Some distance functions in knot theory

Jie CHEN
Division of Mathematics, Graduate School of Information Sciences,
Tohoku University

1 Introduction

In this presentation, we focus on three distance functions in knot theory. They are the Gordian distance, the algebraic Gordian distance and the Alexander polynomial distance. A *knot* is defined to be an oriented circle embedded in the three-sphere S^3. A crossing change on a knot is often called the *unknotting operation*. The *Gordian distance* between two knots is defined to be the minimum number of crossing changes needed to turn one knot into the other. Analogously, the other two distance functions of Seifert matrices and Alexander polynomials respectively are defined. These three distance functions turn the sets of knots, S-equivalence classes of Seifert matrices and Alexander polynomials into metric spaces. We are interested in the question when these distances can be one.

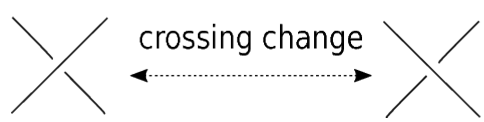


Figure 1: unknotting operation

2 The Gordian distance

We use $d_G(K, K')$ to denote the Gordian distance between two knots K and K' in S^3. The *unknotting number* $u(K)$ of a knot K is defined by $u(K) = d_G(K, O)$, where O is the trivial knot.

The studies of unknotting number and the Gordian distance are often related to the homology groups of covering spaces of knots. Pairing relations between homology classes encode the structures of those covering spaces. There are studies showing that different pairing relations have certain restrictions when the Gordian distance is one for two knots. Lickorish [11] used surgery construction of the double branched cover and showed an unknotting number one knot has an obstruction on the linking form

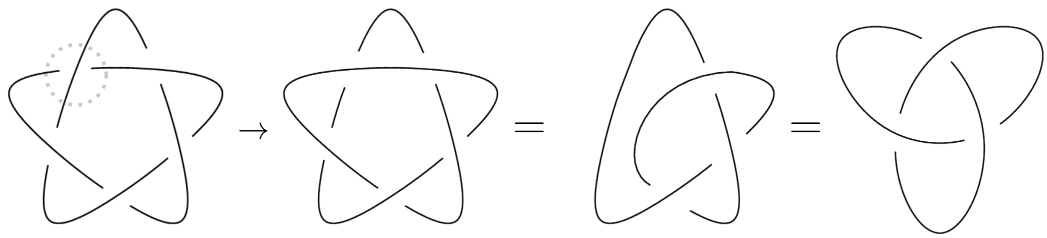


Figure 2: $d_G(3_1, 5_1) = 1$

of its double branched cover. Murakami [12] used a similar technique to generalize Lickorish's result to two knots with Gordian distance one and showed their double branched covers have certain linking forms.

There are studies connecting the Gordian distance with a variety of knot invariants. Murasugi [14] gave a lower bound for the unknotting number in terms of the knot signature. Stoimenow [17] connected the Jones polynomial and the unknotting number. Nakanishi [15] found an obstruction for the Seifert matrices of unknotting number two knots. More algebraic techniques are used to find obstructions for a pair of knots of Gordian distance one. Kawauchi [9] used residue modules and determinant rings to give an obstruction on the Alexander polynomials of a pair of knots with Gordian distance one. With these restrictions, it becomes possible to tell whether two given knots could be transformed into each other by one crossing change.

3 The algebraic Gordian distance

A *Seifert matrix* V is defined to be a square integer matrix satisfying $\det(V - V^T) = 1$. A Seifert matrix V' is said to be *congruent* to V if $V' = PVP^T$ for a unimodular matrix P. A Seifert matrix V' is called an *enlargement* of V if

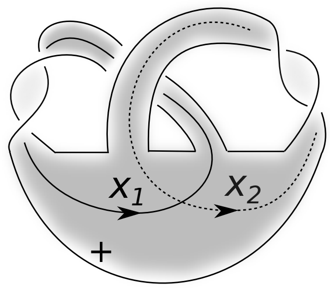
$$V' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & n & M \\ 0 & N^T & V \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & n & M \\ 0 & N^T & V \end{pmatrix},$$

where M and N are row vectors, and n is some integer. Then V is a *reduction* of V'. The S*-equivalence* is an equivalence relation generated by congruences, enlargements and reductions. The S*-equivalence class* of V, denoted by $[V]$, is all Seifert matrices S-equivalent to V; see [16, 19].

Motivated by the unknotting operation, Murakami defined the *algebraic unknotting operation* in [13]. It assigns a Seifert matrix V to $\begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & n & M \\ 0 & N^T & V \end{pmatrix}$ for $\varepsilon = \pm 1$; see [13].

Let $[V]$ and $[V']$ be two S-equivalence classes. The *algebraic Gordian distance* $d_G^a([V], [V'])$ between $[V]$ and $[V']$ is defined to be the minimum number of algebraic unknotting operations needed to deform a matrix in $[V]$ to a matrix in $[V']$.

For a knot K in S^3, a *Seifert surface* of K is a connected orientable surface bounded by K. Given a Seifert surface F, we can choose a generator system $x_1, x_2, \ldots, x_{2g}$ of $H_1(F)$, where g is the genus of F. Let lk denote the linking number. A *Seifert matrix* for F can be calculated by $V = (v_{ij})$ with $v_{ij} = \mathrm{lk}(x_i, x_j^+)$ for $i, j = 1, 2, \ldots, 2g$, where x_j^+ is the result of translating a representative cycle for x_j into $S^3 - F$ along the positive side of F. The *Alexander polynomial* Δ_K of K is defined by the equation $\Delta_K = \det(t^{\frac{1}{2}}V - t^{-\frac{1}{2}}V^T)$. Note that any two S-equivalent Seifert matrices have the same Alexander polynomial. Let $[K]$ denote the S-equivalence class of a Seifert matrix of K. In [13], Murakami proved $d_G(K_1, K_2) \geq d_G^a([K_1], [K_2])$, relating the two distance functions.



$$\begin{array}{cc} & \begin{array}{cc} x_1^+ & x_2^+ \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{array}$$

Figure 3: a Seifert surface and Seifert matrix for 3_1

The *Alexander module* A_V of a Seifert matrix V is defined by $A_V = \Lambda^{2g}/(tV - V^T)\Lambda^{2g}$, where Λ is the *Laurent polynomial ring* $\mathbb{Z}[t, t^{-1}]$. Then we know $A_V \cong H_1(\tilde{X}(K); \mathbb{Z})$, where $\tilde{X}(K)$ is the infinite cyclic cover of the complement of K. We regard A_V as a Λ-module, with t acting on $\tilde{X}(K)$ as the deck transformation. The *Blanchfield pairing* of A_V is a map $\beta_V : A_V \times A_V \longrightarrow Q(\Lambda)/\Lambda$. It is a sesquilinear form, meaning $\beta_V(px, qy) = p\bar{q}\beta_V(x, y)$, where $\bar{q} = q|_{t=t^{-1}}$; see [1]. Note that $(A_V, \beta_V) \cong (A_{V'}, \beta_{V'})$ if V is S-equivalent to V'.

Analogously to the unknotting number, *the algebraic unknotting number* $u_a([V])$ is defined to be $d_G^a([V], [O])$, where $[O]$ is the S-equivalence class of the 0×0 matrix. Murakami proved if $u_a([K]) = 1$, then there exists a generator α for the Alexander module of K such that the Blanchfield pairing $\beta_V(\alpha, \alpha) = \pm\frac{1}{\Delta_K}$. Moreover, the Blanchfield pairing is given by the 1×1-matrix $(\pm\frac{1}{\Delta_K})$; see [13, Theorem 5, p.288].

4 The Alexander polynomial distance

Kawauchi defined the Alexander polynomial distance $\rho(\Delta, \Delta')$ between two Alexander polynomials Δ and Δ' by the equation

$$\rho(\Delta, \Delta') = \min_{K_\Delta, K_{\Delta'}} d_G(K_\Delta, K_{\Delta'}),$$

where K_Δ and $K_{\Delta'}$ are knots with Alexander polynomials Δ and Δ', respectively [9].

Note that $\rho(\Delta, \Delta') \leq 2$; see [9]. This is because there exists an unknotting number one knot for any given Alexander polynomial; see [10]. In [4], it is proved that $d_G^a([K_1], [K_2]) \geq \rho(\Delta_{K_1}, \Delta_{K_2})$.

A question of Jong asks to find two Alexander polynomials Δ and Δ' such that $\rho(\Delta, \Delta') = 2$; see [6–9]. Kawauchi gave a restriction for a pair of Alexander polynomials of degree two such that their distance is one.

5 The Blanchfield pairing of two Seifert matrices of distance one

In [4], by constructing two Seifert matrices of algebraic Gordian distance one and finding the restriction of their Blanchfield pairing, the following theorem is deduced.

Theorem 5.1 ([4]). *Let V and V' be two Seifert matrices. If the algebraic Gordian distance $d_G^a([V], [V']) = 1$, then there exist $a \in A_V$ and $a' \in A_{V'}$ such that $\beta_V(a, a) \equiv \pm\dfrac{\Delta_{V'}}{\Delta_V} \pmod{\Lambda}$ and $\beta_{V'}(a', a') \equiv \pm\dfrac{\Delta_V}{\Delta_{V'}} \pmod{\Lambda}$.*

The following corollaries to Theorem 5.1 give further results on the obstructions of the algebraic Gordian distance and the Alexander polynomial distance.

Corollary 5.2 ([4]). *If $u_a([V]) = d_G^a([V], [V']) = 1$, then there exists $c \in \Lambda$ such that $\pm\Delta_{V'} \equiv c\bar{c} \pmod{\Delta_V}$.*

Corollary 5.3 ([4]). *Let Δ_V and $\Delta_{V'}$ be the Alexander polynomials of Seifert matrices V and V', respectively, with $\Delta_V = h(t + t^{-1}) + 1 - 2h$, $|h|$ being a prime or 1 and $\Delta_{V'} \equiv d \pmod{\Delta_V}$, where $d \in \mathbb{Z}$. If $u_a([V]) = 1$ and if the equation for x and y $h^2x^2 + y^2 + (2h - 1)xy = \pm d$ does not have an integer solution, then the algebraic Gordian distance $d_G^a([V], [V']) \neq 1$.*

If there is an Alexander polynomial that is realized only by unknotting number one matrices, we can use Corollary 5.3 to find many examples to answer Jong's question.

6 Determinant of 2×2 Seifert matrices with algebraic unknotting number one

In [4], the following lemmas on Seifert matrices of algebraic unknotting number one are proven.

Lemma 6.1 ([4]). *If a 2×2 Seifert matrix V has $\det V \in \{1, 2, 3, 5\}$, then $u_a(V) = 1$.*

Lemma 6.2 ([4]). *For a Seifert matrix V, if $\Delta_V = ht + ht^{-1} + 1 - 2h$ with $h \in \{1, 2, 3, 5\}$, then $u_a(V) = 1$.*

The two lemmas are based on Trotter's results in [18, 19].

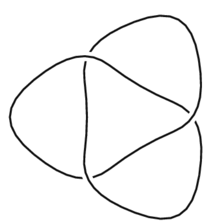
- Any 2×2 Seifert matrix is congruent to a matrix of the form $\begin{pmatrix} a & m+1 \\ m & c \end{pmatrix}$; see [19, p.203].
- Any 2×2 positive definite Seifert matrix is congruent to $\begin{pmatrix} a & m+1 \\ m & c \end{pmatrix}$, where $0 < 2m + 1 \leq \min(a, c)$; see [19, p.204].
- If $\Delta_V = ht + ht^{-1} + 1 - 2h$, V is S-equivalent to a 2×2 Seifert matrix V' with $\det V' = h$; see [18, pp.484-486].

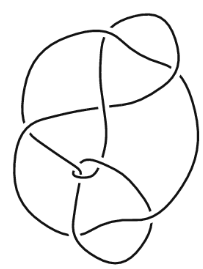
We refer to [2, 5] for more information about the classification of binary quadratic forms, with which we can find more determinants that can be realized only by algebraic unknotting number one Seifert matrices.

By Corollary 5.3, we can find many examples of $\rho(\Delta, \Delta') = 2$.

Corollary 6.3. *The Alexander polynomial distance* $\rho(t+t^{-1}-1, \Delta) = 2$ *if* $\Delta \equiv 4m+2 \pmod{t + t^{-1} - 1}$ *for some* $m \in \mathbb{Z}$.

Now we give an example for this corollary. The following figures are diagrams of the knots 3_1 and 9_{30}, respectively [3].





$\Delta_{3_1} = t + t^{-1} - 1$ $\qquad$ $\Delta_{9_{30}} = -t^3 - t^{-3} + 5t^2 + 5t^{-2} - 12t - 12t^{-1} + 17$

We have $\Delta_{3_1} = t + t^{-1} - 1$ and $\Delta_{9_{30}} = -t^3 - t^{-3} + 5t^2 + 5t^{-2} - 12t - 12t^{-1} + 17$, so $\Delta_{9_{30}} = (-t^2 - t^{-2} + 4t + 4t^{-1} - 7)\Delta_{3_1} + 2$. By Corollary 6.3, we obtain $d_G(K_1, K_2) \geq d_G^a([K_1], [K_2]) \geq \rho(\Delta_{3_1}, \Delta_{9_{30}}) = 2$ for any pair of knots K_1 and K_2 with $\Delta_{K_1} = \Delta_{3_1}$ and $\Delta_{K_2} = \Delta_{9_{30}}$.

Moreover, this example demonstrates how our result helps in calculating the algebraic Gordian distance of two given S-equivalent classes. We know $u_a([9_{30}]) = u_a([3_1]) = 1$. It gives $d_G^a([3_1], [9_{30}]) \leq u_a([3_1]) + u_a([9_{30}]) = 2$. Therefore, we have $d_G^a([3_1], [9_{30}]) = 2$.

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