ARITHMETIC AND DYNAMICAL DEGREES OF SEMIABELIAN VARIETIES

YOHSUKE MATSUZAWA (JOINT WORK WITH KAORU SANO)
松澤陽介 (佐野薫との共同研究)

Let X be a smooth quasi-projective variety and $f\colon X \dashrightarrow X$ a rational self-map, both defined over $\overline{\mathbb{Q}}$. Having studied the arithmetic of the discrete dynamical system $f\colon X \dashrightarrow X \dashrightarrow X \dashrightarrow \cdots$, Silverman introduced the notion of arithmetic degree in [6], which measures the growth rate of height functions along the f-orbits. Take a smooth projectivization $\overline{X}$ of X and fix a Weil height function $h_{\overline{X}}$ on $\overline{X}$ associated with an ample divisor (good references for height functions are [1, 2]). Write $h_X = h_{\overline{X}}|_X$. Consider a point $x \in X$ such that for all $n \geq 0$, $f^n(x)$ is not contained in the indeterminacy locus of f. The *arithmetic degree* of f at x is

$$\alpha_f(x) = \lim_{n\to\infty} \max\{h_X(f^n(x)), 1\}^{1/n}$$

provided that the limit exists. This, of course, measures the exponential growth rate of $h_X(f^n(x))$ as n goes to infinite and is independent of the choice of $\overline{X}$ and $h_{\overline{X}}$. Kawaguchi-Silverman proved the existence of the limit when X is projective and f is a morphism [3]. The convergence in full generality is still open.

When f is dominant, it is conjectured in [6], [4, Conjecture 6] that the arithmetic degree of any Zariski dense orbits are equal to the first dynamical degree δ_f of f. This is the Kawaguchi-Silverman conjecture, and we abbreviate it as KSC. Here, the first dynamical degree is a birational invariant of f which measures the geometric complexity of the dynamical system. When X is projective and f a surjective morphism, δ_f is equal to the spectral radius of the linear map $f^*\colon N^1(X)\otimes_{\mathbb{Z}}\mathbb{R} \longrightarrow N^1(X)\otimes_{\mathbb{Z}}\mathbb{R}$ where $N^1(X)$ is the group of divisors modulo numerical equivalence.

Let $A(f)$ be the set of arithmetic degrees of f, i.e.

$$A(f) = \{\alpha_f(x) \mid P \in X\}$$

when we know $\alpha_f(x)$ exists for all $x \in X$. Keeping the conjecture in mind, we expect that we can describe this set in terms of geometric data of f. When X is a toric variety and f is a self-rational map on X that is induced by a group homomorphism of the algebraic torus, the set $A(f)$ is completely determined by the matrix defining f [6, 5].

Graduate school of Mathematical Sciences, the University of Tokyo.

We prove KSC for self-morphisms of semi-abelian varieties and determine the set $A(f)$.

Theorem 1. *Let X be a semi-abelian variety and $f\colon X \longrightarrow X$ a self-morphism (not necessarily surjective), both defined over $\overline{\mathbb{Q}}$.*

(1) *Suppose f is surjective. Then for any point $x \in X$ with Zariski dense f-orbit, we have $\alpha_f(x) = \delta_f$.*

(2) *For every $x \in X$, the arithmetic degree $\alpha_f(x)$ exists. If we write $f = T_a \circ g$ where T_a is the translation by a point $a \in X$ and g is a group homomorphism, then $A(f) = A(g)$.*

(3) *Suppose f is a group homomorphism. Let $F(t)$ be the monic minimal polynomial of f as an element of $\mathrm{End}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ and*

$$F(t) = t^{e_0}F_1(t)^{e_1}\cdots F_r(t)^{e_r}$$

the irreducible decomposition in $\mathbb{Q}[t]$ where $e_0 \geq 0$ and $e_i > 0$ for $i = 1, \dots, r$. Let $\rho(F_i)$ be the maximum among the absolute values of the roots of F_i. Then we have

$$A(f) \subset \{1, \rho(F_1), \rho(F_1)^2, \dots, \rho(F_r), \rho(F_r)^2\}.$$

More precisely, set

$$X_i = f^{e_0}F_1(f)^{e_1}\cdots F_{i-1}(f)^{e_{i-1}}F_{i+1}(f)^{e_{i+1}}\cdots F_r(f)^{e_r}(X).$$

Define

$$A_i = \begin{cases} \{\rho(F_i)\} & \text{if } X_i \text{ is an algebraic torus,} \\ \{\rho(F_i)^2\} & \text{if } X_i \text{ is an abelian variety,} \\ \{\rho(F_i), \rho(F_i)^2\} & \text{otherwise.} \end{cases}$$

Then we have

$$A(f) = \{1\} \cup A_1 \cup \cdots \cup A_r.$$

Theorem 2. *Let X be a semi-abelian variety and $f\colon X \longrightarrow X$ a surjective morphism both defined over $\overline{\mathbb{Q}}$. Write $f = T_a \circ g$ where T_a is the translation by $a \in X$ and g is an isogeny. Suppose that the minimal polynomial of g has no irreducible factor that is a cyclotomic polynomial. Then there exists a point $b \in X$ such that, for any $x \in X$, the following are equivalent:*

(1) $\alpha_f(x) = 1$;

(2) $\#\, O_f(x) < \infty$;

(3) $x \in b + X(\overline{\mathbb{Q}})_{\mathrm{tors}}$.

Here $X(\overline{\mathbb{Q}})_{\mathrm{tors}}$ is the set of torsion points.

Remark 3. It is easy to see that when f is an isogeny, we can take $b = 0$.

To prove the above theorems, we calculate the first dynamical degrees of self-morphisms of semi-abelian varieties.

Theorem 4. *Let X be a semi-abelian variety over an algebraically closed field of characteristic zero.*

(1) *Let $f\colon X \longrightarrow X$ be a surjective group homomorphism. Let*

$$0 \longrightarrow T \longrightarrow X \xrightarrow{\pi} A \longrightarrow 0$$

be an exact sequence with T a torus and A an abelian variety. Then f induces surjective group homomorphisms

$$\begin{aligned} f_T := f|_T\colon T &\longrightarrow T \\ g\colon A &\longrightarrow A \end{aligned}$$

with $g \circ \pi = \pi \circ f$. Then we have

$$\delta_f = \max\{\delta_g, \delta_{f_T}\}$$

Moreover, let P_T and P_A be the monic minimal polynomials of f_T and g as elements of $\mathrm{End}(T)_{\mathbb{Q}}$ and $\mathrm{End}(A)_{\mathbb{Q}}$ respectively. Then, $\delta_{f_T} = \rho(P_T)$ and $\delta_g = \rho(P_A)^2$.

(2) *Let $f\colon X \longrightarrow X$ be a surjective homomorphism and $a \in X$ a point. Then $\delta_{T_a \circ f} = \delta_f$.*

Remark 5. The description of δ_{f_T} and δ_g in Theorem 4(1) might be well-known.

References

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Graduate school of Mathematical Sciences, the University of Tokyo, Komaba, Tokyo, 153-8914, Japan

E-mail address: `myohsuke@ms.u-tokyo.ac.jp`