Convex property of Wulff shapes and regularity of their convex integrands

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Abstract

In this paper, for a convex closed surface which is called Wulff shape, we investigate the strength of the convexity of the Wulff shape in terms of regularity of its convex integrand.

1 Introduction

It is known that the minimizer of an anisotropic surface energy among all closed surfaces enclosing the same volume is unique (up to translations) and it is called the Wulff shape. An anisotropic surface energy is the integral of an energy density over the considered surface and it is a mathematical model of the surface tension of a crystal. In general, the Wulff shape is not smooth. For example, the Wulff shape is a convex cloesd surface with singularities like a polytope in some cases. The smoothness and the strength of the convexity of the Wulff shape are deeply related to the convexity and the regularity of the energy density. In this article, we investigate the relationship between the strength of the convexity of the Wulff shape and the regularity of the energy density, both locally and globally.

2 Main Results

We prepare some notations. Let $n \in \mathbb{N}$. We denote the unit sphere with center at the origin in $\mathbb{R}^{n+1}$ by S^n. Let $\gamma : S^n \longrightarrow \mathbb{R}_{\geq 0}$ be a continuous function. We use the following symbols.

$$\Gamma_{\gamma,\nu} := \{x \in \mathbb{R}^{n+1} | (x,\nu) \leq \gamma(\nu)\},$$

where (,) stands for the scalar product of x and ν in $\mathbb{R}^{n+1}$.

$$\mathcal{K}_0^n := \{A \in \mathbb{R}^n \mid A \text{ is a compact convex set containing the origin.}\}.$$

For $W \in \mathcal{K}_0^n$ and $\nu \in S^n$,

$$F(W,\nu) := \partial W \cap \partial \Gamma_{\gamma,\nu}.$$

We call the set $F(W,\nu)$ the **ν-way face** of W.

For given $W \in \mathcal{K}_0^{n+1}$, let γ_W be the convex integrand for W (the definition of the integrand, will be given in Section 3).

Theorem 2.1. *Let $W \in \mathcal{K}_0^{n+1}$ and $\nu \in S^n$. Then, the following (a) and (b) are equivalent.*

(a) γ_W *is differentiable at* ν.
(b) $F(W,\nu) = \{$*one point*$\}$.

Corollarly 2.1 (H.Han and T.Nishimura 2016 [1])**.** *Let $W \in \mathcal{K}_0^{n+1}$. Then, the following (a) and (b) are equivalent.*

(a) γ_W *is of* C^1
(b) W *is strictly convex.*

3 Preliminaries

Let $\gamma : S^n \longrightarrow \mathbb{R}_{\geq 0}$ be continuous function. Then, we define the **Wulff shape** for γ as follows :

$$W_\gamma \stackrel{\text{def}}{=} \bigcap_{\nu \in S^n} \Gamma_{\gamma,\nu}$$

and we call γ a **support function** for the Wulff shape W_γ.
By definition, it is obvious to hold the following property.

Proposition 3.1. *For a continuous function $\gamma : S^n \longrightarrow \mathbb{R}_{\geq 0}$, W_γ is a compact convex set containing the origin.*

Conversely, for a given compact convex set W containing the origin, we can construct $\gamma : S^n \longrightarrow \mathbb{R}_{\geq 0}$ which satisfies $W = W_\gamma$.
In fact, putting

$$\gamma(\nu) := \max_{x \in W}(x, \nu),$$

we have $W = W_\gamma$.
We call $\gamma : S^n \longrightarrow \mathbb{R}_{\geq 0}$ constructed as above the **convex integrand** (or energy density) for W and denote it by γ_W.

Definition 3.1. *Let $\gamma : S^n \longrightarrow \mathbb{R}_{\geq 0}$.*
Then, the homogeneous extension $\tilde{\gamma} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}_{\geq 0}$ of γ is defined by

$$\tilde{\gamma}(x) = \begin{cases} \|x\|\gamma(\frac{1}{\|x\|}x), & (x \neq \mathbf{0}), \\ 0, & (x = \mathbf{0}). \end{cases}$$

Proposition 3.2 ([2])**.** *Let $W \in \mathcal{K}_0^{n+1}$, and γ be the convex integrand for W.*
Let $x \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$. Then, the following (a) and (b) are equivalent.

(a) $\tilde{\gamma}$ *is differentiable at* x.
(b) $\partial W \cap \partial \Gamma_{\tilde{\gamma}(x),x} = \{one\ point\}$,

where $\Gamma_{\tilde{\gamma}(x),x} = \{y \in \mathbb{R}^{n+1} | (y, x) \leq \tilde{\gamma}(x)\}$.

Definition 3.2. *Let $W \subset \mathbb{R}^n$. Then, if $\overline{xy} \setminus \{x, y\} \subset Int(W)$ holds for any distinct two points $x, y \in \partial W$, we say that W is* **strictly convex**.

Proposition 3.3. *Let $W \in \mathcal{K}_0^{n+1}$, and γ be the convex integrand for W. Then, the following (a) and (b) are equivalent.*

(a) *W is strictly convex.*
(b) $F(W, \nu) = \{one\ point\}$ $(\forall \nu \in S^n)$

4 Sketch of the proof of Main Theorem (Theorem2.1)

Let $W \in \mathcal{K}_0^{n+1}$, and γ be the convex integrand for W. First, since $\gamma = \tilde{\gamma}|_{S^n}$ holds, the following (a) and (b) are equivalent for any $\nu \in S^n$ and $\lambda > 0$.

(a) γ *is differentiable at* ν *(resp.* C^1*).*
(b) $\tilde{\gamma}$ *is differentiable at* $\lambda\nu$ *(resp.* C^1*).*

Moreover, it is easy to verify that the following (a) and (b) are equivalent for any $\nu \in S^n$ and $\lambda > 0$.

(a) $F(W, \nu) = \{one\ point\}$.

(b) $\partial W \cap \partial \Gamma_{\tilde{\gamma}(\lambda\nu), \lambda\nu} = \{one\ point\}$.

By using these facts and the results given in Section 3, we can prove the desired result.

Reference

[1] H.Han and T.Nishimura. Strictly convex Wulff shapes and C^1 convex integrands. preprint (available from arXiv:1507.05162v2 [math.MG] (2016)).

[2] R.Schneider. Convex Bodies: The Brunn-Minkowski Theory 2nd edition. *Encyclopedia of Mathematics and its Applications* **44**. Cambridge University Press, Cambridge, 2013