Constant mean curvature surfaces in hyperbolic 3-space with curvature lines on horospheres*

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1 Introduction

The study of constant mean curvature (CMC) H surfaces in hyperbolic 3-space depend greatly on the value of H. For a CMC $H \geq 1$ surface in $\mathbb{H}^3$, Lawson showed in [8] that there is a corresponding CMC surface or minimal surface in the Euclidean 3-space or the 3-sphere $\mathbb{S}^3$, called the Lawson correspondence. Consequently, there have been many studies on CMC $H \geq 1$ surfaces. For example, Bryant constructed a Weierstrass type representation for CMC 1 surfaces using the aforementioned correspondence in [2], while Umehara and Yamada expanded upon this result in [12]. However, there have been relatively few papers regarding CMC $0 \leq H < 1$ surfaces in $\mathbb{H}^3$, where the existence and regularity of such surfaces has been investigated in papers such as [1], [7], [9], [10], or [11].

Meanwhile, Dorfmeister, Pedit, and Wu gave a generalized form of Weiertrass-type representations for CMC surfaces in $\mathbb{R}^3$ in [5] by using loop group methods to construct integrable surfaces, frequently called the DPW method. The DPW method gave rise to numerous applications in constructing CMC surfaces in various space forms, including CMC surfaces in $\mathbb{H}^3$. In fact, the DPW method allowed for a unified approach to constructing CMC surfaces in $\mathbb{H}^3$, as shown in results such as [4] and [6].

On the other hand, Wente investigated CMC surfaces with spherical curvature lines, or CMC surfaces of Enneper type in [14]. After examining CMC 1/2 surfaces and minimal surfaces in Euclidean 3-space $\mathbb{R}^3$ with this property, he proceeded to study minimal surfaces of Enneper type in hyperbolic 3-space $\mathbb{H}^3$. In fact, he considered minimal surfaces with one family of curvature lines lying on any type of sphere, including compact spheres, horospheres, and open pseudospheres. By obtaining a partial differential equation from the spherical curvature line condition, he was able to show that the family of such surfaces depends on four parameters, up to shifts of parameters in the domain.

The result of this presentation is primarily motivated by the work of Wente in [14]; however, instead of considering minimal surfaces in $\mathbb{H}^3$ with curvature lines lying on any type of sphere, we consider CMC $H \geq 0$ surfaces, but limit the type of sphere, on which one family of curvature lines lies, to horospheres. By looking at the horosphericity condition of one family of curvature lines, we obtain and solve a system of partial differential equations for the metric function to show that any CMC surface with $H \geq 0$ in $\mathbb{H}^3$ with one family of curvature lines on horospheres must be a rotation surface of $\mathbb{H}^3$, as defined in [3]. However, it turns out that the converse is not true; namely, there are CMC rotation surfaces whose curvature lines do not lie on horospheres. Therefore, we identify the exact condition on the metric function where one family of curvature lines lie on horospheres. Finally, we investigate the geometric properties of CMC rotation surfaces with one family of curvature lines lying on horospheres.

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2 Preliminaries

We use the Minkowski model of hyperbolic 3-space $\mathbb{H}^3$. Let $\mathbb{R}^{3,1}$ be the Minkowski space equipped with metric

$$\langle (x_1, x_2, x_3, x_0), (y_1, y_2, y_3, y_0)\rangle := x_1y_1 + x_2y_2 + x_3y_3 - x_0y_0,$$

and consider the hyperbolic 3-space as

$$\mathbb{H}^3 = \{x = (x_1, x_2, x_3, x_0) \in \mathbb{R}^{3,1} : \langle x, x\rangle = -1, x_0 > 0\}.$$

Following [14], we define a sphere $S[m, q]$ in $\mathbb{H}^3$ as

$$S[m, q] := \{x \in \mathbb{H}^3 : \langle x, m\rangle = q\}$$

for a constant vector $m \in \mathbb{R}^{3,1}$ and a constant $q \in \mathbb{R}$, assuming the set is non-empty. Furthermore, a sphere $S[m, q]$ is

- a compact sphere if $\langle m, m\rangle < 0$ and $q < -1$,
- a horosphere if $\langle m, m\rangle = 0$ and $q < -1$, or
- an open pseudosphere if $\langle m, m\rangle > 0$ for any arbitrary q.

Note that a plane in $\mathbb{H}^3$ may be treated as an open pseudosphere with $q = 0$.

Let $\Sigma \subset \mathbb{R}^2$ be a simply-connected domain with coordinates (u, v), and let $X : \Sigma \to \mathbb{H}^3$ be a conformally immersed surface. Since $X(u, v)$ is conformal,

$$\mathrm{d}s^2 = \rho^{-2}(\mathrm{d}u^2 + \mathrm{d}v^2)$$

for some $\rho : \Sigma \to \mathbb{R}$.

We choose the unit normal vector field $\xi : \Sigma \to \mathbb{S}^{2,1}$ of X, where $\mathbb{S}^{2,1} = \{x \in \mathbb{R}^{3,1} : \langle x, x\rangle = 1\}$. Since we are interested in constant mean curvature (CMC) surfaces, we let the mean curvature $H \geq 0$ be any constant. Totally umbilic surfaces, either a plane or a sphere, trivially have curvature lines on horospheres; therefore, we assume that $X(u, v)$ is not totally umbilic, and that (u, v) are conformal curvature line (isothermic) coordinates on Σ. Then we can normalize the Hopf differential factor such that $Q = -\frac{1}{2}$, and calculate the integrability condition, or the Gauss equation, as

$$\rho \cdot \Delta\rho - \rho_u^2 - \rho_v^2 + \rho^4 + 1 - H^2 = 0.$$

We first reformulate the results for minimal surfaces in [14] to be applicable to CMC $H \geq 0$ surfaces.

Lemma 1 (cf. Theorem 5.2 of [14])**.** *Let $X(u, v)$ be an umbilic-free CMC surface with isothermic coordinates. If the v-curvature lines lie on spheres $S[m_1(u), q_1(u)]$, then there are real functions $\alpha(u)$ and $\beta(u)$ such that*

$$2\rho_u(u, v) = \alpha(u) + \beta(u)\rho^2. \tag{2.1}$$

In fact, for $N_1 := m_1 + \langle m_1, X\rangle X$ and $\langle N_1, \xi\rangle =: |N_1||X| \cos\theta_1$,

$$\alpha(u) = -\frac{2q_1}{|N_1| \sin\theta_1} - 2H \cot\theta_1 \quad \text{and} \quad \beta(u) = -2 \cot\theta_1.$$

Next, we describe how we can determine the type of sphere $S[m_1(u), q_1(u)]$ from (2.1).

Lemma 2 (cf. Theorem 5.3 of [14])**.** *Let $X(u, v)$ be as in Lemma 1, further satisfying* (2.1)*. Then, $S[m_1(u), q_1(u)]$ is*

- *a compact sphere if and only if $(\alpha - H\beta)^2 - \beta^2 > 4$,*
- *a horosphere if and only if $(\alpha - H\beta)^2 - \beta^2 = 4$, or*
- *an open pseudosphere if and only if $(\alpha - H\beta)^2 - \beta^2 < 4$.*

Therefore, finding CMC surfaces with v-curvature lines lying on horospheres is equivalent to solving for $\rho(u,v)$ satisfying the following system of partial differential equations

$$\begin{cases} \rho\cdot\Delta\rho-\rho_u^2-\rho_v^2+\rho^4+1-H^2=0 & \text{(CMC condition)}, \quad (2.2a)\\ 2\rho_u=\alpha+\beta\rho^2 & \text{(spherical } v\text{-curvature line condition)}, \quad (2.2b)\end{cases}$$

for some functions $\alpha(u)$ and $\beta(u)$ satisfying

$$(\alpha-H\beta)^2-\beta^2=4 \qquad \text{(horosphericity condition for } v\text{-curvature lines)}. \tag{2.3}$$

3 CMC surfaces with curvature lines on horospheres

Now, to find CMC surfaces with one family of curvature lines on horospheres, we describe how we can solve (2.2) satisfying (2.3). To do this, we first refer to the following useful fact from [14].

Fact 1 (cf. Theorem 2.3 of [14])**.** *Let $\rho(u,v)$ be a solution to the system* (2.2). *If $\rho_v\not\equiv 0$, then the functions $\alpha(u)$ and $\beta(u)$ are solutions to the system*

$$\begin{cases}\alpha_{uu}=a\alpha-2\alpha^2\beta-2(H^2-1)\beta,\\ \beta_{uu}=a\beta-2\alpha\beta^2-2\alpha,\end{cases} \tag{3.1}$$

where a (resp. b) is some constant. Furthermore,

$$4\rho_v^2=-(4+\beta^2)\rho^4+4\beta_u\rho^3+(6\alpha\beta-4a)\rho^2-4\alpha_u\rho-(4(H^2-1)+\alpha^2) \tag{3.2}$$

From this point, assume that $X(u,v)$ is an umbilic-free surface.

3.1 CMC surface with v-curvature lines on horospheres

Now assume that the v-curvature lines lie on horospheres. Using Fact 1, we know that

$X(u,v)$ is a CMC surface with v-curvature lines on horospheres
$\iff \rho\not\equiv 0$ satisfies (2.2a), (2.2b), and (2.3) for some appropriate α and β
$\iff \rho\not\equiv 0,\alpha,\beta$ satisfies (2.2b), (2.3), (3.1), and (3.2).

Therefore, finding CMC surfaces with v-curvature lines on horospheres is equivalent to executing the following steps:

1. Find α and β satisfying (2.3) and (3.1).
2. Find ρ satisfying (2.2b) and (3.2) using α and β from the previous step.

Now, to better use the condition (2.3), define

$$y(u):=\alpha(u)+(1-H)\beta(u) \quad \text{and} \quad z(u):=\alpha(u)-(1+H)\beta(u), \tag{3.3}$$

which allow us to rewrite (2.3) as

$$y(u)\cdot z(u)=4. \tag{3.4}$$

Using these relations, we may simplify (3.1), as shown in the following lemma.

Lemma 3. *Let $y(u)$ and $z(u)$ be as in* (3.3) *and* (3.4). *Then, finding $\alpha(u)$ and $\beta(u)$ satisfying* (3.1) *with* (2.3) *is equivalent to finding $y(u)$ satisfying*

$$(y_u(u))^2=-\frac{1}{4}(H+1)y(u)^4+(a+4H)y(u)^2+4(1-H). \tag{3.5}$$

Using (3.5), we may prove the following proposition.

Proposition 1. *Let $X(u,v)$ be a CMC surface with isothermic coordinates. If one family of curvature lines lies on horospheres, then $X(u,v)$ must be a rotation surface.*

3.2 CMC rotation surfaces with curvature lines on horospheres

By Proposition 1, we understand that if one family of curvature lines of a CMC surface lies on horospheres, then the surface must be a rotation surface. However, the converse may not be true; namely, there may be CMC rotation surfaces which curvature lines do not lie on horospheres. Therefore, we now find the exact condition for a CMC rotation surface to satisfy the horosphericity condition.

Let $X(u, v)$ be a CMC rotation surface with isothermic coordinates. Then, either $\rho_u \equiv 0$ or $\rho_v \equiv 0$; first, assume that $\rho_v \equiv 0$. Then, (2.2a) becomes

$$\rho_u^2 = -\rho^4 - c_1\rho^2 - H^2 + 1, \tag{3.6}$$

for some real constant of integration c_1.

Considering the right side of (3.6) as a polynomial P_2 of ρ, we understand that

$$\begin{cases} c_1 \in \mathbb{R} & \text{if } 0 \le H < 1, \\ c_1 < -2\sqrt{H^2-1} & \text{if } H = 1, \\ c_1 \le -2\sqrt{H^2-1} & \text{if } H > 1. \end{cases} \tag{3.7}$$

Considering the case $\rho_u \equiv 0$ in a similar fashion, we find the exact condition such that one family of curvature lines lie on horospheres as follows.

Theorem 1. *$X(u, v)$ is a CMC surface with one family of curvature lines on horospheres if and only if it is a CMC rotation surface where the metric function ρ satisfies either*

- *$\rho_v \equiv 0$ and $\rho_u^2 = -\rho^4 - c_1\rho^2 - H^2 + 1$ for* $\begin{cases} c_1 \le 2H & \text{if } 0 \le H < 1, \\ c_1 < -2\sqrt{H^2-1} & \text{if } H = 1, \\ c_1 \le -2\sqrt{H^2-1} & \text{if } H > 1, \end{cases}$ *or*
- *$\rho_u \equiv 0$ and $\rho_v^2 = -\rho^4 - c_2\rho^2 - H^2 + 1$ for $c_2 \le -2H$*

for some constants c_1 and c_2 (see also the following Figure 1).

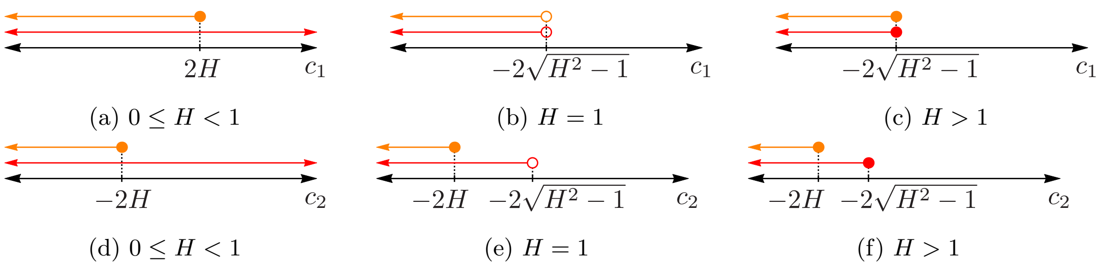


Figure 1: Relationship between c_1 and c_2 and the resulting surface. Values in the red region define CMC rotation surfaces, while values in the orange region define CMC rotation surfaces with one family of curvature lines on horospheres.

4 Geometric meaning of the bifurcation

Theorem 1 tells us that while some rotation surfaces have one family of curvature lines on horospheres, some CMC rotation surfaces do not have this condition. In this section, we consider the geometric meaning of the bifurcation. First, for any rotation surfaces we show the existence of two fixed 2-dimensional spans in $\mathbb{R}^{3,1}$ (i.e. fixed hyperplanes), as in [3] and [13].

Proposition 2 ([3], [13])**.** *Let $X(u, v)$ be a CMC rotation surface in $\mathbb{H}^3$ with $\rho_v \equiv 0$. Then, for $A^1(v) := \rho(u)X_v(u, v)$*

$$P^1 = \operatorname{span}\left\{A^1(v), A^1_v(v)\right\} \tag{4.1}$$

becomes a fixed hyperplane in $\mathbb{R}^{3,1}$ *for all* (u,v)*. Moreover, the orthogonal complement* $(P^1)^\perp$ *also becomes a fixed hyperplane such that*

$$(P^1)^\perp = \text{span}\left\{B^1(u), B^1_u(u)\right\} \tag{4.2}$$

for $B^1(u) := \frac{\rho_u}{1-H-\rho^2}X + \rho X_u + \frac{\rho_u}{1-H-\rho^2}\xi$*. We call* $(P^1)^\perp$ *the* ***generating hyperplane*** *of a CMC rotation surface* $X(u,v)$ *in* $\mathbb{H}^3$*.*

Note that if $\rho_v \equiv 0$, then v is in fact the parameter describing the rotation of the surface $X(u,v)$. However, $B^1(u)$ and $(B^1)_u(u)$ are vectors independent of v. Since the rotations in $\mathbb{R}^{3,1}$ leave two directions fixed, we can deduce that $(P^1)^\perp = \text{span}\left\{B^1(u), B^1_u(u)\right\}$ is indeed the *generating hyperplane* for the rotation surface $X(u,v)$ as defined in [3].

Now, following [3], we define the types of CMC rotation surfaces and the (rotation) axis.

Definition (cf. p.688 of [3])**.** Following the notation from Proposition 2, a CMC rotation surface $X(u,v)$ is

- **spherical** if $(P^1)^\perp$ has $(+\,-)$ signature,
- **hyperbolic** if $(P^1)^\perp$ has $(+\,+)$ signature, or
- **parabolic** if $(P^1)^\perp$ has $(+\,0)$ signature.

Moreover, for a spherical CMC rotation surface, we call $\vec{v}_1 := (P^1)^\perp \cap \mathbb{H}^3$ the **(rotation) axis** of $X(u,v)$.

By calculating the signature of the basis of the generating hyperplane, we understand the geometric meaning of the bifurcation in Theorem 1 and Figure 1 as explained in the following Figure 2.

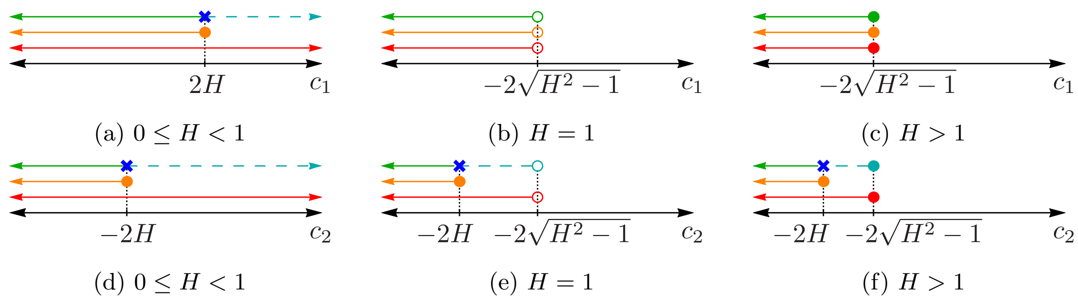
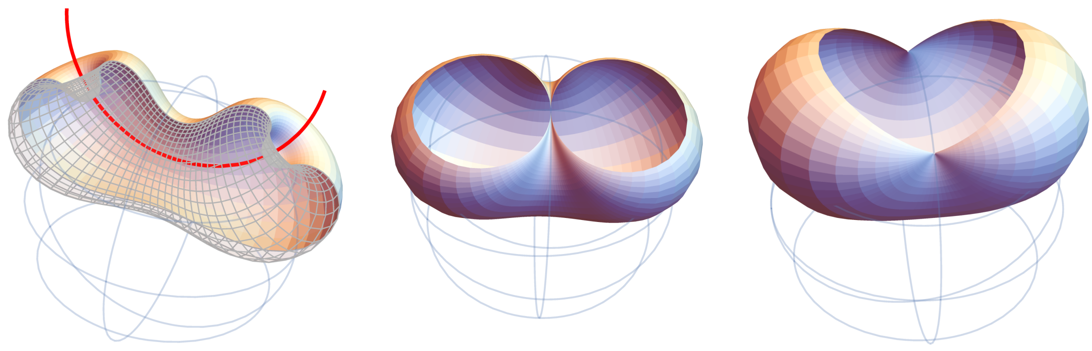


Figure 2: Relationship between the horosphericity condition and types of CMC rotation surface. In addition to the information in Figure 1, values in the green, blue, and dashed cyan regions define CMC spherical, parabolic, and hyperbolic rotation surfaces, respectively.

Therefore, in summary, we obtain the following main theorem.

Theorem 2. *If* $X(u,v)$ *is a constant mean curvature surface in* $\mathbb{H}^3$ *with one family of curvature lines lying on horospheres, then it must be a piece of one, and only one, of the following:*

- *a totally umbilic surface,*
- *a CMC parabolic rotation surface, or*
- *a CMC spherical rotation surface.*



(a) Spherical rotation surface (b) Parabolic rotation surface (c) Hyperbolic rotation surface

Figure 3: CMC 1/2 rotation surfaces in $\mathbb{H}^3$ visualized using the Poincaré ball model. The first two surfaces have one family of curvature lines on horospheres, while the last surface does not.

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