On the uniform birationality of the pluriadjoint line bundles

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Abstract

We show a uniform birationality for multiadjoint maps. That is, for any integer n and κ with $1 \leq \kappa \leq n$, there exists a positive integer $m_{n,\kappa}$ such that for any n-dimensional polarized manifold (X, L) with $\kappa(K_X + L) = \kappa$, the linear system $|m(K_X + L)|$ gives the Iitaka fibration associated to $K_X + L$ for every $m \geq m_{n,\kappa}$.

1 Introduction

Let X be a smooth projective variety defined over $\mathbf{C}$ and L an ample line bundle over X. Then the pair (X, L) is called a **polarized manifold**.

In the classification theory of polarized manifolds, it is important to study a condition on the integer m for which $|K_X + mL|$ is free. Fujita's freeness conjecture ([9]) predicts that $|K_X + mL|$ is free for every $m \geq \dim X + 1$. It is known that this conjecture is true in the case of $\dim X \leq 4$ (cf. [27], [8], [17]). In higher dimensional case, H. Tsuji ([31]) proved that $|K_X + mL|$ is free for any $m \geq \dim X(\dim X + 1)/2 + 1$ (see also [2]).

On the other hand when $K_X + L$ is nef, by virtue of the nonvanishing theorem due to V. Shokurov ([28]), $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$ holds for $m \gg 0$. In particular, $\kappa(X, K_X + L) \geq 0$ holds. Then it is important to find an integer m with $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$. Concerning this, F. Ambro ([1]) and Y. Kawamata ([19]) proposed the following conjecture:

Conjecture 1.1. *Let X be a normal projective variety and B an effective $\mathbf{Q}$-divisor on X such that (X, B) is a KLT pair (cf. Definition 2.6). Let D be a nef Cartier divisor on X such that $D - (K_X + B)$ is nef and big. Then $H^0(X, \mathcal{O}_X(D)) \neq 0$ holds.*

We note that if X is smooth, $B = 0$ and $D := K_X + L$ is nef, then this conjecture implies that $H^0(X, \mathcal{O}_X(K_X + L)) \neq 0$ holds for every polarized manifold (X, L) with $K_X + L$ nef. In [19], Kawamata solved the conjecture above when X is 2-dimensional and when X is a minimal 3-fold. In [16], A. Höring solved it in the case where X is a normal projective 3-fold with at most $\mathbf{Q}$-factorial canonical singularities, $B = 0$, and $D - K_X$ is a nef and big Cartier divisor. These results are immediate consequences of the Hirzebruch-Riemann-Roch theorem and some classical results on surfaces and 3-folds. In constrast in higher dimensional case, it is rather difficult to calculate the dimension of $H^0(X, \mathcal{O}_X(D))$. Indeed, Conjecture 1.1 is still widely open for the case of $\dim X \geq 4$.

In [10, Problem 3.2], Y. Fukuma proposed the following problem:

Problem 1.2. *For a fixed positive integer n, find the smallest positive integer m_n depending only on n such that $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$ holds for every polarized manifold (X, L) of dimension n with $\kappa(K_X + L) \geq 0$.*

It is known that $m_1 = 1$ and $m_2 = 1$ (cf. [10, Theorem 2.8]). Also, it was proved that $m_3 = 1$ (See [16, 1.5 Theorem] and [12]). In [12], the case of $\dim X = 4$ was treated. Concerning higher dimensional case, in [3] the author showed that $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$ for every $m \geq n(n + 1)/2 + 2$ for any n-dimensional polarized manifold (X, L) with $K_X + L$ nef.

In this report, we consider the case where $K_X + L$ is not necessarily nef; we prove the following theorem.

Theorem 1.3. *Fix integers n and κ with $1 \leq \kappa \leq n$. Then there exists a positive integer $m_{n,\kappa}$ such that, if (X, L) is an n-dimensional polarized manifold with $\kappa(X, K_X + L) = \kappa$, then $\Phi_{|m(K_X+L)|}$ is birationally equivalent to Φ_∞ for every integer $m \geq m_{n,\kappa}$, where Φ_∞ denotes the Iitaka fibration associated to $K_X + L$ (cf. Section 2.2). In particular, for any n-dimensional polarized manifold (X, L) with $\kappa(X, K_X + L) = \kappa$,*

$$H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$$

holds for every $m \geq m_{n,\kappa}$.

We show the theorem above in Section 4. In the proof, we reduce the problem to uniform birationality for multiples of big adjoint bundles $K_Y + B$ over the base space Y of Φ_∞. Thus we need to show the following:

Theorem 1.4. *Fix a positive integer n. Then there exists a positive integer m_n such that, for a smooth projective variety Y of dimension n and for a big Cartier divisor B on Y with $K_Y + B$ big, the linear system $|m(K_Y + B)|$ gives a birational map for every $m \geq m_n$.*

We prove the theorem above in Section 3 by induction on n. Also we use the techniques developed by Hacon-McKernan ([13]), Takayama ([29]) and Tsuji ([33]) in their study of pluricanonical systems of projective varieties of general type. We notice that the method does not lead to an effective constant m_n.

As a consequence of Theorem 1.4, we have the following:

Corollary 1.5. *Fix a positive integer n. Then there exists a positive constant $C(n)$ such that, for a smooth projective variety Y of dimension n and for a big Cartier divisors B on Y with $K_Y + B$ big,*

$$\mu(Y, K_Y + B) \geq C(n)$$

holds, where $\mu(Y, K_Y + B)$ denotes the volume of Y with respect to $K_Y + B$ (cf. Definition 2.18).

We use the corollary in the induction procedure in the proof of Theorem 1.4. In fact, assuming that there exists such a lower bound $C(k)$ in the case of $k < n$, we can deduce that Theorem 1.4 is true in dimension n (see Lemma 3.2 and Section 3.3).

2 Preliminaries

In this section, we review some algebraic and analytic notions.

2.1 Nef and big line bundles

In this subsection, we shall recall some properties of nef and big line bundles.

Definition 2.1. Let X be a normal variety and D a Cartier divisor on X. Then

1. D is said to be *nef*, if $D \cdot C \geq 0$ holds for every irreducible curve C on X.

2. D is said to be *big*, if $\kappa(X, D) = \dim X$ holds,

where $\kappa(X, D)$ denotes the Iitaka-Kodaira dimension of D defined by

$$\kappa(X, D) = \limsup_{m \to \infty} \frac{\log h^0(X, \mathcal{O}_X(mD))}{\log m}.$$

By the Riemann-Roch theorem, we have the following:

Proposition 2.2 ([6, Corollary 4.3])**.** *Let X be a smooth projective variety of dimension n and D a nef and big divisor on X. Then*

$$h^0(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1})$$

holds.

2.2 Iitaka fibration

Let L be a line bundle on a projective variety X. We define the set $\mathbf{N}(L)$ of L by

$$\mathbf{N}(L) := \{m \geq 0 \mid H^0(X, \mathcal{O}_X(mL)) \neq 0\}.$$

If $\kappa(X, L) \geq 1$, then $\dim \Phi_{|mL|}(X) = \kappa(X, L)$ holds for every sufficiently large integer $m \in \mathbf{N}(L)$. Here we denote by $\Phi_{|mL|}$ the rational map associated with the linear system $|mL|$. The following theorem by S. Iitaka is fundamental:

Theorem 2.3 ([23, 2.1.C])**.** *Let X be a normal projective variety and L a line bundle on X with $\kappa(X, L) > 0$. Then for all sufficiently large $k \in \mathbf{N}(L)$, the rational maps $\Phi_{|kL|} : X \dashrightarrow Y_k$ are birationally equivalent to a fixed algebraic fiber space*

$$\Phi_\infty : X_\infty \longrightarrow Y_\infty$$

of normal spaces such that the restriction of L to a very general fiber of Φ_∞ has Iitaka dimension zero, and that $\dim Y_\infty = \kappa(X, L)$ *holds.* (We call Φ_∞ the **Iitaka fibration** associated to L.)

2.3 Singularities of divisors

In this subsection, we shall introduce the notion of singularities of pairs.

Definition 2.4. Let X be a normal variety and $U := X_{\rm reg}$ the nonsingular locus of X. Since $\operatorname{codim}(X \setminus U) \geq 2$ holds, every divisor on X is uniquely determined by its restriction to U (cf. [15, Chapter II]). Then we can define the *canonical sheaf* $\omega_X = \mathcal{O}_X(K_X)$ of X by

$$\omega_X := i_*\mathcal{O}_U(K_U),$$

where $i : U \hookrightarrow X$ denotes the inclusion.

Definition 2.5. Let (X, D) be a pair of a normal variety X and an effective $\mathbf{Q}$-divisor D on X. A proper birational morphism $\mu : Y \longrightarrow X$ is said to be a **log resolution** of (X, D), if Y is smooth and $\operatorname{Exc}(\mu) \cup \mu_*^{-1}D$ has a simple normal crossing support, where $\operatorname{Exc}(\mu)$ denotes the exceptional locus of μ, and $\mu_*^{-1}D$ denotes the strict transform of D.

Definition 2.6. Let (X, D) be a pair of a normal variety and an effective $\mathbf{Q}$-divisor on X. Suppose that $K_X + D$ is $\mathbf{Q}$-Cartier. Let $\mu : Y \to X$ be a log resolution of (X, D). Then we have the formula:

$$K_Y = \mu^*(K_X + D) + \sum_i a_i E_i,$$

where E_i is a prime divisor and $a_i \in \mathbf{Q}$. Then the pair (X, D) is said to have only **Kawamata log terminal singularities** (**KLT**, for short) (resp. **log canonical singularities** (**LC**, for short)), if $a_i > -1$ (resp. $a_i \geq -1$) holds for every i. We call a_i the **discrepancy coefficient** for E_i. The pair (X, D) is said to be KLT (resp. LC) at a point $x \in X$, if $(U, D|_U)$ is KLT (resp. LC) for some neighborhood U of x.

Definition 2.7. Let (X, D) be a pair of a normal variety and an effective $\mathbf{Q}$-divisor on X. A subvariety W of X is said to be a **center of log canonical singularities** (or **LC-center**) for (X, D), if there exists a log resolution $\mu : Y \longrightarrow X$ of (X, D) and a prime divisor E on Y with discrepancy coefficient $e \leq -1$ and $\mu(E) = W$. We denote by $CLC(X, D)$ the set of all centers of log canonical singularities for (X, D). For a point $x \in X$, we set $CLC(X, x, D) := \{W \in CLC(X, D) \mid x \text{ belongs to } W\}$.

Proposition 2.8 ([17, Proposition 1.5])**.** *Let (X, D) be a pair of a normal variety and an effective $\mathbf{Q}$-Cartier divisor such that $K_X + D$ is $\mathbf{Q}$-Cartier. Assume that X is KLT and (X, D) is LC. If W_1 and W_2 are the elements of $CLC(X, D)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. This implies that if (X, D) is LC but not KLT at a point $x \in X$, there exists the unique minimal element of $CLC(X, x, D)$.* (We call the minimal element of $CLC(X, x, D)$ the **minimal center of log canonical singularities** of (X, D) at x.)

2.4 Singular hermitian metrics and multiplier ideal sheaves

Our basic tool is singular hermitian metrics as in [6]. Here we shall recall the notions of singular hermitian metrics and multiplier ideal sheaves.

Definition 2.9. Let L be a holomorphic line bundle over a complex manifold X. A **singular hermitian metric** h on L is given by

$$h = h_0 \cdot e^{-\varphi},$$

where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{\rm loc}(X)$. We call φ the **weight function** of h with respect to h_0, and we say that the hermitian line bundle (L, h) is the **singular hermitian line bundle** over X. The **curvature current** Θ_h of h is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\partial\overline{\partial}\varphi,$$

where $\Theta_{h_0} = \sqrt{-1}\overline{\partial}\partial \log h_0$ is the curvature form of h_0 and $\partial\overline{\partial}\varphi$ is taken in the sense of currents.

Example 2.10. Let L be a holomorphic line bundle over a complex manifold X. Suppose that there exists a positive integer m such that $\Gamma(X, \mathcal{O}_X(mL)) \neq 0$. Let $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$ be a nontrivial section. Then

$$h := \frac{1}{|\sigma|^{2/m}} = \frac{h_0}{h_0^{\otimes m}(\sigma, \sigma)^{1/m}}$$

is a singular hermitian metric on L, where h_0 is an arbitrary C^∞-hermitian metric on L (the right hand side is independent of h_0). By Poincaré-Lelong's formula, we have $\Theta_h = 2\pi/m(\sigma)$, where (σ) denotes the current of integration over the divisor of σ. In particular, Θ_h is a positive current.

Definition 2.11. Let (L, h) be a singular hermitian line bundle over a complex manifold X. The **L^2-sheaf** $\mathcal{L}^2(L, h)$ of (L, h) is defined by

$$\mathcal{L}^2(L, h)(U) := \{\sigma \in \Gamma(U, \mathcal{O}_X(L)) \mid h(\sigma, \sigma) \in L^1_{\mathrm{loc}}(U)\},$$

where U runs over the open subsets of X.

Now we shall write h as $h = h_0 \cdot e^{-\varphi}$, where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{\mathrm{loc}}(X)$ is the weight function of h with respect to h_0. Then we define the **multiplier ideal sheaf** $\mathcal{I}(h)$ of (L, h) by

$$\mathcal{I}(h) := \mathcal{L}^2(\mathcal{O}_X, e^{-\varphi}).$$

Note that $\mathcal{L}^2(L, h) = \mathcal{O}_X(L) \otimes \mathcal{I}(h)$ holds.

The notion of multiplier ideal sheaves is very useful in investigating singularities of pairs as in the previous section. Using the above notation, we shall define multiplier ideal sheaves of divisors as follows.

Definition 2.12. Let $D = \sum_i a_i D_i$ be an effective $\mathbf{Q}$-divisor on X. Let σ_i be a global section of $\mathcal{O}_X(D_i)$ with divisor D_i and let h_i be a C^∞-hermitian metric on $\mathcal{O}_X(D_i)$. Then we define the multiplier ideal sheaf $\mathcal{I}(D)$ associated with D by

$$\mathcal{I}(D) := \mathcal{L}^2\left(\mathcal{O}_X, \frac{1}{\prod_i h_i(\sigma_i, \sigma_i)^{a_i}}\right).$$

The following proposition reveals a relation between multiplier ideal sheaves and singularities of pairs.

Proposition 2.13 ([22, Proposition 3.20])**.** *Let X be smooth projective variety of dimension n and D an effective $\mathbf{Q}$-divisor on X. Then (X, D) is KLT at a point x of X if and only if $\mathcal{I}(D)_x = \mathcal{O}_{X,x}$ holds. In particular, if the multiplicity of D at x is greater than or equal to n, then $\mathcal{I}(D)_x \neq \mathcal{O}_{X,x}$.*

The following vanishing theorem due to A. Nadel ([25]) is crucial.

Theorem 2.14. *Let (L, h) be a singular hermitian line bundle over a compact Kähler manifold X and ω a Kähler form on X. Suppose that the curvature current Θ_h of h is strictly positive, i.e., there exists a constant $\varepsilon > 0$ such that $\Theta_h - \varepsilon\omega$ is a positive $(1,1)$-current. Then $\mathcal{I}(h)$ is a coherent sheaf on X, and*

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h)) = 0$$

holds for every $q \geq 1$.

2.5 Analytic Zariski Decompositions

In this subsection, we recall the notion of pseudoeffective line bundles, and introduce the notion of analytic Zariski decompositions, which is used in the proof of Theorem 1.4 (Section 3). Using analytic Zariski decompositions, we can handle pseudoeffective line bundles like nef line bundles.

Definition 2.15. A line bundle L over a smooth projective variety X is said to be **pseudoeffective**, if there exists an ample line bundles A over X such that

$$H^0(X, \mathcal{O}_X(mL + A)) \neq 0$$

holds for every integer $m \geq 1$.

Remark 2.15.1. By [6, Proposition 4.2], we see that L is pseudoeffective if and only if L has a singular hermitian metric h with $\Theta_h \geq 0$. Then we call (L, h) a pseudoeffective singular hermitian line bundle.

The following notion of analytic Zariski decompositions was introduced by Tsuji ([30]).

Definition 2.16. Let L be a holomorphic line bundle over a compact complex manifold X. A singular hermitian metric h on L is said to be an **analytic Zariski decomposition** (**AZD**, for short), if the following properties hold:

1. Θ_h is a positive current;
2. for any positive integer m, the natural inclusion:

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)) \longrightarrow H^0(X, \mathcal{O}_X(mL))$$

is an isomorphism.

By definition, it follows that a line bundle equipped with an AZD is pseudoeffective. Conversely, we claim that every pseudoeffective line bundle has an AZD as follows:

Theorem 2.17 ([7, Theorem 1.5])**.** *Let X be a smooth projective variety and L a pseudoeffective line bundle over X. Then L has an AZD.*

2.6 Volumes of pseudoeffective line bundles

In order to measure the positivity of pseudoeffective line bundles, we introduce the notion of volume of a smooth projective variety with respect to a singular hermitian metrics. First we shall recall the notion of volumes with respect to line bundles:

Definition 2.18. Let M be a n-dimensional complex manifold and L a holomorphic line bundle over M . We define the **volume** of M with respect to L by

$$\mu(M, L) = n! \cdot \limsup_{m\to\infty} m^{-n} h^0(M, \mathcal{O}_M(mL)).$$

We shall define the volumes with respect to pseudoeffective singular hermitian line bundles:

Definition 2.19. Let X be a smooth projective variety X of dimension n and (L, h) a pseudoeffective singular hermitian line bundle over X. We define the volume of X with respect to (L, h) by

$$\mu(X,(L, h)) = n! \cdot \limsup_{m\to\infty} m^{-n} h^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)).$$

Let Y be a subvariety of X and $\pi_Y : \tilde{Y} \longrightarrow Y$ its desingularization. Then we define $\mu(Y,(L, h)|_Y)$ as

$$\mu(Y,(L, h)|_Y) := \mu(\tilde{Y}, \pi_Y^*(L, h)|_Y),$$

where $\pi_Y^*(L, h) := (\pi_Y^* L, \pi_Y^* h)$. The right hand side is independent of the choice of the desingularization π_Y of Y (see [33, Remark 2.23]).

2.7 A subadjunction theorem

In this subsection, we review a subadjunction theorem, which is used in the proof of Lemma 3.7 later. Lemma 3.7 is a key step for the proof of Theorem 1.4.

Let M be a smooth projective variety and (L, h_L) a pseudoeffective singular hermitian line bundle over M. Suppose that h_L is lower-semicontinuous. (Then the local potential of the curvature current Θ_{h_L} is plurisubharmonic, and so the restriction of h_L to any subvariety of M is well-defined.) Let m_0 be a positive integer. Suppose that

$$\Gamma(M, \mathcal{O}_M(m_0L) \otimes \mathcal{I}_\infty(h_L^{m_0})) \neq 0$$

holds. Then we take a non-trivial section $\sigma \in \Gamma(M, \mathcal{O}_M(m_0L) \otimes \mathcal{I}_\infty(h_L^{m_0}))$. Let α be a rational number with $0 < \alpha \leq 1$. Let S be an irreducible subvariety of M such that $(M, \alpha(\sigma))$ is LC but not KLT on the generic point of S, and $(M,(\alpha - \varepsilon)(\sigma))$ is KLT on the generic point of S for any rational number $0 < \varepsilon \ll 1$. Then we define the function $\Psi_S : M \longrightarrow [-\infty, 0)$ by

$$\Psi_S := \alpha \log h_L^{m_0}(\sigma, \sigma).$$

For simplicity, we suppose that S is smooth. (When S is not smooth, we need to take an embedded resolution in order to apply Theorem 2.20 below.) We also assume that S is not contained in the singular locus of h.

Let dV be a C^∞-volume form on M. Then we can define the residue volume form $dV[\Psi_S]$ on S as follows: Let $f : N \longrightarrow M$ be a log resolution of $(M, \alpha\Delta)$. Then we can define the residue volume form $f^*dV_N[f^*\Psi_S]$ on the divisorial component of $f^{-1}(S)$.

Then we have the following extension theorem, which is a generalization of Theorem 2.24 in [33].

Theorem 2.20. *Let M, S, Ψ_S, φ, L, h_L be as above. Let (B, h_B) be a pseudoeffective singular hermitian line bundle such that the singular locus of h_B does not contain S. Let d be an integer with $d > \alpha m_0$. Suppose that there exists an AZD h_S of*

$$(K_M + dL + B|_S, e^{-\varphi} \cdot (dV^{-1} \cdot h_L^d \cdot h_B)|_S).$$

Then any element of

$$A^2(S, \mathcal{O}_S(m(K_M + dL + B)),(dV^{-1} \cdot h_L^d \cdot h_B)|_S \cdot h_S^{m-1}, dV[\Psi_S])$$

extends to an element of $H^0(M, \mathcal{O}_M(m(K_M + dL + B)))$ for every integer $m \geq 1$.

The proof is parallel to that of Theorem 2.24 in [33], so we omit it.

3 Proof of Theorem 1.4

The organization of this section is as follows: In Section 3.1, we show Corollary 1.5. As mentioned in Section 1, we use this in the proof of Theorem 1.4. In Section 3.2, we prove a weaker version of Theorem 1.4 (see Lemma 3.2) by using Angehrn-Siu's method. Finally in Section 3.3, by using these lemmas, we show Theorem 1.4.

3.1 Proof of Corollary 1.5

In this subsection, we show Corollary 1.5. First we recall the following result:

Lemma 3.1 (cf. [26, Corollary 6.1])**.** *Let X be a smooth projective n-fold and L a big line bundle over X. Let m be a positve integer. Suppose that $\Phi_{|mL|} : X \dashrightarrow \mathbf{P}^N$ is a birational map onto its image. Then*

$$\deg_{\mathbf{P}^N} \Phi_{|mL|} \leq m^n \cdot \mu(X, L)$$

holds, where $N := h^0(X, mL) - 1$.

Proof of Corollary 1.5. Let Y be a smooth projective n-folds and B a big Cartier divisors on Y with $K_Y + B$ big. Let m_n be the positive constant as in Theorem 1.4. Then since $\Phi_{|m_n(K_Y+B)|}$ gives a birational map onto its image, by Lemma 3.1, we obtain

$$1 \leq \deg_{\mathbf{P}^N} \Phi_{|m_n(K_Y+B)|} \leq m_n^n \cdot \mu(Y, K_Y + B).$$

Therefore putting $C(n) := m_n^{-n}$, we have completed the proof of Corollary 1.5. □

3.2 Point separation for big pluriadjoint systems

We prove the following weaker version of Theorem 1.4:

Lemma 3.2. *Let n be a positive integer. Suppose that there exists a positive constant v such that, if V is a smooth projective variety with $\dim V < n$ and D is a big Cartier divisor on V with $K_V + D$ big, then $\mu(K_V + D) \geq v$ holds. Then there exists positive constants a_n and b_n such that, for any smooth projective n-fold Y and for any big Cartier divisors B such that $K_Y + B$ is big, the map $\Phi_{|m(K_Y+B)|}$ is a birational map for every $m \geq a_n \cdot \mu(K_Y + B)^{-1/n} + b_n$.*

Before going on the proof of the lemma above, we shall construct a filtration of Y as follows: Let Y and B be as above. Let h be an AZD of $K_Y + B$ as in Theorem 2.17. Then we may assume that h is lower semicontinuous, and so the restriction of h to any subvariety of Y is well-defined. We denote by Y° the set of points y on Y such that $|m(K_Y + B)|$ is base-point-free at y and $\Phi_{|m(K_Y+B)|}$ is an isomorphism on a neighborhood of y for some $m \geq 1$. Then it suffices to show the following:

Lemma 3.3. *Let x and y be distinct points on Y°. Then there exist a filtration of Y:*

$$Y \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_r \supsetneq Y_{r+1} = \{x\} \text{ or } \{y\},$$

by a strictly decreasing sequence of subvarieties $\{Y_i\}_{i=0}^{r+1}$ for some r, and invariants:

$$\alpha_0, \alpha_1, \cdots, \alpha_r > 0,$$

$$\mu_0, \mu_1, \cdots, \mu_r \quad (\mu_i := \mu(Y_i, (K_Y + B, h)|_{Y_i}))$$

with the estimates:

$$\alpha_i \leq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + \delta, \tag{3.1}$$

where δ is a fixed positive number with $\delta \ll 1/n$ and $n_i := \dim Y_i$. We also obtain global sections:

$$\sigma_i \in H^0(Y, \mathcal{O}_Y((m_i + l_i)(K_Y + B)))$$

for some positive integers $m_0, \ldots, m_r$ and $l_1, \ldots, l_r$, where $l_0 := 0$. Letting

$$D_i := \frac{1}{m_i + l_i}(\sigma_i),$$

α_i is defined inductively by

$$\alpha_i := \inf \left\{ \alpha > 0 \mid \left(Y, \sum_{j=0}^{i-1} (\alpha_j - \varepsilon_j) D_j + \alpha D_i \right) \text{ is KLT at neither } x \text{ nor } y \right\},$$

where $\varepsilon_0, \ldots, \varepsilon_{i-1} > 0$ are small rational numbers. Here each Y_i ($i = 2, \ldots, r$) is the minimal center of log canonical singularities of the pair

$$\left(Y, \sum_{j=0}^{i-2} (\alpha_j - \varepsilon_j) D_j + \alpha_{i-1} D_{i-1} \right)$$

at x or y.

If we have already constructed the filtration above, we obtain the following:

Lemma 3.4. $\Phi_{|m(K_Y+B)|}$ *separates* x *and* y *for every integer* $m > \sum_{i=0}^{r} \alpha_i + 1$.

Proof. For each integer i with $0 \le i \le r$, we define the singular hermitian metric h_i on $K_Y + B$ by $h_i := |\sigma_i|^{-2/(m_i+l_i)}$. Since B is big, there exists an $\mathbf{Q}$-effective divisor E on Y such that $B - E$ is ample. We may assume that both x and y are not contained in the support of E. Now we fix a C^∞-hermitian metric h'_B on $B - E$ with strictly positive curvature. Let $E = \sum_k e_k E_k$ be the irreducible decomposition of E and $\sigma_k \in \Gamma(X, E_k)$ a global section of E_k with $(\sigma_k) = E_k$. Then we define the singular hermitian metric h_B on B by

$$h_B := h'_B \cdot \frac{1}{\prod_k |\sigma_k|^{2e_k}}.$$

We fix an integer m with $m > \sum_{i=0}^{r} \alpha_i + 1$. Let $h_{x,y}$ be the singular hermitian metric on $(m-1)(K_Y + B) + B$ defined by

$$h_{x,y} := \left(\prod_{i=0}^{r-1} h_i^{\alpha_i - \varepsilon_i} \right) \cdot h_r^{\alpha_r} \cdot h^{m-1-\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i) - \alpha_r} \cdot h_B,$$

where h is an AZD of $K_Y + B$ as above. Then it follows that the curvature current of $h_{x,y}$ is strictly positive. Furthermore, by the construction of the filtration (Lemma 3.3), we see that the support of $\mathcal{O}_Y/\mathcal{I}(h_{x,y})$ contains both x and y and it is isolated at least at one of x or y. Then by virtue of Theorem 2.14, we obtain the surjection:

$$H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \longrightarrow H^0(Y, \mathcal{O}_Y(m(K_Y + B)) \otimes \mathcal{O}_Y/\mathcal{I}(h_{x,y})).$$

Hence we can take a section $\sigma \in H^0(Y, \mathcal{O}_Y(m(K_Y + B)))$ with $\sigma(x) \neq 0$ and $\sigma(y) = 0$, thereby completing the proof of Lemma 3.4. □

3.2.1 Proof of Lemma 3.3

Proof of Lemma 3.3. We set $\mu_0 := \mu(Y, (K_Y + B, h))$ (cf. Definition 2.19). Since $K_Y + B$ is big and h is an AZD, we get $\mu_0 > 0$. Then by the Riemann-Roch theorem and the dimension-counting argument, we have the following:

Lemma 3.5. *Fix a positive number* $\varepsilon < 1$. *Then*

$$H^0(Y, \mathcal{O}_Y(m(K_Y + B)) \otimes \mathcal{I}(h^m) \otimes \mathfrak{m}_{x,y}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)m/\sqrt[n]{2} \rceil}) \neq 0$$

holds for every $m \gg 1$, *where* $\mathfrak{m}_{x,y} := \mathfrak{m}_x \cdot \mathfrak{m}_y$.

We take a number $0 < \varepsilon < 1$ and an integer $m_0 \gg 0$ as in Lemma 3.5, and take a non-trivial section:

$$\sigma_0 \in H^0(Y, \mathcal{O}_Y(m_0(K_Y + B)) \otimes \mathcal{I}(h^{m_0}) \otimes \mathfrak{m}_{x,y}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)m_0/\sqrt[n]{2} \rceil}).$$

Then we define the $\mathbf{Q}$-divisor D_0 on Y by

$$D_0 := \frac{1}{m_0}(\sigma_0),$$

and the singular hermitian metric h_0 on $K_Y + B$ by

$$h_0 := \frac{1}{|\sigma_0|^{2/m_0}}.$$

Let α_0 be the positive number defined by

$$\alpha_0 := \inf\{\alpha > 0 \mid (Y, \alpha D_0) \text{ is KLT at neither } x \text{ nor } y.\}.$$

(In other words, α_0 is the infimum of positive numbers α such that both x and y belong to the support of $\mathcal{O}_Y/\mathcal{I}(h_0^{\alpha})$.) Then by Proposition 2.13, we have the following estimate:

Lemma 3.6.

$$\alpha_0 \le \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}} + \delta,$$

where δ *is a positive number with* $\delta \ll 1/n$.

Let Y_1 be the minimal center of log canonical singularities of $(Y, \alpha_0 D_0)$ at x or y. If $Y_1 = \{x\}$, then we stop constructing the filtration. If not, we repeat the same process. Finally we obtain the desired filtration of Y. This completes the proof of Lemma 3.3. □

3.2.2 Proof of Lemma 3.2

In order to complete the proof of Lemma 3.2, we need to show the following estimate:

Lemma 3.7. *Let $Y \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_r \supsetneq Y_{r+1} = \{x\}$ or $\{y\}$ be the filtration as constructed in Lemma 3.3. Let j be an integer with $1 \leq j \leq r$. Let $\varpi : W_j \longrightarrow Y_j$ be a resolution of singularities. Then*

$$\mu(W_j, K_{W_j} + B_{W_j}) \leq \left(\left\lceil 1 + \sum_{i=0}^{j-1} \alpha_i \right\rceil\right)^{n_j} \cdot \mu_j$$

holds, where $B_{W_j} := \varpi^* B|_{Y_j}$.

The proof is as follows: Let D_i, α_i, ε_i be as in Lemma 3.3. Fix an integer j with $1 \leq j \leq r$. Then we set

$$D := \sum_{i=0}^{j-2} (\alpha_i - \varepsilon_i) D_i + \alpha_{j-1} D_{j-1}.$$

Let $\pi : Z \longrightarrow Y$ be a log resolution of (Y, D) which factors through ϖ. By taking a suitable modification, we may assume that there exists a unique irreducible component F_j of the exceptional divisor of π with discrepancy -1 such that F_j dominates Y_j by π. Let $\pi_j : F_j \longrightarrow W_j$ be the morphism induced by the construction of F_j. We set $\beta_j := \sum_{i=0}^{j-2} (\alpha_i - \varepsilon_i) + \alpha_{j-1}$ and $\gamma_j := 1 + \lceil \beta_j \rceil$. Then we obtain the following :

Lemma 3.8.

$$\mu(W_j, K_{W_j} + B_{W_j}) \leq n_j! \limsup_{m \to \infty} m^{-n_j} \dim \mathrm{Image}\{H^0(Y, \mathcal{O}_Y(m\gamma_j(K_Y + B))) \longrightarrow H^0(Y_j, \mathcal{O}_{Y_j}(m\gamma_j(K_Y + B)))\}$$

holds.

This lemma follows from the subadjunction theorem (Theorem 2.20), but the proof is technical. So we omit it.

Proof of Lemma 3.7. Recall that h is an AZD of $K_Y + B$ as in Theorem 2.17. Then, we have

$$H^0(Y, \mathcal{O}_Y(m\gamma_j(K_Y + B)) \otimes \mathcal{I}_\infty(h^{m\gamma_j})) \cong H^0(Y, \mathcal{O}_Y(m\gamma_j(K_Y + B)))$$

for every $m \geq 0$. Thus for every section $\sigma \in H^0(Y, \mathcal{O}_Y(m\gamma_j(K_Y + B)))$, we see that

$$\sigma|_{Y_j} \in H^0(Y_j, \mathcal{O}_{Y_j}(m\gamma_j(K_Y + B)))$$

holds. Therefore by virtue of Lemma 3.8, we obtain

$$\begin{aligned}\mu(W_j, K_{W_j} + B_{W_j}) &\leq n_j! \cdot \limsup_{m \to \infty} m^{-n_j} \cdot h^0(Y_j, \mathcal{O}_{Y_j}(m\gamma_j(K_Y + B)) \otimes \mathcal{I}(h^{m\gamma_j})) \\ &\leq \gamma_j^{n_j} \cdot \mu(Y_j, (K_Y + B, h)|_{Y_j}).\end{aligned}$$

Since $\beta_j \leq \sum_{i=0}^{j-1} \alpha_i$, we get the desired inequality. This completes the proof of Lemma 3.7. □

In order to prove Lemma 3.2, we use the following lemma:

Lemma 3.9 ([26, Lemma 6.4])**.** *Let Y be a smooth projective variety and M an effective $\mathbf{Q}$-divisor such that $K_Y + M$ is big. Let V be a subvariety passing through a very general point on Y and $\varphi : V' \longrightarrow V$ a desingularization. Then the $\mathbf{Q}$-divisor $K_{V'} + \varphi^* M|_V$ is big.*

Proof of Lemma 3.2. Let x and y be distinct points on Y°. Let

$$Y \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_r \supsetneq Y_{r+1} = \{x\} \text{ or } \{y\}$$

be the filtration as in Lemma 3.3. Since x and y are belong to Y°, by virtue of Lemma 3.9, we see that $K_{W_j} + B_{W_j}$ is big for every j. Then by assumption and Lemma 3.7, we see that

$$v \leq \mu(W_j, K_{W_j} + B_{W_j}) \leq \left(\left\lceil 2 + \sum_{i=0}^{j-1} \alpha_i \right\rceil\right)^{n_j} \cdot \mu_j$$

holds for every $1 \leq j \leq r$. Then by (3.1), we have

$$\frac{1}{\sqrt[n_j]{\mu_j}} \leq \left(\left\lceil 2 + \sum_{i=0}^{j-1} \alpha_i \right\rceil\right) \cdot \frac{1}{\sqrt[n_j]{v}} \leq \left(3 + \sum_{i=0}^{j-1} \frac{\sqrt[n_i]{2} n_i}{\sqrt[n_i]{\mu_i}}\right) \cdot \frac{1}{\sqrt[n_j]{v}}.$$

Then we have

$$\left(1+\sum_{i=0}^{r}\alpha_i <\right) 3+\sum_{i=0}^{r}\frac{\sqrt[n_i]{2}n_i}{\sqrt[n_i]{\mu_i}} \leq \left(3+\sum_{i=0}^{r-1}\frac{\sqrt[n_i]{2}n_i}{\sqrt[n_i]{\mu_i}}\right)\left(1+\frac{n_r\sqrt[n_r]{2}}{\sqrt[n_r]{v}}\right).$$

By repeating the same process and using Lemma 3.4, we finally obtain positive constants a_n and b_n depending only on n such that $\Phi_{|m(K_Y+L)|}$ separates x and y for every m with

$$m \geq \frac{a_n}{\sqrt[n]{\mu_0}} + b_n.$$

This completes the proof of Lemma 3.2. □

3.3 Completion of the proof of Theorem 1.4

Proof of Theorem 1.4. The proof follows the argument in [26, 6.2] (see also [29, p. 584]). We use induction on the dimension of the varieties. We fix a positive integer n. Suppose that Theorem 1.4 and so Theorem 1.5 hold for $n-1$. Let a_n and b_n be positive constants as in Lemma 3.2. Let Y be a smooth projective variety of dimension n and B a big Cartier divisors on Y with K_Y+B big. If $\mu(K_Y+B) \geq 1$ holds, then $\Phi_{|m(K_Y+B)|}$ is birational for every $m \geq a_n+b_n$. On the other hand, by the Hibert-scheme type argument, we obtain a positive constant $C(n)$ such that, for every a smooth projective n-fold Y and for every big Cartier divisor B on Y with K_Y+B big and $\mu(Y, K_Y+B) < 1$, the inequality $\mu(Y, K_Y+B) \geq C(n)$ holds. This completes Theorem 1.4 . □

4 Proof of Theorem 1.3

We shall prove Theorem 1.3. Let us fix positive integers n and κ. Fix a polarized manifold (X, L) of dimension n with $\kappa(X, K_X+L) = \kappa$. Let $f := \Phi_\infty : X \longrightarrow Y$ be the Iitaka fibration associated with K_X+L. Taking a suitable modification, we may also suppose that Y is smooth. (Notice that L is not ample but big because of the modification above in this case.) Since $\kappa(F, K_F+L|_F) = 0$ holds for a general fiber F of f, by [4, Lemma 3.3.2], we see that $K_F+L|_F$ is linearly equivalent to $\mathcal{O}_F$. Hence the reflexive sheaf:

$$B := f_*\mathcal{O}_X(K_{X/Y}+L)^{**}$$

is an invertible sheaf on Y. Note that L has a singular hermitian metric h_L with strictly positive curvature current since L is big. Then we define the singular hermitian metric h_B on B by

$$h_B(\sigma,\sigma) := \int_{X/Y} h_L \cdot \sigma \wedge \overline{\sigma},$$

where σ is a local section of B. By [32, Theorem 1.5] (see also [5, Theorem 0.1]), we see that h_B has strictly positive curvature current and so B is big. Further by definition, we have

$$H^0(X, \mathcal{O}_X(m(K_X+L))) \cong H^0(Y, \mathcal{O}_Y(m(K_Y+B))) \tag{4.1}$$

for every $m \geq 1$, and hence K_Y+B is big. Therefore, applying Theorem 1.4, we deduce that $\Phi_{|m(K_Y+B)|}$ is a birational map onto its image for every $m \geq m_\kappa$, where m_κ is the positive integer as in Theorem 1.4. As a result, letting $m_{n,\kappa} := m_\kappa$, we conclude that $\Phi_{|m(K_X+L)|}$ is birationally equivalent to f for every $m \geq m_{n,\kappa}$. This completes the proof of Theorem 1.3.

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