take g to be the parallel surface to f that is a constant Gaussian curvature $K = 1$ surface, i.e. $g = f - n$. Following Bianchi's construction in [2], we construct a new constant Gaussian curvature $K = 1$ from g as follows.

First, we define the complexified tangential line congruence g^N of g as

$$g^N := g + \lambda(\cos\varphi \, e_1 + \sin\varphi \, e_2)$$

for some constant $\lambda \in \mathbb{C} \setminus \{0\}$ and some function $\varphi : \Sigma \to \mathbb{C}$. Then we demand that

1. The vector $g^N - g$ are tangent to both surfaces at their respective points, and
2. The normal vectors e_3 and e_3^N have a constant angle σ with each other at corresponding points.

Using these two conditions and (4), we calculate that

$$\coth^2 \sigma + \frac{1}{\lambda^2} = 1.$$

Hence, we can define β and ϑ so that

$$i \sinh \beta = \coth \sigma, \quad \cosh \beta = \frac{1}{\lambda}, \quad i\vartheta = \varphi,$$

and obtain the Bianchi partial differential equations (PDE):

$$\begin{cases} (\vartheta - \omega)_z = \frac{1}{2} e^{\beta} \sinh(\vartheta + \omega) \\ (\vartheta + \omega)_{\bar{z}} = \frac{1}{2} e^{-\beta} \sinh(\vartheta - \omega) \end{cases} \tag{5}$$

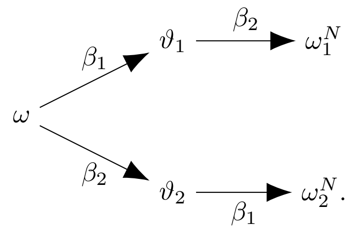
where $z := u + iv \in \mathbb{C}$. Then a direct calculation gives us the following theorem.

Proposition 1. *Let ω be a solution to (4), and let ϑ be defined via (5). Then ϑ is also a solution to (4), i.e.*

$$\Delta \vartheta - \sinh \vartheta \cosh \vartheta = 0.$$

4 Bianchi permutability theorem

We now aim to show that following Bianchi in [1] and [2], Bianchi permutability theorem also holds for the transformation as defined in Proposition 1. Let ω_1^N and ω_2^N be twice successive transformations satisfying the following schematic diagram:



Through direct calculation, we can show that

$$\omega^N = \omega_1^N = \omega_2^N$$

if and only if

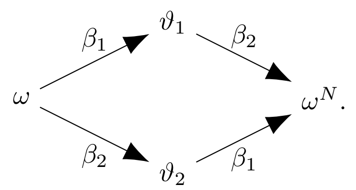
$$\tanh\left(\frac{\omega^N - \omega}{2}\right) = \coth\left(\frac{\beta_1 - \beta_2}{2}\right) \tanh\left(\frac{\vartheta_1 - \vartheta_2}{2}\right).$$

and hence we have the following theorem.

Theorem 2. *Let ω be a solution to*

$$\Delta \omega - \sinh \omega \cosh \omega = 0,$$

and let ϑ_1 and ϑ_2 be defined via Bianchi PDE using β_1 and β_2, respectively. Then there exists a fourth solution ω^N such that the following schematic diagram holds:



Moreover, ω^N can be found using the following algebraic expression:

$$\tanh\left(\frac{\omega^N - \omega}{2}\right) = \coth\left(\frac{\beta_1 - \beta_2}{2}\right) \tanh\left(\frac{\vartheta_1 - \vartheta_2}{2}\right).$$

5 Bianchi-Bäcklund transformation for spacelike CMC surfaces in $\mathbb{R}^{2,1}$

In general, since we used complexifed tangential line congruence, the new solution constructed via Bianchi PDE is complex, even if the original solution is real. However, using Theorem 2 and performing twice successive transformations, we may force the solution to be real by letting $\beta_1 = \beta$ and $\beta_2 = -\bar{\beta}$ for some complex $\beta \in \mathbb{C}\backslash\{0\}$. We call such an ω^N a *Bianchi-Bäcklund transformation* of ω. Therefore, given any spacelike CMC surface, we obtain a new spacelike CMC surface via the following recipe:

1. From a given spacelike CMC surface in $\mathbb{R}^{2,1}$, recover the metric function ω satisfying (4).
2. Choose any nonzero β and perform a Bianchi-Bäcklund transformation via Theorem 2 to obtain a new real solution ω^N.
3. Construct a new spacelike CMC surface having ω^N as its metric function.

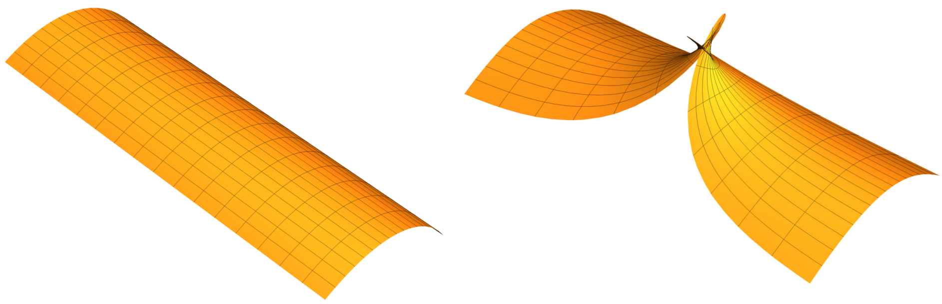


Figure 1: Example of a Bianchi-Bäcklund transformation for spacelike CMC surfaces in $\mathbb{R}^{2,1}$. On the left is hyperbolic cylinder, corresponding to the vacuum solution $\omega \equiv 0$; on the right is a Bianchi-Bäcklund transformation of hyperbolic cylinder, an analogue of bubbletons in $\mathbb{R}^3$.

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