Let $X$ be a smooth quasi-projective variety and $f: X \to X$ a rational self-map, both defined over $\mathbb{Q}$. Having studied the arithmetic of the discrete dynamical system $f: X \to X \to X \to \cdots$, Silverman introduced the notion of arithmetic degree in [3], which measures the growth rate of height functions along the $f$-orbits. Take a smooth projectivization $\overline{X}$ of $X$ and fix a Weil height function $h_{\overline{X}}$ on $\overline{X}$ associated with an ample divisor (good references for height functions are [1, 2]). Write $h_X = h_{\overline{X}}|_X$. Consider a point $x \in X$ such that for all $n \geq 0$, $f^n(x)$ is not contained in the indeterminacy locus of $f$. The arithmetic degree of $f$ at $x$ is

$$\alpha_f(x) = \lim_{n \to \infty} \max \{ h_X(f^n(x)), 1 \}^{1/n}$$

provided that the limit exists. This, of course, measures the exponential growth rate of $h_X(f^n(x))$ as $n$ goes to infinite and is independent of the choice of $\overline{X}$ and $h_{\overline{X}}$. Kawaguchi-Silverman proved the existence of the limit when $X$ is projective and $f$ is a morphism [3]. The convergence in full generality is still open.

When $f$ is dominant, it is conjectured in [6, Conjecture 6] that the arithmetic degree of any Zariski dense orbits are equal to the first dynamical degree $\delta_f$ of $f$. This is the Kawaguchi-Silverman conjecture, and we abbreviate it as KSC. Here, the first dynamical degree is a birational invariant of $f$ which measures the geometric complexity of the dynamical system. When $X$ is projective and $f$ a surjective morphism, $\delta_f$ is equal to the spectral radius of the linear map $f^*: N^1(X) \otimes \mathbb{Z} \mathbb{R} \to N^1(X) \otimes \mathbb{Z} \mathbb{R}$ where $N^1(X)$ is the group of divisors modulo numerical equivalence.

Let $A(f)$ be the set of arithmetic degrees of $f$, i.e.

$$A(f) = \{ \alpha_f(x) \mid P \in X \}$$

when we know $\alpha_f(x)$ exists for all $x \in X$. Keeping the conjecture in mind, we expect that we can describe this set in terms of geometric data of $f$. When $X$ is a toric variety and $f$ is a self-rational map on $X$ that is induced by a group homomorphism of the algebraic torus, the set $A(f)$ is completely determined by the matrix defining $f$ [6, 8].

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We prove KSC for self-morphisms of semi-abelian varieties and determine the set $A(f)$.

**Theorem 1.** Let $X$ be a semi-abelian variety and $f: X \rightarrow X$ a self-morphism (not necessarily surjective), both defined over $\overline{\mathbb{Q}}$.

1. Suppose $f$ is surjective. Then for any point $x \in X$ with Zariski dense $f$-orbit, we have $\alpha_f(x) = \delta_f$.
2. For every $x \in X$, the arithmetic degree $\alpha_f(x)$ exists. If we write $f = T_a \circ g$ where $T_a$ is the translation by a point $a \in X$ and $g$ is a group homomorphism, then $A(f) = A(g)$.
3. Suppose $f$ is a group homomorphism. Let $F(t)$ be the monic minimal polynomial of $f$ as an element of $\text{End}(X) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}$ and
   \[
   F(t) = t^{e_0} F_1(t)^{e_1} \cdots F_r(t)^{e_r}
   \]
   the irreducible decomposition in $\mathbb{Q}[t]$ where $e_0 \geq 0$ and $e_i > 0$ for $i = 1, \ldots, r$. Let $\rho(F_i)$ be the maximum among the absolute values of the roots of $F_i$. Then we have
   \[
   A(f) \subset \{1, \rho(F_1), \rho(F_1)^2, \ldots, \rho(F_r), \rho(F_r)^2\}.
   \]
   More precisely, set
   \[
   X_i = f^{e_0} F_1(f)^{e_1} \cdots F_{i-1}(f)^{e_{i-1}} F_{i+1}(f)^{e_{i+1}} \cdots F_r(f)^{e_r}(X).
   \]
   Define
   \[
   A_i = \begin{cases} \{\rho(F_i)\} & \text{if } X_i \text{ is an algebraic torus,} \\ \{\rho(F_i)^2\} & \text{if } X_i \text{ is an abelian variety,} \\ \{\rho(F_i), \rho(F_i)^2\} & \text{otherwise.} \end{cases}
   \]
   Then we have
   \[
   A(f) = \{1\} \cup A_1 \cup \cdots \cup A_r.
   \]

**Theorem 2.** Let $X$ be a semi-abelian variety and $f: X \rightarrow X$ a surjective morphism both defined over $\overline{\mathbb{Q}}$. Write $f = T_a \circ g$ where $T_a$ is the translation by $a \in X$ and $g$ is an isogeny. Suppose that the minimal polynomial of $g$ has no irreducible factor that is a cyclotomic polynomial. Then there exists a point $b \in X$ such that, for any $x \in X$, the following are equivalent:

1. $\alpha_f(x) = 1$;
2. $\# O_f(x) < \infty$;
3. $x \in b + X(\overline{\mathbb{Q}})_{\text{tors}}$.

Here $X(\overline{\mathbb{Q}})_{\text{tors}}$ is the set of torsion points.

**Remark 3.** It is easy to see that when $f$ is an isogeny, we can take $b = 0$.

To prove the above theorems, we calculate the first dynamical degrees of self-morphisms of semi-abelian varieties.
Theorem 4. Let $X$ be a semi-abelian variety over an algebraically closed field of characteristic zero.

(1) Let $f : X \to X$ be a surjective group homomorphism. Let

$$0 \to T \to X \xrightarrow{\pi} A \to 0$$

be an exact sequence with $T$ a torus and $A$ an abelian variety. Then $f$ induces surjective group homomorphisms

$$f_T := f|_T : T \to T$$

$$g : A \to A$$

with $g \circ \pi = \pi \circ f$. Then we have

$$\delta_f = \max \{ \delta_g, \delta_{f_T} \}$$

Moreover, let $P_T$ and $P_A$ be the monic minimal polynomials of $f_T$ and $g$ as elements of $\text{End}(T)_{\mathbb{Q}}$ and $\text{End}(A)_{\mathbb{Q}}$ respectively. Then, $\delta_{f_T} = \rho(P_T)$ and $\delta_g = \rho(P_A)^2$.

(2) Let $f : X \to X$ be a surjective homomorphism and $a \in X$ a point. Then $\delta_{T_{\cdot}a} = \delta_f$.

Remark 5. The description of $\delta_{f_T}$ and $\delta_g$ in Theorem 4(1) might be well-known.

References